

Optimal decoherence control in non-Markovian open dissipative quantum systems

Wei Cui, Zai Rong Xi,^{*} and Yu Pan

Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China

(Received 30 December 2007; published 26 March 2008)

We investigate the optimal control problem for a non-Markovian open, dissipative quantum system. Optimal control using the Pontryagin maximum principle is specifically derived. The influences of ohmic reservoir with Lorentz-Drude regularization are numerically studied in a two-level system under the following three conditions: $\omega_0 \ll \omega_c$, $\omega_0 \approx \omega_c$, or $\omega_0 \gg \omega_c$, where ω_0 is the characteristic frequency of the quantum system of interest, and ω_c the cutoff frequency of the ohmic reservoir. The optimal control process shows its remarkable influences on the decoherence dynamics. The temperature is a key factor in the decoherence dynamics. We analyze the optimal decoherence control in high temperature, intermediate temperature, and low temperature reservoirs, respectively. It implies that designing some engineered reservoirs with the controlled coupling and state of the environment can slow down the decoherence rate and delay the decoherence time. Moreover, we compare the non-Markovian optimal decoherence control with the Markovian one and find that with non-Markovian the engineered artificial reservoirs are better than with the Markovian approximation in controlling the open, dissipative quantum system's decoherence.

DOI: [10.1103/PhysRevA.77.032117](https://doi.org/10.1103/PhysRevA.77.032117)

PACS number(s): 03.65.Yz, 03.67.Lx, 03.67.Pp

I. INTRODUCTION

The theory of open quantum systems deals with the systems that interact with their surrounding environments [1–7]. Such systems are of great interest, and these open quantum systems have been extensively studied since the origin of quantum theory [8]. Despite of the noticeable progresses in the theory, many fundamental difficulties still remain. One of the problems is decoherence (or loss of coherence) due to the interactions between system and environment. Recently, it received intense considerations in quantum information and quantum computation, where decoherence is regarded as a bottleneck to the construction of a quantum information processor [2,9,10]. The persistence of quantum coherence is relied on in quantum computer, quantum cryptography, and quantum teleportation, and it is also fundamental in understanding the quantum world for the interpretation that the emergence of the classical world from the quantum world can be seen as a decoherence process due to the interaction between system and environment.

Various methods have been proposed to reduce this unexpected effect, such as the quantum error-correction code [3,11], error-avoiding code [4,5], minimal decoherence model [6], bang-bang techniques [7] (where the system is constantly flipped to suppress decoherence), quantum Zeno effect (QZE) [12], and decoherence-free subspaces (DFS) [13]. Unfortunately, all of these schemes cannot suppress the unexpected effect successfully for accessorial conditions are needed. Altafini [14] pointed out that the irreversible decohering dynamics is uncontrollable under coherent control. Optimal control technique, which has been successfully studied in chemical systems [15–17] and classical systems [18], has been exploited to control the quantum decoherence [2,19,20], where an optimal control law was designed to ef-

fectively suppress decoherence effects in Markovian open quantum systems, dynamic coupling in the spin-boson model, and time optimal control, respectively. In this paper, we consider the optimal decoherence control problem in a non-Markovian quantum open system.

Markovian approximation is used under the assumption that the correlation time between the systems and environments is infinitely short [1,2,21]. For neglecting the memory effect, the Lindblad master equation has been built. However, in some cases, such as quantum Brownian motion (QBM) [22] and a two-level atom interacting with a thermal reservoir with Lorentzian spectral density [23], an exactly analytic description of the open quantum system dynamic is needed. Especially in high-speed communication the characteristic time scales become comparable with the reservoir correlation time, and in solid state devices memory effects are typically non-negligible. So it is necessary to extensively study the non-Markovian master equation. We briefly compare the non-Markovian dynamics (non-Markovian master equation) with the Markovian process (Markovian master equation) in the Appendix. For details one can refer to Gardiner's book [21] or/and Breuer's book [1].

In this paper the focus will be on the optimal decoherence control of a non-Markovian quantum system, particularly the simplest system possible, a two-level system governed by the time-convolutionless (TCL) equation. We determine control fields which minimize the cost functional suppressing the decoherence process by applying the Pontryagin maximum principle (PMP) in ohmic reservoir with Lorentz-Drude regularization in the following three conditions: $\omega_0 \ll \omega_c$, $\omega_0 \approx \omega_c$, and $\omega_0 \gg \omega_c$, where ω_0 is the characteristic frequency of the quantum system of interest and ω_c the cutoff frequency of ohmic reservoir. Thus $\omega_c \ll \omega_0$ implies that the spectrum of the reservoir does not completely overlap with the frequency of the system oscillator and $\omega_0 \gg \omega_c$ implies the converse case. With this it is possible to engineer different types of artificial reservoirs, and couple them to the system in a controlled way. We also compare our results with

^{*}zrxi@iss.ac.cn

no-control system evolution and the optimal control of the open system with Markovian approximation. The main result of the paper is that decoherence phenomenon can be successfully suppressed in the $\omega_0 \ll \omega_c$ case. Then this explores the coupling of the system to engineered reservoirs [24,25], in which the coupling and state of the environment are controllable. This may pave a way to the realization of the first basic elements of quantum computers.

The paper is organized as follows. We first introduce quantum decoherence and the quantum master equation for driven open quantum systems. In Sec. III we formulate the optimal control formalism and deduced PMP with a minimum cost functional. Moreover, we consider the non-Markovian two-level optimal control problem. In Sec. IV, we numerically analyze the optimal control of decoherence to the two-level system and analyze the difference between Markovian optimal control and non-Markovian optimal control from both the system time evolution and the power spectrum. Conclusions and prospective views are given in Sec. V.

II. MODELING THE QUANTUM DECOHERENCE CONTROL SYSTEM

Consider a quantum system S embedded in a dissipative environment B and interacting with a time-dependent classical external field, i.e., the control field. The total Hamiltonian has the general form

$$H_{tot} = H_0 + H_B + H_{int} = H_S + H_C(t) + H_B + H_{int}, \quad (1)$$

where H_S is the Hamiltonian of the system, $H_C(t)$ the Hamiltonian of the control field, H_B the bath, and H_{int} their interaction that is responsible for decoherence. The operators H_S and H_B act on \mathcal{H}_S and \mathcal{H}_B , respectively. The operator $H_C(t)$ contains a time-dependent external field to adjust the quantum evolution of the system. One of the central goals of the theoretical treatment is then the analysis of the dynamical behavior of the populations and coherences, which are given by the elements of the reduced density matrix, defined as

$$\rho_S(t) = \text{tr}_B[\rho_{tot}(t)], \quad (2)$$

where ρ_{tot} is the total density matrix for both the system and the environment, and tr_B the partial trace taken over the environment. The driven model consists of an N -level system interacting with a thermal bath in the presence of external control field [2,26], and the Hamiltonian is

$$H_C(t) = \sum_i u_i(t) H_i, \quad (3)$$

H_i is the control Hamiltonian adjusted by the control parameters $u_i(t)$, and $u_i(t)$ represents the control field. The Hamiltonian of the environment is assumed to be composed of harmonic oscillators with natural frequencies ω_i and masses m_i ,

$$H_B = \sum_{i=1}^N \left(\frac{p_i^2}{2m_i} + \frac{m_i}{2} x_i^2 \omega_i^2 \right), \quad (4)$$

where $(x_1, x_2, \dots, x_N, p_1, p_2, \dots, p_N)$ are the coordinates and their conjugate momenta, and the Planck constant \hbar is as-

signed to be 1. The interaction Hamiltonian between the system S and the environment B is assumed to be bilinear [1],

$$H_{int} = \alpha \sum_n A_n \otimes B_n. \quad (5)$$

The interaction Hamiltonian in the interaction picture therefore takes the form

$$H_{int}(t) = e^{i(H_S+H_B)t} H_{int} e^{-i(H_S+H_B)t} = \alpha \sum_n A_n(t) \otimes B_n(t), \quad (6)$$

where

$$A_n(t) = e^{iH_S t} A_n e^{-iH_S t},$$

$$B_n(t) = e^{iH_B t} B_n e^{-iH_B t}.$$

The effect of the environment on the dynamics of the system can be seen as an interplay between the dissipation and fluctuation phenomena; and it is the general environment that makes the quantum system loss of coherence (decoherence). In general the decoherence can be demonstrated as the interaction between the system and environment. Then the reduced density matrix of the system can evolve into the form

$$\rho_T \simeq \sum_n |c_n|^2 |a_n\rangle\langle a_n|, \quad (7)$$

which describes a statistical mixture of noninterfering states. Thus a commonly proposed way to analyze decoherence is by examining how the nondiagonal elements of the reduced density matrix evolve under the master equation.

In the present work, we shall concentrate on optimal control of the decoherence effect in an open quantum system. The kinetic equation of a strong coupling non-Markovian quantum system is the following exact time-convolutionless (TCL) form of the master equation:

$$\frac{d}{dt} \mathcal{P}\rho(t) = \sum_i u_i \tilde{\mathcal{K}}_i(t) \mathcal{P}\rho(t) + \mathcal{K}(t) \mathcal{P}\rho(t) + \mathcal{I}(t) \mathcal{Q}\rho(t_0), \quad (8)$$

with the time-local generator, called the TCL generator

$$\tilde{\mathcal{K}}_i(t) = e^{iH_S t} H_i e^{-iH_S t}, \quad \mathcal{K}(t) = \alpha \mathcal{P}\mathcal{L}(t) [1 - \Sigma(t)]^{-1} \mathcal{P}, \quad (9)$$

and the inhomogeneity

$$\mathcal{I}(t) = \alpha \mathcal{P}\mathcal{L}(t) [1 - \Sigma(t)]^{-1} g(t, t_0) \mathcal{Q}, \quad (10)$$

where $\Sigma(t)$ is the superoperator

$$\Sigma(t) = \alpha \int_{t_0}^t ds \mathcal{G}(t, s) \mathcal{Q}\mathcal{L}(s) \mathcal{P}\mathcal{G}(t, s).$$

For details see the Appendix and/or [1].

In order to facilitate the calculations, we will convert the differential equation (8) from the complex density matrix representation into the so-called coherent vector representation [2,14,27]. First, we choose an orthonormal basis of $N \times N$ matrices $\{(I, \Omega_j)\}_{j=1,2,\dots,N^2-1}$ with respect to the inner product $\langle X, Y \rangle = \text{tr}(X^\dagger Y)$, where I is the N -dimensional iden-

tity matrix and Ω_j are $N \times N$ Hermitian traceless matrices. In particular, the Hermitian density matrix ρ can be represented as $\rho = \frac{1}{N}I + \sum_i x_i \Omega_i$, where $\vec{x} = (x_1, x_2, \dots, x_{N^2-1})^T$ is a real (N^2-1) dimensional vector, called the coherent vector of ρ . This is the well-known Bloch vector representation of quantum systems. Thus the master equation (8) can be rewritten as a differential equation of the coherent vector:

$$\dot{x}(t) = O_0 x(t) + \sum_{i=1}^k u_i(t) O_i x(t) + L_1(t)x(t) + L_2(t), \quad (11)$$

with the initial condition

$$x(t_0) = x_0,$$

where $O_0, O_i \in \text{so}(N^2-1)$ are the adjoint representation matrices of $-iH_0, -iH_i$, respectively, and x_0 is the coherence vector of ρ_0 , and the term $L_1(t)x(t)$ represents the decoherence process, k is the number of control fields, and $\sum_i^k u_i H_i$ adjusts the quantum evolution such that the coherence is conserved.

III. QUANTUM OPTIMAL CONTROL PROBLEM

A. General formalism

As is well-known, the evolution of the state variable $x(t)$ governed by the master equation (11) depends not only on the initial state x_0 but also on the choice of the time-dependent control variable $u(t)$. Some earlier works to these control problems are listed in Refs. [28–30]. Especially, the exact result was considered of the quantum two-state dynamics driven by stationary non-Markovian discrete noise in [31]. In this section, we are going to suppress the unexpected effect of decoherence by the optimal control technique that wants to force the system evolving along some prescribed cohering trajectories. The target state chosen is the free evolution of the closed system:

$$\dot{\rho}_T(t) = -i[H_0, \rho_T(t)], \quad (12)$$

which is equivalent to $x^0(t) = e^{O_0(t-t_0)}x_0$. The cost functional is

$$J[u(t)] = \Psi[x(t_f), x^0(t_f)] + \int_{t_0}^{t_f} \Theta(x(t), x^0(t_f), u(t)) dt, \quad (13)$$

where the functional $\Psi[x(t_f), x^0(t_f)]$ represents distance between the system and objects at final time and the functional $\int_{t_0}^{t_f} \Theta(x(t), x^0(t_f), u(t)) dt$ accounts for the transient response with $\Theta(x(t), x^0(t_f), u(t)) \geq 0$.

The optimal control problem considered in this paper is to minimize the cost functional $J[u(t)]$ with some dynamical constraints. That is, our problem is

$$\min_{u \in \mathcal{U}_{[t_0, t_f]}} J[u(t)] = \Psi[x(t_f), x^0(t_f)] + \int_{t_0}^{t_f} \Theta(t) dt,$$

$$\dot{x}(t) = O_0 x(t) + \sum_{i=1}^k u_i(t) O_i x(t) + L_1(t)x(t) + L_2(t),$$

$$x(t_0) = x_0, \quad t \in [t_0, t_f], \quad (14)$$

where $\mathcal{U}_{[t_0, t_f]} = \{u(\cdot) : [t_0, t_f] \rightarrow \mathbb{R}^k\}$ and $u(\cdot)$ piecewise continuous.

Using the Pontryagin's maximum principle [18], the optimal solution to this problem is characterized by the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{\partial J}{\partial t} + \min_{u \in \mathcal{U}_{[t_0, t_f]}} \left\{ O_0 x(t) + \sum_{i=1}^k u_i(t) O_i x(t) + L_1(t)x(t) + L_2(t) + \Theta(x(t), x^0(t_f), u(t)) \right\} = 0,$$

$$J(x(t_f), t_f) = \Psi[x(t_f)]. \quad (15)$$

In general, it is usually difficult to obtain the analytic solution. Nevertheless, one can always have the numerical solution. To illustrate this method and give more insight, we will consider this problem for the non-Markovian two-level system in the following.

B. Optimal control of non-markovian two-level system

In this section we consider the decoherence of a two-level system whose controlled Hamiltonian is

$$H_0 = \frac{1}{2} \{ \omega_0 \sigma_z + u_x(t) \sigma_x + u_y(t) \sigma_y \}, \quad (16)$$

where σ_k with $k=x, y, z$ are the Pauli matrices; ω_0 is the transition frequency of the two-level system, and $u(t)$ is the modulation by the time-dependent external control field. In fact, the free Hamiltonian is $H_S = \frac{1}{2} \omega \sigma_z$. Then the control Hamiltonian can be described by σ_x, σ_y according to Cartan decomposition of the Lie algebra $\text{su}(2)$, which was discussed by Zhang *et al.* in detail [26]. This is the standard model for atom-field interaction [32–35].

In our two-level system the assumed bilinear interaction between the system S and the environment B can be written as

$$H_{int} = \alpha (\sigma_+ \otimes B + \sigma_- \otimes B^\dagger) \quad \text{with } B = \sum_i k_i a_i, \quad (17)$$

where $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$, the raising and lowering operator, respectively, k_i is the coupling constant between the spin coordinate and the i th environment oscillator, and a_i is the annihilation operator of the i th harmonic oscillators of the environment. The coupling constants enter the spectral density function $J(\omega)$ of the environment defined by

$$J(\omega) = \frac{\pi}{2} \sum_i \frac{k_i}{m_i \omega_i} \delta(\omega - \omega_i) \quad (18)$$

and the index i labels the different field models of the reservoir with frequencies ω_i . In the continuum limit the spectral density has the form

$$J(\omega) = \eta\omega \left(\frac{\omega}{\omega_c}\right)^{n-1} \exp\left(-\frac{\omega}{\omega_c}\right), \quad (19)$$

where ω_c is a cutoff frequency, and η a dimensionless coupling constant. The environment is classified as ohmic, sub-ohmic, and sup-ohmic according to $n=1$, $0 < n < 1$, and $n > 1$, respectively [36–38].

In this case, the open quantum system can be written as follows [1,39,40]:

$$\begin{aligned} \dot{\rho}_S = & -\frac{i}{2}\omega_0[\sigma_z, \rho_S] - \frac{i}{2}u_x(t)[\sigma_x, \rho_S] - \frac{i}{2}u_y(t)[\sigma_y, \rho_S] \\ & + \frac{\Delta(t) + \gamma(t)}{2}\{2\sigma_- \rho_S \sigma_+ - \sigma_+ \sigma_- \rho_S - \rho_S \sigma_+ \sigma_-\} \\ & + \frac{\Delta(t) - \gamma(t)}{2}\{2\sigma_+ \rho_S \sigma_- - \sigma_- \sigma_+ \rho_S - \rho_S \sigma_- \sigma_+\}. \end{aligned} \quad (20)$$

For convenience we map the density matrix of the two-level system onto the Bloch vector $x(t) = [x_1(t), x_2(t), x_3(t)]^T \in \mathbb{R}^3$ defined by $x(t) = \text{Tr}[\sigma \rho(t)]$, which implies that

$$\begin{aligned} x_1(t) & \equiv \rho_{01}(t) + \rho_{10}(t), \\ x_2(t) & \equiv i[\rho_{01}(t) - \rho_{10}(t)], \\ x_3(t) & \equiv \rho_{00}(t) - \rho_{11}(t). \end{aligned} \quad (21)$$

Then the explicit equations of motion for the components of the Bloch vector read

$$\begin{aligned} \dot{x}_1(t) & = -\Delta(t)x_1(t) - \omega_0 x_2(t) + x_3(t)u_y(t), \\ \dot{x}_2(t) & = \omega_0 x_1(t) - \Delta(t)x_2(t) - x_3(t)u_x(t), \\ \dot{x}_3(t) & = -2\Delta(t)x_3(t) - 2\gamma(t) + x_2(t)u_x(t) - x_1(t)u_y(t), \end{aligned} \quad (22)$$

where the expressions for the relevant time dependent coefficients, up to the second order in the system-reservoir coupling constant, are given by [1,39]

$$\begin{aligned} \Delta(t) & = \int_0^t d\tau k(\tau) \cos(\omega_0 \tau), \\ \gamma(t) & = \int_0^t d\tau \mu(\tau) \sin(\omega_0 \tau), \end{aligned} \quad (23)$$

with

$$k(\tau) = 2 \int_0^\infty d\omega J(\omega) \coth[\hbar\omega/2k_B T] \cos(\omega\tau),$$

$$\mu(\tau) = 2 \int_0^\infty d\omega J(\omega) \sin(\omega\tau) \quad (24)$$

being the noise and the dissipation kernels, respectively. Equation (22) can be written compactly as

$$\dot{x}(t) = A(t)x(t) + B(t), \quad (25)$$

where

$$A(t) = \begin{pmatrix} -\Delta(t) & -\omega_0 & u_y(t) \\ \omega_0 & -\Delta(t) & -u_x(t) \\ -u_y(t) & u_x(t) & -2\Delta(t) \end{pmatrix}$$

and

$$B(t) = \begin{pmatrix} 0 \\ 0 \\ -2\gamma(t) \end{pmatrix}.$$

Let the ohmic spectral density with a Lorentz-Drude cut-off function

$$J(\omega) = \frac{2\gamma_0}{\pi} \omega \frac{\omega_c^2}{\omega_c^2 + \omega^2}, \quad (26)$$

where γ_0 is the frequency-independent damping constant and it is usually assumed to be 1. ω is the frequency of the bath, and ω_c is the high-frequency cutoff. For this type of spectral density the bath correlations can be determined analytically as

$$k(\tau) = 4k_B T \omega_c^2 \sum_{n=-\infty}^{+\infty} \frac{\omega_c e^{-\omega_c |\tau|} - |\nu_n| e^{-|\nu_n| |\tau|}}{\omega_c^2 - \nu_n^2}, \quad (27)$$

where $\nu_n = 2\pi n k_B T$ and

$$\mu(\tau) = 2\hbar \omega_c^2 e^{-\omega_c |\tau|} \text{sgn}(\tau). \quad (28)$$

Then the analytic expression for the dissipation coefficient $\gamma(t)$ appearing in Eq. (23) is

$$\gamma(t) = \frac{\alpha^2 \omega_0 r^2}{1+r^2} [1 - e^{-r\omega_0 t} \cos(\omega_0 t) - r e^{-r\omega_0 t} \sin(\omega_0 t)], \quad (29)$$

and the closed analytic expression for $\Delta(t)$ is [40]

$$\begin{aligned} \Delta(t) = & \alpha^2 \omega_0 \frac{r^2}{1+r^2} \left(\coth(\pi r_0) - \cot(\pi r_c) e^{-\omega_c t} [r \cos(\omega_0 t) - \sin(\omega_0 t)] + \frac{1}{\pi r_0} \cos(\omega_0 t) [\bar{F}(-r_c, t) + \bar{F}(r_c, t) - \bar{F}(ir_0, t) - \bar{F}(-ir_0, t)] \right. \\ & \left. - \frac{1}{\pi} \sin(\omega_0 t) \left\{ \frac{e^{-\nu_1 t}}{2r_0(1+r_0^2)} [(r_0 - i)\bar{G}(-r_0, t) + (r_0 + i)\bar{G}(r_0, t)] + \frac{1}{2r_c} [\bar{F}(-r_c, t) - \bar{F}(r_c, t)] \right\} \right), \end{aligned} \quad (30)$$

where $r_0 = \omega_0/2\pi k_B T$, $r_c = \omega_c/2\pi k_B T$, $r = \omega_c/\omega_0$, and

$$\bar{F}(x,t) \equiv {}_2F_1(x, 1, 1+x, e^{-\nu t}), \quad (31)$$

$$\bar{G}(x,t) \equiv {}_2F_1(2, 1+x, 2+x, e^{-\nu t}). \quad (32)$$

${}_2F_1(a, b, c, z)$ is the hypergeometric function and takes the form

$$\begin{aligned} {}_2F_1(a, b, c, z) &= 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)z^2}z^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!}, \end{aligned}$$

where $(a)_n$ is a Pochhammer symbol. Under the high temperature limit, we have

$$\Delta(t) = 2\alpha^2 \kappa T \frac{r^2}{1+r^2} \left\{ 1 - e^{-r\omega_0 t} \left[\cos(\omega_0 t) - \frac{1}{r} \sin(\omega_0 t) \right] \right\}. \quad (33)$$

In the following we consider the optimal control formalism of our two-level system. For simplicity we define the cost functional as

$$J[u(t)] = \int_{t_0}^{t_f} \{ [x(t) - x^0(t)]^2 + \theta u^T(t)u(t) \} dt, \quad (34)$$

where $\theta > 0$ is a weighting factor used to achieve a balance between the tracking precision and the control constraints. The corresponding Hamiltonian function is

$$\begin{aligned} \mathcal{H}[x(t), u(t), \lambda(t), t] &= \{ [x(t) - x^0(t)]^2 + \theta u^T(t)u(t) \} + \lambda(t)^T [A(t)x(t) + B(t)] \\ &= \{ [x_1(t) - x_1^0(t)]^2 + [x_2(t) - x_2^0(t)]^2 + [x_3(t) - x_3^0(t)]^2 + \theta(u_1^2(t) + u_2^2(t)) \} \\ &\quad + \lambda_1(t) [-\Delta(t)x_1(t) - \omega_0 x_2(t) + x_3(t)u_y(t)] + \lambda_2(t) [\omega_0 x_1(t) - \Delta(t)x_2(t) - x_3(t)u_x(t)] \\ &\quad + \lambda_3(t) [-2\Delta(t)x_3(t) - 2\gamma(t) + x_2(t)u_x(t) - x_1(t)u_y(t)], \end{aligned}$$

where $\lambda(t) = [\lambda_1(t), \lambda_2(t), \lambda_3(t)]^T$ is the so-called Lagrange multiplier and $x^0(t) = [x_1^0(t), x_2^0(t), x_3^0(t)]$ is the target trajectory defined by $\dot{\rho} = -\frac{i}{2}[H_0, \rho]$. It is easy to see that $x^0(t) = (x_1^0 \cos \omega t - x_2^0 \sin \omega t, x_1^0 \sin \omega t + x_2^0 \cos \omega t, x_3^0)$. The optimal solution can be solved by the following differential equation with two-sided boundary values:

$$\dot{x}^*(t) = \frac{\partial \mathcal{H}}{\partial \lambda} = A(t)x(t) + B(t),$$

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x} = -2[x(t) - x^0(t)] - A(t)^T \lambda(t),$$

$$x^*(0) = x_0,$$

$$\lambda(t_f) = 0, \quad (35)$$

together with

$$\left. \frac{\partial \mathcal{H}}{\partial u} \right|_* = \frac{\partial \mathcal{H}[x^*(t), u^*(t), \lambda(t), t]}{\partial u} = 0,$$

$$\left. \frac{\partial^2 \mathcal{H}}{\partial u^2} \right|_* = \frac{\partial^2 \mathcal{H}[x^*(t), u^*(t), \lambda(t), t]}{\partial u^2} \leq 0, \quad (36)$$

which implies that

$$u_x(t) = \frac{1}{2\theta} \{ \lambda_2 x_3 - \lambda_3 x_2 \},$$

$$u_y(t) = \frac{1}{2\theta} \{ \lambda_3 x_1 - \lambda_1 x_3 \}. \quad (37)$$

The minimum principle requires the solution of the complicated nonlinear equations. When there is one and only one solution $\{x(t), \lambda(t)\}$ it is the required optimal solution [18]. In general, it is difficult to obtain the analytic solution, if possible existence, to the above optimal control problem. So numerical demonstration to this problem will be considered in the next section.

IV. NUMERICAL DEMONSTRATION AND DISCUSSIONS

In this section, we use the formalism of the preceding section to determine the optimal control of the decoherence. Though the spin-bath models of real systems are expected to be more complicated than the two-level Hamiltonians considered here, we study the system in various aspects to understand the effect of this simple system on the decoherence control.

In our simulations, the system parameters are chosen as follows: $x(0) = (\frac{\sqrt{3}}{2}, \frac{-\sqrt{2}}{4}, \frac{-\sqrt{2}}{4})$, strong coupling constant $\alpha^2 = 0.01$, weighting factor $\theta = 1$, and $\omega_0 = 1$ as the norm unit. Moreover, we regard the temperature as a key factor in the decoherence process. For high temperature $k_B T = 300\omega_0$, intermediate temperature $k_B T = 3\omega_0$, and low temperature $k_B T = 0.3\omega_0$. Another reservoir parameter playing a key role in the dynamics of the system is the ratio $r = \omega_c/\omega_0$ between the reservoir cutoff frequency ω_c and the system oscillator frequency ω_0 . As we will see in this section, by varying these

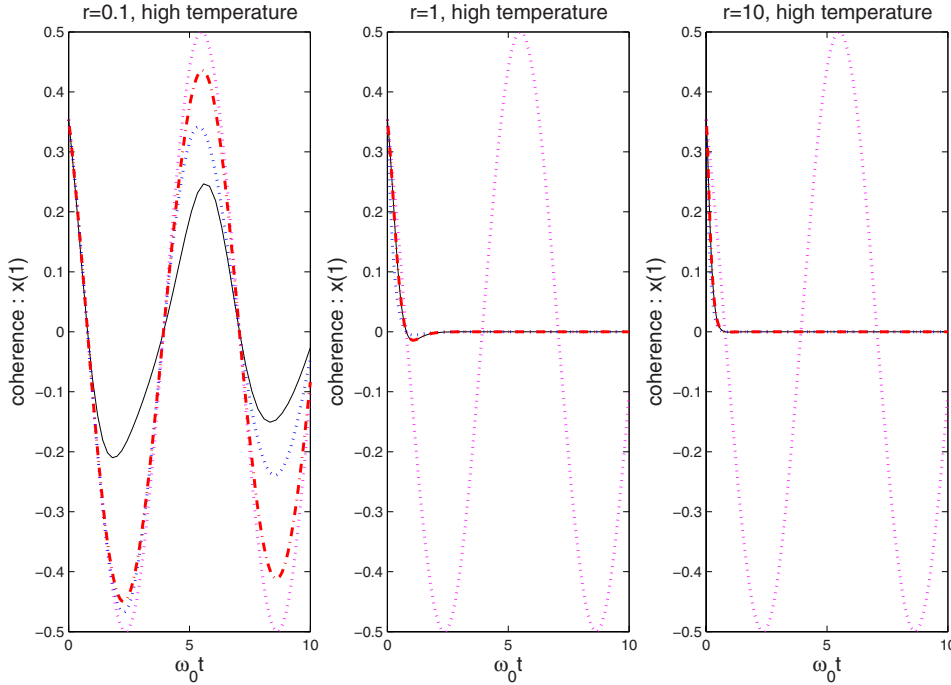


FIG. 1. (Color online) Surviving coherence in off-diagonal matrix elements vs time t [Eq. (35)] under a high temperature environment, without control action (black solid line), Markovian optimal control (blue dashed line), non-Markovian optimal control (red dotted line), and target trajectory (crimson dash-dotted line) at $r=0.1$, $r=1$, and $r=10$, respectively.

two parameters $k_B T$ and $r = \omega_c / \omega_0$, both the time evolution and the optimal control of the open system vary prominently from Markovian to non-Markovian.

A. High temperature reservoir

For high reservoir temperature, diffusion coefficient $\Delta(t)$ (30) has the approximation form (33), which plays a dominant role since $\Delta(t) \gg \gamma(t)$. Note that, for time t large enough, the coefficients $\Delta(t)$ and $\gamma(t)$ can be approximated by their Markovian stationary values $\Delta_M = \Delta(t \rightarrow \infty)$ and $\gamma_M = \gamma(t \rightarrow \infty)$. From Eqs. (29) and (30) we have

$$\gamma_M = \frac{\alpha^2 \omega_0 r^2}{1 + r^2}, \quad (38)$$

and

$$\Delta_M = \alpha^2 \omega_0 \frac{r^2}{1 + r^2} \coth(\pi r_0). \quad (39)$$

Then, under high temperature,

$$\coth(\pi r_0) \approx 1 + \frac{1}{\pi r_0} \approx \frac{2kT}{\omega_0}, \quad (40)$$

$$\Delta_M^{HT} = 2\alpha^2 kT \frac{r^2}{1 + r^2}.$$

Inserting Eqs. (38) and (40) into Eqs. (35) one can easily get the Markovian optimal decoherence control.

Figure 1 shows optimal control of decoherence for $r \ll 1$, $r=1$, and $r \gg 1$ in the high temperature reservoir. All of these contain a solid line for free evolution, a dashed line for Markovian optimal control, a dotted line for non-Markovian optimal control, and a dash-dotted line for target trajectory. We

can see clearly that the decoherence can be controlled perfectly in the $r \ll 1$ reservoir. From Fig. 2 we can see that the decoherence time τ_D can be delayed for a long time and its amplitude amplified heavily with the non-Markovian control. On the other hand, Fig. 3 shows that the non-Markovian

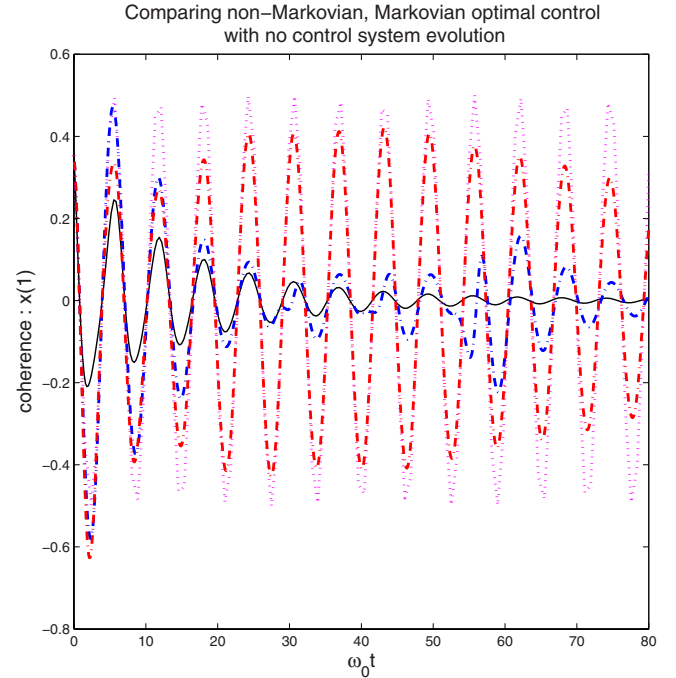


FIG. 2. (Color online) Comparing Markovian optimal control, non-Markovian optimal control with no control under high temperature environment, without control action (black solid line), Markovian optimal control (blue dashed line), non-Markovian optimal control (red dotted line), and target trajectory (crimson dash-dotted line) at $r=0.1$.

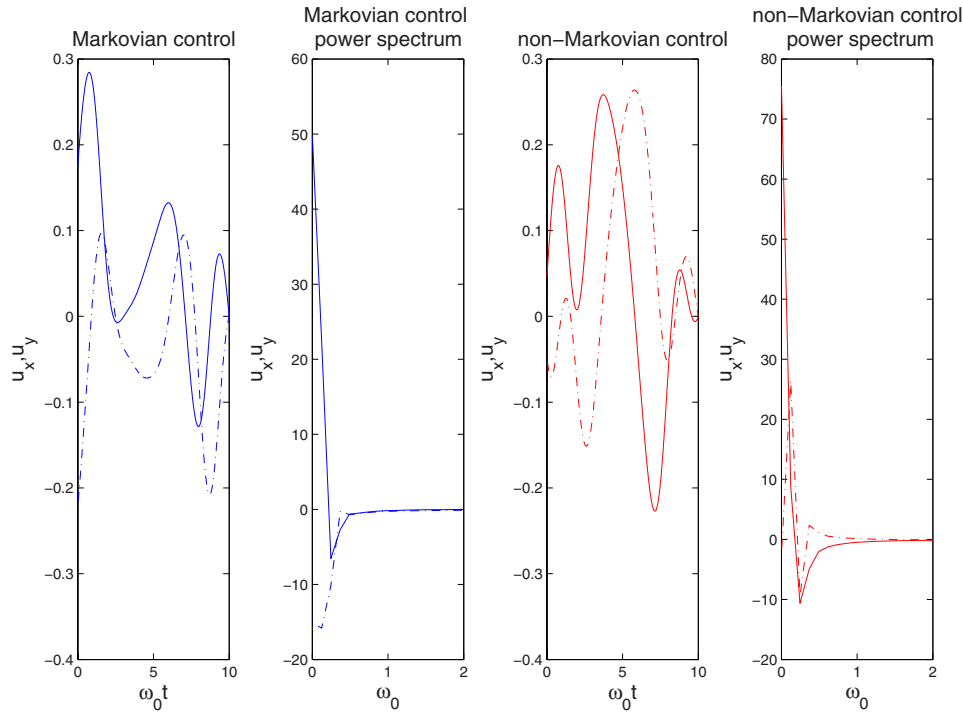


FIG. 3. (Color online) Comparing Markovian optimal control with non-Markovian optimal control in a $r=0.1$ high temperature reservoir. Markovian optimal control u_x (blue solid line), u_y (blue dash-dotted line) and non-Markovian optimal control u_x (red solid line), u_y (red dash-dotted line).

control field is changed more rapidly than the Markovian control field and the frequency of non-Markovian is more plenty than the Markovian, which helps to understand that the non-Markovian case is done better than the Markovian case and implies that it is necessary to consider the non-Markovian case.

From Fig. 1 we can also see that either Markovian or non-Markovian optimal control cannot do well when $r=1$ or 10. As we discussed before, diffusion is always dominant under the high temperature. In the case $r \ll 1$, $\Delta(t) > 0$ is always true [40]. However, Maniscalco *et al.* [40] showed

that if $r > 0.27$ the diffusion coefficient $\Delta(t) < 0$, and the system becomes non-Lindblad. It implies that the environment induced fluctuations will be large enough. So our control field is negligible when comparing with the high-frequency harmonic oscillators of the reservoir.

B. Lower temperature reservoir

As temperature decreases, the amplitude of $\Delta(t)$ becomes smaller and smaller and $\gamma(t)$ becomes larger and larger, which is not negligible anymore. There exists a time which

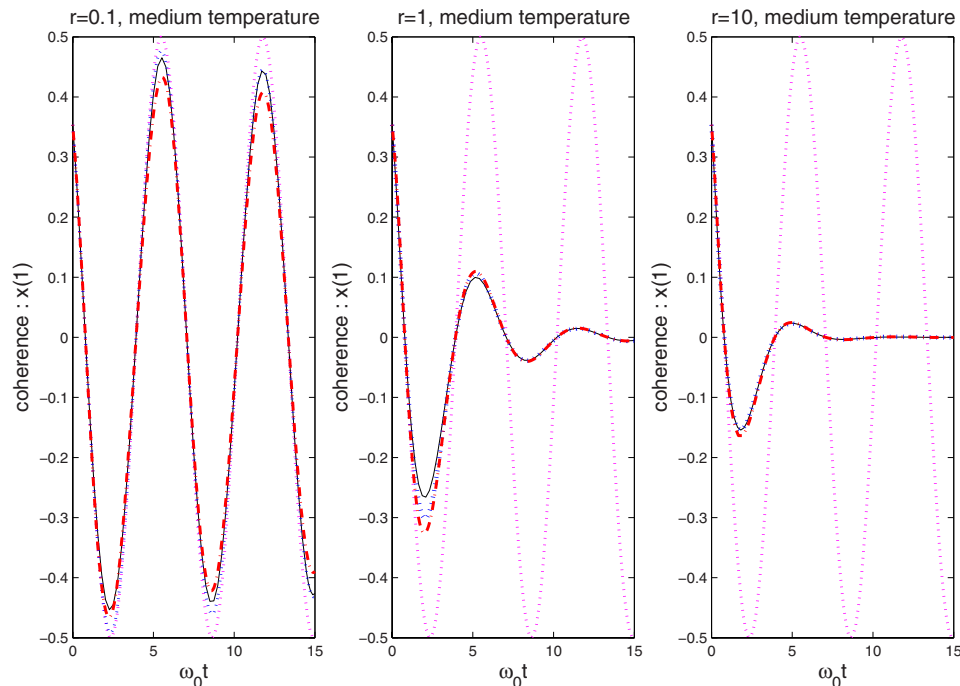


FIG. 4. (Color online) Surviving coherence in off-diagonal matrix elements vs time t [Eq. (35)] under medium temperature environment, without control action (black solid line), Markovian optimal control (blue dashed line), non-Markovian optimal control (red dotted line), and target trajectory (crimson dash-dotted line) at $r=0.1$, $r=1$, and $r=10$, respectively.

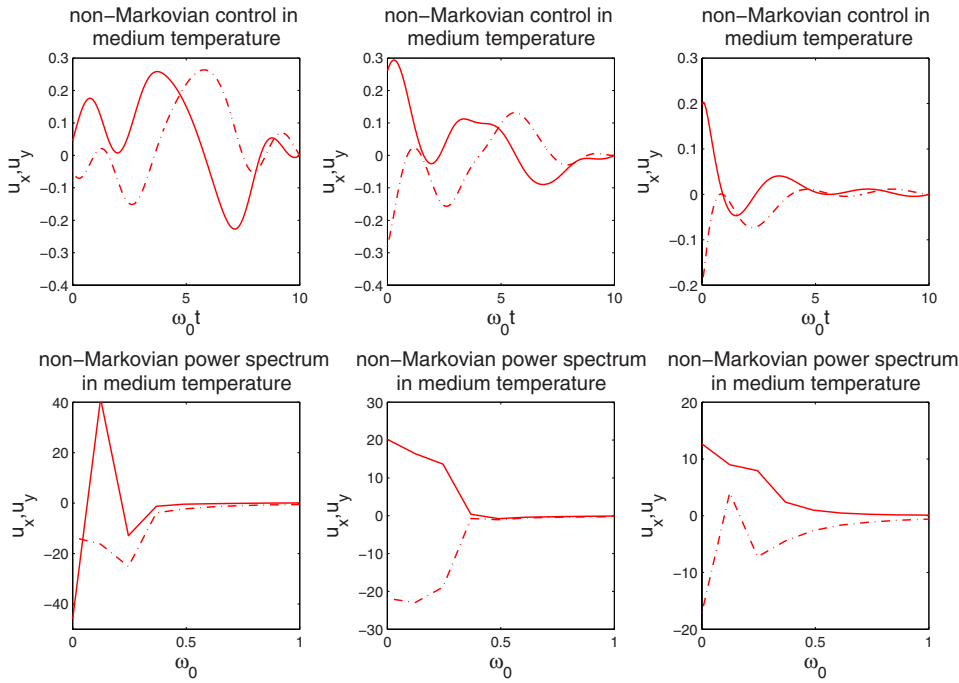


FIG. 5. (Color online) Non-Markovian optimal controls and their power spectrum in medium temperature reservoir for $r=0.1$, $r=1$, and $r=10$, respectively. Non-Markovian optimal control u_x (red solid line), u_y (red dash-dotted line).

relates to both the temperature and the ratio such that after the time the combination of dissipation and diffusion coefficient $\Delta(t) - \gamma(t) < 0$, which changes the properties of the control system (35).

Figure 4 shows the non-Markovian optimal control and Fig. 5 their power spectrum for intermediate temperature, and Figs. 6 and 7 for low temperature. At intermediate temperature the non-Markovian optimal control plays a little role especially in Figs. 4(b) and 4(c). Note that in Figs. 4(a) and 6(a) the free evolution is with little decoherence. We note that the optimal control does well at low temperature in Fig. 6. In Fig. 6, both Markovian and non-Markovian play an

important role in controlling the decoherence in both $r=1$ and 10. They can make the quantum coherence persistent for a long time.

C. Engineering reservoirs

During the last two decades, great advances in laser cooling and trapping experimental techniques have made it possible to trap a single ion and cool it down to very low temperature. These cold trapped ions are the favorite candidates for a physical implementation of quantum computers and realization of the quantum cryptography and quantum tele-

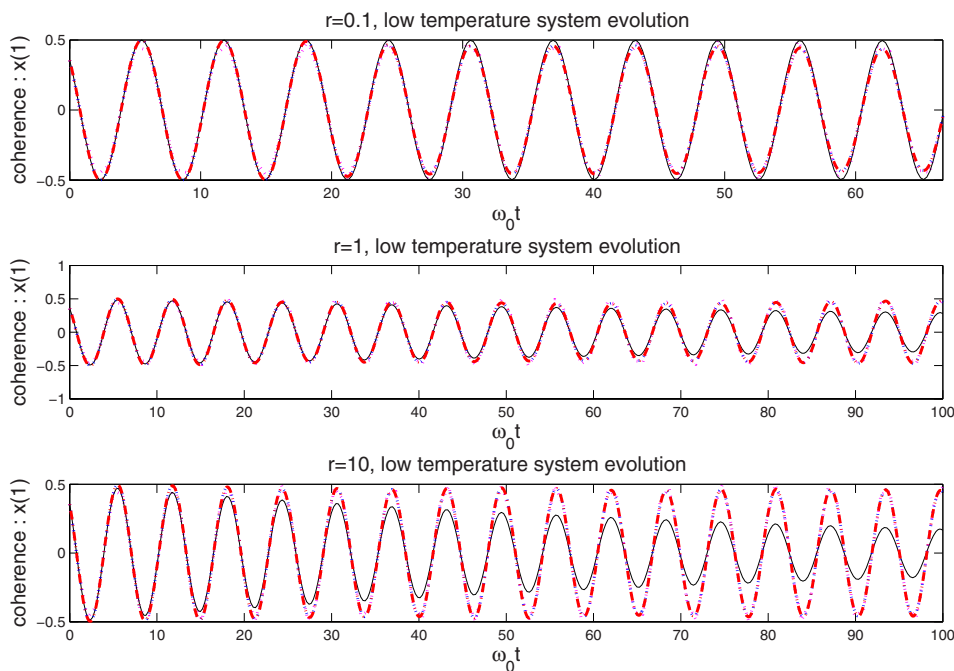


FIG. 6. (Color online) Surviving coherence in off-diagonal matrix elements vs time t [Eq. (35)] under low temperature environment, without control action (black solid line), Markovian optimal control (blue dashed line), non-Markovian optimal control (red dotted line), and target trajectory (crimson dash-dotted line) at $r=0.1$, $r=1$, and $r=10$, respectively.

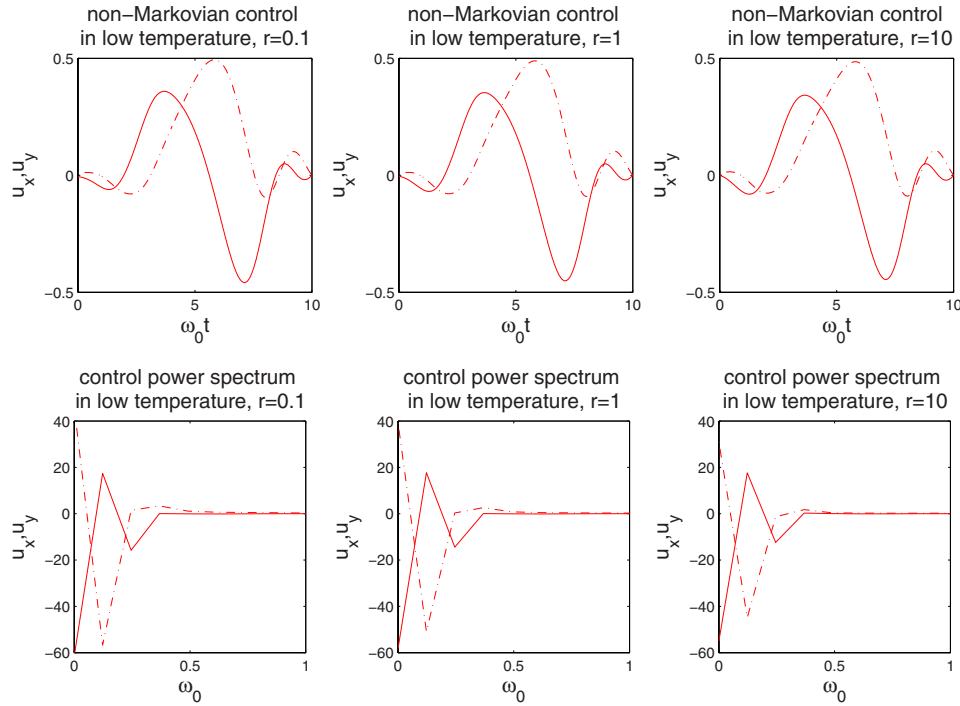


FIG. 7. (Color online) Non-Markovian optimal controls and their power spectrum in a low temperature reservoir for $r=0.1$, $r=1$, and $r=10$, respectively. Non-Markovian optimal control u_x (red solid line), u_y (red dash-dotted line).

portation. All of these rely on the persistence of quantum coherence. References [24,25] are the recent experimental procedures for engineering artificial reservoirs. They showed how to couple properly engineered reservoirs with the trapped atomic ion's harmonic motion. They measured the decoherence of superpositions of coherent states and two-Fock-state superpositions in the engineering artificial reservoirs. Several types of engineering artificial reservoirs are simulated, e.g., a high-temperature amplitude reservoir, a zero-temperature amplitude reservoir, and a high-temperature phase reservoir. From above discussions we find that our optimal decoherence control fields do well in the engineering artificial reservoirs.

Table I shows the controllable property of a non-Markovian open, dissipative quantum system. When $r \ll 1$ the system free evolution is with little decoherence at low and intermediate temperature and our optimal control plays an important role in controlling the decoherence phenomenon at high temperature. Moreover, when $r \gg 1$ and $r=1$ our optimal control also plays an important role in controlling the decoherence phenomenon at low temperature. They indicate that these engineered reservoirs could be designed that

the coupling and state of the environment can be controlled to slow down the decoherence rate and delay decoherence time.

V. CONCLUSIONS

In the present work, we have studied the optimal control of the decoherence for the non-Markovian open quantum system. In the general formalism we proposed the optimal control problem and derived the corresponding Hamilton-Jacobi-Bellman equation. Usually this kind of problem is difficult to be analytically solved. Then we considered this problem in the non-Markovian two-level system. Through transforming its master equation into the Bloch vector representation we obtained the corresponding differential equation with two-sided boundary values.

Finally, we numerically studied the non-Markovian decoherence control for three different conditions, i.e., $\omega_0 \ll \omega_c$, $\omega_0 \approx \omega_c$, and $\omega_0 \gg \omega_c$ in the ohmic environment whose spectral density is with a Lorentz-Drude cutoff function. Our numerical results indicated that the decoherence dynamics behaves differently for the different environmental condition which leads to significant distinctness in the time dependent behavior of the dissipation function $\gamma(t)$ and $\Delta(t)$. We regarded temperature as a key factor in the decoherence effect and showed that the decoherence cannot be controlled effectively in high temperature for both the Markovian and non-Markovian. Comparing with the Markovian approximation we believed that it is necessary to consider the non-Markovian quantum system. Most of all, we analyzed the short time, moderate time, and long time decoherence control behaviors for $r=0.1$, which implies $\omega_c \ll \omega_0$. In this case the decoherence can be controlled effectively, which may

TABLE I. Controllability.

$r \setminus T$	Low T	Med T	High T
$r=0.1$	Slow decay	Slow decay	Controllable(non)
$r=1$	Controllable	Uncontrollable	Uncontrollable
$r=10$	Controllable	Uncontrollable	Uncontrollable

indicate that the decoherence rate can be slowed down and decoherence time can be delayed through designing some engineered reservoirs proposed by Myatt *et al.* [24] and Turcotte *et al.* [25].

ACKNOWLEDGMENTS

This research was supported by the National Natural Science Foundation of China (Grants No. 60774099 and No. 60221301) and by the Chinese Academy of Sciences (Grant No. KJCX3-SYW-S01). The first author would like to thank Dr. J. Zhang for many fruitful discussions.

APPENDIX: COMPARING THE NON-MARKOVIAN DYNAMICS WITH THE MARKOVIAN DYNAMICS

1. Quantum Markovian process and Markovian master equation

Quantum Markovian process or Markovian approximation is widely used in an open quantum system, typically in interaction of radiation with matter (weak coupling); quantum optics and cavity-QED (weak damping); quantum decoherence; quantum Brownian motion (high temperatures); quantum information; quantum error correction; stochastic unravelling (Monte Carlo simulations); laser cooling (Lévy statistics of quantum jumps), and so on. The essence of the quantum Markovian process contains three assumptions.

(i) The initial factorization ansatz (Feynman-Vernon approximation). At time $t=0$ the bath B is in thermal equilibrium and uncorrelated with the system S ,

$$\rho_{tot}(0) = \rho_S(0) \otimes \rho_B. \quad (\text{A1})$$

(ii) Weak system-bath interaction (Born approximation).

(iii) Markovian approximation. The relaxation time τ_B of the heat bath is much shorter than the time scale τ_R ($\tau_B \ll \tau_R$) over which the state of the system varies appreciably.

Then it induced the dynamical map Φ_t :

$$\rho_S(0) \rightarrow \rho_S(t) = \Phi_t \rho_S(0) = \text{tr}_B \{ U_t [\rho_S(0) \otimes \rho_B] U_t^\dagger \}. \quad (\text{A2})$$

With some conditions, like completely positive, Hermiticity, and trace preservation we get a quantum dynamical semigroup: $\Phi_t = \exp[\mathcal{L}t]$, which implies the Markovian master equation:

$$\frac{d}{dt} \rho_S(t) = \mathcal{L} \rho_S(t), \quad (\text{A3})$$

where the generator of time evolution is in Lindblad form:

$$L \rho_S(t) = -\frac{i}{\hbar} [H_S, \rho_S] + \sum_i \gamma_i \left[a_i \rho_S a_i^\dagger - \frac{1}{2} \{ a_i^\dagger a_i, \rho_S \} \right]. \quad (\text{A4})$$

2. Non-Markovian dynamics and non-markovian master equation

A non-Markovian dynamics system is not a new research problem, but recently it received considerable consideration [40–43]. Comparing with the Markovian dynamics it has three properties: (i) semigroup property violated: slow decay of correlations, strong memory effects; (ii) initial correlations: classically correlated or entangled initial states; and (iii) strong couplings and low temperatures, with which we can study the short-time behavior and exact evolution of quantum decoherence. With the help of these three properties we can derive effective equations (master equations). As far as we know, there are two ways to derive the master equation. One is called the path-integral method by Halliwell *et al.* [44], Hu *et al.* [45], Ford *et al.* [46], and Karrlein *et al.* [47], the other is the projection operator method by Breuer [1,41–43].

The projection operator method is also called the Nakajima-Zwanzig projection. The basic idea of the technique is to define a map \mathcal{P} as

$$\mathcal{P} \rho = \text{tr}_B \{ \rho \} \otimes \rho_B, \quad (\text{A5})$$

where ρ_B is a fixed environment state and the map \mathcal{P} is a projection superoperator acting on operators, i.e., $\mathcal{P}^2 = \mathcal{P}$. Its complementary projection is

$$\mathcal{Q} = \mathcal{I} - \mathcal{P}, \quad (\text{A6})$$

where \mathcal{I} is the identity map. Thus the Nakajima-Zwanzig equation can be derived [43]:

$$\frac{d}{dt} \mathcal{P} \rho(t) = \int_0^t ds K(t,s) \mathcal{P} \rho(s) + \mathcal{I}(t) \mathcal{Q} \rho(0), \quad (\text{A7})$$

where $K(t,s)$ is the memory kernel. To second order in the coupling constant the general form of the master equation can be approximated by

$$\frac{d}{dt} \mathcal{P} \rho(t) = \mathcal{K}(t) \mathcal{P} \rho(t) + \mathcal{I}(t) \mathcal{Q} \rho(0). \quad (\text{A8})$$

In general, the TCL generator is

$$\begin{aligned} \mathcal{K}(t) \rho_S = & -\frac{i}{\hbar} [H_S(t), \rho_S] + \sum_i [C_i(t) \rho_S D_i^\dagger(t) + D_i(t) \rho_S C_i^\dagger(t)] \\ & - \frac{1}{2} \sum_i \{ D_i^\dagger(t) C_i(t) + C_i^\dagger D_i(t), \rho_S \}, \end{aligned} \quad (\text{A9})$$

where $C_i(t) \neq D_i(t)$, which means that it is not in the Lindblad form [1,21].

- [1] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
- [2] J. Zhang, C. W. Li, R. B. Wu, T. J. Tarn, and X. S. Liu, *J. Phys. A* **38**, 6587 (2005).
- [3] S. Lloyd and Jean-Jacques E. Slotine, *Phys. Rev. Lett.* **80**, 4088 (1998).
- [4] P. Zanardi and M. Rasetti, *Phys. Rev. Lett.* **79**, 3306 (1997).
- [5] I. L. Chuang and Y. Yamamoto, *Phys. Rev. A* **52**, 3489 (1995).
- [6] R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. A* **65**, 062101 (2002).
- [7] P. Zanardi, *Phys. Rev. A* **63**, 012301 (2000).
- [8] J. Von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University, Princeton, NJ, 1955).
- [9] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
- [10] M. B. Mensky, *Quantum Measurements and Decoherence: Models and Phenomenology* (Kluwer Academic, Dordrecht, 2000).
- [11] P. W. Shor, *Phys. Rev. A* **52**, R2493 (1995).
- [12] L. Viola and S. Lloyd, *Phys. Rev. A* **58**, 2733 (1998).
- [13] B. Elattari and S. A. Gurvitz, *Phys. Rev. Lett.* **84**, 2047 (2000).
- [14] C. Altafini, *J. Math. Phys.* **44**, 2357 (2003).
- [15] H. Rabitz, R. de ViVie-Riedle, M. Motzkus, and K. Kompa, *Science* **288**, 824 (2000).
- [16] R. J. Levis, G. M. Menkir, and H. Rabitz, *Science* **292**, 709 (2001).
- [17] H. Rabitz, *Science* **299**, 525 (2003).
- [18] V. F. Krotov, *Global Methods in Optimal Control Theory* (Marcel Dekker, New York, 1996).
- [19] H. Jirari and W. Pötz, *Phys. Rev. A* **74**, 022306 (2006).
- [20] D. Sugny, C. Kontz, and H. R. Jauslin, *Phys. Rev. A* **76**, 023419 (2007).
- [21] C. W. Gardiner and P. Zoller, *Quantum Noise*, 2nd ed. (Springer-Verlag, Berlin, 2000).
- [22] W. H. Zurek, *Rev. Mod. Phys.* **75**, 715 (2003).
- [23] B. M. Garraway, *Phys. Rev. A* **55**, 2290 (1997).
- [24] C. J. Myatt, B. E. King, Q. A. Turchette, C. A. Sackett, D. Kielpinski, W. M. Itano, C. Monroe, and D. J. Wineland, *Nature (London)* **403**, 269 (2000).
- [25] Q. A. Turchette, C. J. Myatt, B. E. King, C. A. Sackett, D. Kielpinski, W. M. Itano, C. Monroe, and D. J. Wineland, *Phys. Rev. A* **62**, 053807 (2000).
- [26] J. Zhang, R. B. Wu, C. W. Li, T. J. Tarn, and J. W. Wu, *Phys. Rev. A* **75**, 022324 (2007).
- [27] K. Blum, *Density Matrix Theory and Applications* (Plenum Press, New York, 1981).
- [28] M. Thorwart, L. Hartmann, I. Goychuk, and P. Hänggi, *J. Mod. Opt.* **47**, 2905 (2000).
- [29] M. Grifoni and P. Hänggi, *Phys. Rep.* **304**, 229 (1998).
- [30] J. S. Shao, C. Zerbe, and P. Hänggi, *Chem. Phys.* **235**, 81 (1998).
- [31] I. Goychuk and P. Hänggi, *Chem. Phys.* **324**, 160 (2006).
- [32] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, England, 1997).
- [33] P. Meystre, *Atom Optics* (Springer-Verlag, New York, 2001).
- [34] C. Anastopoulos and B. L. Hu, *Phys. Rev. A* **62**, 033821 (2000).
- [35] S. Shresta, C. Anastopoulos, A. Dragulescu, and B. L. Hu, *Phys. Rev. A* **71**, 022109 (2005).
- [36] U. Weiss, *Quantum Dissipative System* (World Scientific, Singapore, 1993).
- [37] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, *Rev. Mod. Phys.* **59**, 1 (1987).
- [38] J. H. An and W. M. Zhang, *Phys. Rev. A* **76**, 042127 (2007).
- [39] S. Maniscalco, S. Olivares, and M. G. A. Paris, *Phys. Rev. A* **75**, 062119 (2007).
- [40] S. Maniscalco, J. Piilo, F. Intravaia, F. Petruccione, and A. Messina, *Phys. Rev. A* **70**, 032113 (2004).
- [41] H. P. Breuer, B. Kappler, and F. Petruccione, *Ann. Phys. (N.Y.)* **291**, 36 (2001).
- [42] H. P. Breuer, D. Burgarth, and F. Petruccione, *Phys. Rev. B* **70**, 045323 (2004).
- [43] H. P. Breuer, *Phys. Rev. A* **75**, 022103 (2007).
- [44] J. J. Halliwell and T. Yu, *Phys. Rev. D* **53**, 2012 (1996).
- [45] B. L. Hu, J. P. Paz, and Y. Zhang, *Phys. Rev. D* **45**, 2843 (1992).
- [46] G. W. Ford and R. F. O'Connell, *Phys. Rev. D* **64**, 105020 (2001).
- [47] R. Karleim and H. Grabert, *Phys. Rev. E* **55**, 153 (1997).