Multiple-state Feshbach resonances mediated by high-order couplings

Christopher J. Hemming* and Roman V. Krems†
Department of Chemistry, University of British Columbia, 2036 Main Mall, Vancouver, British Columbia, Canada V6T 1Z1
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We present a study of multistate Feshbach resonances mediated by high-order couplings. Our analysis focuses on a system with one open scattering state and multiple bound states. The scattering state is coupled to one off-resonant bound state and multiple Feshbach resonances are induced by a sequence of indirect couplings between the closed channels. We derive a general recursive expression that can be used to fit the experimental data on multistate Feshbach resonances involving one continuum state and several bound states and present numerical solutions for several model systems. Our results elucidate general features of multistate Feshbach resonances induced by high-order couplings and suggest mechanisms for controlling collisions of ultracold atoms and molecules with external fields.

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I. INTRODUCTION

Following the paper of Tiesinga, Verhaar, and Stoof [1], Feshbach resonances have been used as an important tool for controlling interactions of ultracold atoms [2], the creation of ultracold molecules [3–7], and experimental studies of correlated phenomena in ultracold gases [8–10]. A Feshbach resonance occurs when a scattering state of two colliding particles interacts with a metastable bound state of the two-particle system. The coupling between the scattering state and the bound state leads to a resonant enhancement of the scattering cross section as the energy of the scattering state approaches the energy of the bound state. Collisions at ultracold temperatures are entirely determined by single partial-wave scattering: *s*-wave scattering for collisions of bosons or distinct atoms and *p*-wave scattering for collisions of identical fermions. If the scattering state is coupled directly to a bound state, Feshbach resonances in ultracold collisions can be described by a two-state model where a single partial-wave state interacts with an isolated bound state (see [3,11] and references in [11]). Atomic and molecular systems with anisotropic interactions may also exhibit resonances induced by indirect couplings. For example, the magnetic dipole-dipole interaction in collisions of chromium atoms [12] or the second-order spin-orbit interaction in collisions of Cs or Rb atoms [13,14] couple the *s*-wave scattering state with a *d*-wave state and the *d*-wave state with a *g*-wave state, while there is no direct coupling between *s*-wave and *g*-wave collision channels. A resonance in the *g*-wave channel may affect the *s*-wave scattering amplitude of ultracold atoms through the sequence of two indirect couplings. Similar resonances occur in collisions of distinct atoms in the presence of electric fields [15]. Electric fields couple *s*-wave collision channels with *p*-wave scattering states. The numerical calculations of Refs. [15,16] showed that the *s*-wave collision cross section may undergo a resonant variation in the presence of electric fields if the corresponding *p*-wave collision channel is coupled resonantly with a *p*-wave bound state. Resonances induced by high-order couplings may also occur in chemical dynamics of molecules [17] and electron-molecule or positron-molecule scattering [18,19]. The symmetry of the interactions involving charged particles limits the number of directly coupled scattering states and leads to a sequence of indirect couplings.

Feshbach resonances induced by direct couplings and three-state Feshbach resonances induced by second-order couplings have been analyzed in numerous previous studies [11]. The studies of multistate Feshbach resonances involving three or more bound states have been limited to processes induced by the interaction of molecules or atoms with laser light such as multiphoton ionization [20]. The properties of multistate Feshbach resonances in atomic and molecular collisions may however be different. Resonances induced by a sequence of several indirect couplings may become particularly important as the studies of ultracold collision physics begin to focus on large polyatomic molecules [21]. One proposed method of cooling polyatomic molecules to ultracold temperatures is sympathetic cooling. In the absence of reaction processes, molecules can potentially be cooled by elastic collisions in a reservoir of ultracold atoms [22–24]. The experimental realization of this method may, however, be complicated by naturally occurring Feshbach resonances. The density of molecular states near zero point energy is very large in polyatomic molecules [25] and these bound states may give rise to Feshbach resonances in collisions between molecules and ultracold atoms. The probability of three-body recombination and other loss processes in collisions involving large polyatomic molecules may thus be enhanced. Not all of the bound states may, however, give rise to Feshbach resonances. The interaction between an atom and a large molecule usually probes only a limited number of molecular states. The other molecular states can interact with the atom-molecule scattering state by indirect couplings and it is important to understand whether they can generate Feshbach resonances with significant widths. A general description of multistate Feshbach resonances occurring in molecular collisions is necessary to understand the prospects for controlling collisions of polyatomic molecules at low temperatures with external electromagnetic fields.

*chemming@chem.ubc.ca
†rkrems@chem.ubc.ca

Two-state Feshbach resonances as well as resonances induced by indirect couplings can be described using the projection operator method introduced by Feshbach [26,27]. The approach is based on partitioning the Hilbert space of the total Hamiltonian for the collision system into two orthogonal subspaces of open and closed channels. For resonances induced by indirect couplings, the closed channel subspace contains more than one state and the solution of the Schrödinger equation can be obtained by first diagonalizing the closed part of the Hamiltonian and then considering the coupling to the open subspace ([28], p. 157). This can be done analytically for three-state resonances involving two closed channels [28,29]. The procedure may, however, be cumbersome for resonances induced by a sequence of more than two indirect couplings. In the present paper, we use the approach of Feshbach to derive a recursive expression describing Feshbach resonances induced by a sequence of indirect couplings through bound states and analyze the properties of such resonances. This analysis is a special case of the multichannel collision problem considered by Feshbach ([28], p. 157). We show that the Feshbach formalism for systems with tridiagonal Hamiltonians can be reduced to finding the roots of recursive polynomials. Our derivation provides a general form that can be used to fit the experimental data on multistate Feshbach resonances involving one continuum state and several bound states. We present the numerical solutions for the roots of the recursive polynomials for several model systems that represent atomic or molecular collisions with anisotropic interactions. Our analysis elucidates general features of multistate Feshbach resonances induced by high-order couplings and demonstrates the analogy with multistate quantum optics systems.

II. THEORY

We consider a scattering state coupled to a resonant bound state by $n-1$ sequential couplings. The collision system is described by the following set of coupled equations:

$$(E-H_{11})|1\rangle = V_{12}|2\rangle,$$

$$(E-H_{22})|2\rangle = V_{21}|1\rangle + V_{32}|3\rangle,$$

$$\vdots$$

$$(E-H_{nn})|n^+\rangle = V_{n-1,n}|n-1\rangle, \quad (1)$$

where channel n is open and contains the incident flux and channels 1 through $n-1$ are closed. We assume that each closed channel k contains a bound state $|\phi_k\rangle$ satisfying the Schrödinger equation $(\epsilon_k - H_{kk})|\phi_k\rangle = 0$, and that each such state is well separated from other states in its channel. We adopt the procedure of Feshbach to repeatedly remove the last closed channel [26,29]. After eliminating channels 1 and 2, we obtain the following equation for channel 3:

$$(E-H_{33}-V_{32}\mathcal{G}_{2(1)}V_{23})|3\rangle = V_{34}|4\rangle, \quad (2)$$

where we use Green's operators of the form

$$\mathcal{G}_{\nu(\rho)}(E) \equiv \frac{1}{E-H_{\nu\nu}-V_{\nu\rho}G_\rho(E)V_{\rho\nu}}. \quad (3)$$

This notation can be extended to define general Green's operators that appear in solving the system (1), such as that for the pseudo-Hamiltonian on the left-hand side of Eq. (2). If σ_k with $k=1$ to n are channel indices,

$$\mathcal{G}_{\sigma_n(\sigma_{n-1}(\cdots(\sigma_1)\cdots))}(E) \equiv \frac{1}{E-H_{\sigma_n\sigma_n}-V_{\sigma_n,\sigma_{n-1}}\mathcal{G}_{\sigma_{n-1}(\sigma_{n-2}(\cdots(\sigma_1)\cdots))}(E)V_{\sigma_{n-1}\sigma_n}}. \quad (4)$$

The inverse of the operator in Eq. (2) is denoted by $\mathcal{G}_{3(2(1))}(E)$. Repeating this process we obtain for channel n an effective Schrödinger equation

$$(E-H_{nn}-V_{n,n-1}\mathcal{G}_{n-1(n-2(\cdots(1)\cdots))}V_{n-1,n})|n^+\rangle = 0, \quad (5)$$

with the effective potential determined by

$$V_{\rm eff}(E) = V_{nn} + V_{n,n-1}\mathcal{G}_{n-1(n-2(\cdots(1)\cdots))}(E)V_{n-1,n}, \quad (6)$$

where the operator $\mathcal{G}_{n-1(n-2(\cdots(1)\cdots))}(E)$ is

$$\mathcal{G}_{n-1(n-2(\cdots(1)\cdots))}(E) = \cfrac{1}{E-H_{n-1,n-1}-V_{n-1,n-2}\cfrac{1}{E-H_{n-2,n-2}-V_{n-2,n-3}\cfrac{1}{\cfrac{\vdots}{E-H_{22}-V_{21}\cfrac{1}{E-H_{11}}V_{12}}}V_{n-3,n-2}}V_{n-2,n-1}}. \quad (7)$$

Using the isolated state approximation for $k=1,\ldots,n-1$, we obtain

$$\mathcal{G}_{k(k-1(\cdots))}(E) \approx |\phi_k\rangle\langle\phi_k|\frac{P_{k-1}(E)}{P_k(E)}, \quad (8)$$

where $P_k(E)$ are polynomials in E satisfying the recursion relation

$$P_0(E) = 1,$$

$$P_1(E) = E - \epsilon_1,$$

$$P_k(E) = (E - \epsilon_k)P_{k-1}(E) - A_k P_{k-2}(E), \quad k \geq 2, \tag{9}$$

where

$$A_k \equiv |\langle \phi_k | V_{k,k-1} | \phi_{k-1} \rangle|^2. \tag{10}$$

This gives the implicit equation for $|n^+\rangle$,

$$|n^+\rangle = |\phi_n^+\rangle + G_n V_{n,n-1}|\phi_{n-1}\rangle\langle\phi_{n-1}|V_{n-1,n}|n^+\rangle \frac{P_{n-2}}{P_{n-1}}, \tag{11}$$

from which we can obtain the resonant contribution to the T-matrix element for elastic scattering in the open channel,

$$T_{nn}^r = \langle \phi_n^- | V_{n,n-1} | \phi_{n-1}\rangle\langle\phi_{n-1}|V_{n,n-1}|n^+\rangle \frac{P_{n-2}}{P_{n-1}}. \tag{12}$$

We evaluate the matrix element $\langle\phi_{n-1}|V_{n,n-1}|n^+\rangle$ by multiplying Eq. (11) on the left-hand side by $\langle\phi_{n-1}|V_{n-1,n}$,

$$\langle\phi_{n-1}|V_{n,n-1}|n^+\rangle = \frac{\langle\phi_{n-1}|V_{n-1,n}|\phi_n^+\rangle}{1 - \langle\phi_{n-1}|V_{n-1,n}G_n V_{n,n-1}|\phi_{n-1}\rangle \frac{P_{n-2}}{P_{n-1}}}. \tag{13}$$

Using this result, Eq. (12) yields

$$T_{nn}^r(E) = \frac{\langle\phi_n^-|V_{n,n-1}|\phi_{n-1}\rangle\langle\phi_{n-1}|V_{n-1,n}|\phi_n^+\rangle P_{n-1}(E)}{P_{n-1}(E) - \langle\phi_{n-1}|V_{n-1,n}G_n(E)V_{n,n-1}|\phi_{n-1}\rangle P_{n-2}(E)} = \frac{\Gamma(E)P_{n-1}(E)}{2\pi Q_{n-1}(E)}, \tag{14}$$

with $Q_{n-1}(E)$ defined by

$$Q_{n-1}(E) = P_{n-1}(E) - \left(\Delta(E) - \frac{i\Gamma(E)}{2}\right)P_{n-2}(E), \tag{15}$$

where

$$\Delta(E) - \frac{i\Gamma(E)}{2} \equiv \langle\phi_{n-1}|V_{n-1,n}G_n^+(E)V_{n,n-1}|\phi_{n-1}\rangle. \tag{16}$$

Resonances are associated with the roots of the equation $Q_{n-1}(E)=0$, which correspond to poles of the S matrix. The root which approaches ϵ_k [or $\epsilon_{n-1}+\Delta(\epsilon_{n-1})$, when $k=n-1$] when the coupling strengths tend to zero is E_k. The real part of the root gives the resonance energy, and the resonance width is $-2\,\mathrm{Im}\,E_k$. A real root of Q_{n-1}, for example, when one of the couplings $A_k=0$, must be a root of both P_{n-1} and P_{n-2}, and by Eq. (14) does not give rise to a pole in T_{nn}^r. Physically, the resonance width approaches zero as the root approaches the real axis. A factorization

$$Q_{n-1}(E) = \prod_{k=1}^{n-1} E - \mathcal{E}_k(E) + \frac{i\Gamma_k(E)}{2} \tag{17}$$

exists, where $\mathcal{E}_k(E)$ and $\Gamma_k(E)$ are real for real E, $E_k = \mathcal{E}_k(E_k) - \frac{i\Gamma_k(E_k)}{2}$, and

$$\sum_{k=1}^{n-1}\left(\mathcal{E}_k(E) - \frac{i\Gamma_k(E)}{2}\right) = \Delta(E) - \frac{i\Gamma(E)}{2} + \sum_{k=1}^{n-1}\epsilon_k. \tag{18}$$

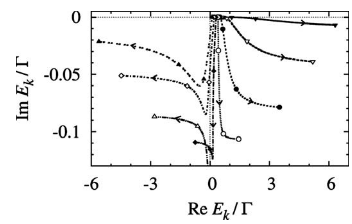

FIG. 1. Trajectories of the roots of an eighth-degree polynomial $Q_8(E)$ for a model (i) system in the complex E plane as A increases. Calculations are performed with all $A_k=A$ and ϵ_k given by Eq. (19) with $\delta\epsilon/\Gamma=0.1$, on a fine grid of A/Γ values. Symbols indicate the points corresponding to $A/\Gamma=0$, 0.1, 1, and 10.

III. NUMERICAL ANALYSIS

In order to elucidate the properties of resonances induced by indirect couplings, we consider several model problems and analyze the roots of the polynomials giving rise to multistate Feshbach resonances (14). The parameters Δ and Γ are assumed to be independent of energy. $Q_{n-1}(E)$ is then a polynomial in E and may be solved numerically. $\mathcal{E}_k - i\Gamma_k/2 = E_k$ in Eqs. (17) and (18) implies that the roots of $Q_{n-1}(E)$ sum to $\Delta - i\Gamma/2 + \Sigma\epsilon_k$.

Model (i). All couplings A_k have the same magnitude A and the energies of the bound states ϵ_k are closely and regularly spaced,

$$\epsilon_k = \delta\epsilon(n-1-k), \quad k \leq n-2,$$

$$\epsilon_{n-1} + \Delta = 0, \tag{19}$$

where $\delta\epsilon$ is the spacing between levels. Figure 1 illustrates the trajectories of the roots of an eighth-degree polynomial $Q_8(E)$ in the complex energy plane as A increases. For all roots, $\mathrm{Im}\,E_k<0$ when $A>0$, and $\mathrm{Im}\,E_k$ approaches a constant as $A\to\infty$. We have found that these are generic properties, observed in all models, and that they are independent of the order of the bound states, the number of bound states and the regularity and the magnitude of their spacing. Resonances correspond to poles of the S matrix typically below the positive real axis in the complex E plane ([30], p. 240), and we expect the roots to lie in the half-plane $\mathrm{Im}\,E \leq 0$. This would imply that the resonance widths sum to Γ and every resonance must have width less than Γ. For the polynomials $Q_{n-1}(E)$ with Γ and Δ constant, it can be proven that a real root can only occur when at least one $A_k=0$, and hence each root must remain on one side of the real axis when all A_k

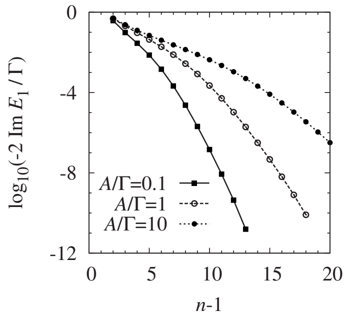

FIG. 2. Width of the E_1 resonance for a model (i) system with $n-1$ bound states calculated with all $A_k = A$ and ϵ_k given by Eq. (19) with $\delta\epsilon/\Gamma = 0.1$.

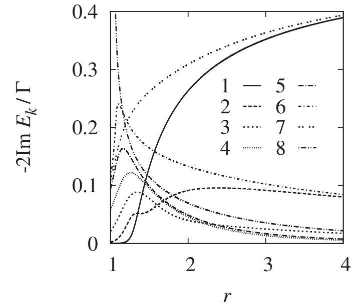

FIG. 4. Width of resonances versus r for a model (iii) system with $n-1=8$ bound states, $A_2/\Gamma=0.1$ and ϵ_k given by Eq. (19) with $\delta\epsilon/\Gamma=0.1$. Numbers indicate the index k of the root E_k.

>0. From Eq. (15), a real root of Q_{n-1} must be a root of both P_{n-1} and P_{n-2}. From Eq. (9), $A_k P_{k-2} = (E-\epsilon_k)P_{k-1} - P_k$ for $k \geq 2$, hence if every $A_k \neq 0$, a common root of P_{n-1} and P_{n-2} must be a root of all P_k, $k \leq n-1$, including P_0. However, this would imply that P_0 has a root, which is not possible since $P_0 = 1$. We conclude that in a solution of Eq. (9) with all $A_k \neq 0$, no two consecutive P_k may have a common root. As a consequence, there can be no real roots of Q_{n-1} unless a coupling $A_k = 0$. An analysis of the resonance widths shows that the width of the resonance associated with the root E_1, which for weak coupling strengths is primarily due to the last bound state $|\phi_1\rangle$, decreases rapidly as the number of intermediate bound states increases (Fig. 2).

Model (ii). The couplings $A_2 < A_3 < \cdots < A_{n-1}$ form the arithmetic series $A_k = A_2 + (k-2)\Delta A$ and the energies ϵ_k are regularly spaced in the sequence order. This is a generalization of model (i). We find that, in general, the widths of all resonances tend to constants as ΔA becomes large. The limiting values of Im E_k are close to each other, however one or two resonances in the middle of the energy range typically approach smaller limiting values separated from the others. These general trends do not change when the energies of the ϵ_k are randomly permuted.

Figure 3 presents a characteristic dependence of the resonance widths on the number of intermediate states in the coupling scheme for several values of ΔA. Interestingly, the width of the resonance is a slowly varying function of the number of intermediate states for large ΔA and decreases rapidly with the number of intermediate states (or the order of coupling) when ΔA is small.

Model (iii). The couplings $A_2 < A_3 < \cdots < A_{n-1}$ form the geometric series $A_k = r^{k-2}A_2$ and the energies ϵ_k are regularly spaced in the sequence order. The widths of the resonances with this coupling scheme depend on the magnitude of A_2. Figure 4 presents the dependence of the resonance widths on r for $A_2 = 0.1/\Gamma$. The results are qualitatively the same for any $A_2 > 0.1/\Gamma$. Surprisingly, the widths of most resonances tend to zero as r increases. The widths of two resonances ($k=1$ and $k=7$) tend to the same nonzero constant. The bound state $k=1$ is the last and the bound state $k=7$ is the second in the sequence of the indirectly coupled bound states. The dependence of the resonance widths on the coupling order (Fig. 5) is thus very simple in the limit of strong couplings. This implies that the number of Feshbach resonances with significant widths in a system with strong indirect couplings should be much smaller than the density of the bound states as certain bound states will not give rise to Feshbach resonances.

Figure 6 presents the resonance widths calculated with $A_2 = 0.001/\Gamma$. The dependence of the resonance widths on r is qualitatively the same but the resonance corresponding to $k=1$ is no longer dominant. Many of the intermediate resonances vanish as the coupling strength becomes large.

To understand the influence of the resonance positions on the resonance widths, we repeated the calculation of Fig. 5 with the following order of the bound state energies: $\epsilon_5 < \epsilon_3 < \epsilon_8 < \epsilon_1 < \epsilon_9 < \epsilon_2 < \epsilon_{10} < \epsilon_6 < \epsilon_4 < \epsilon_7$. The widths of the

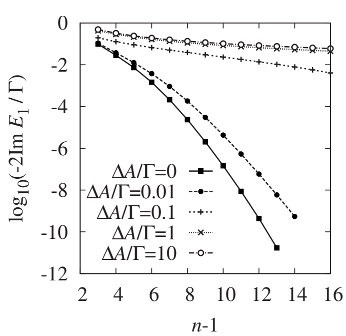

FIG. 3. Width of the E_1 resonance for a model (ii) system with $n-1$ bound states calculated with $A_2/\Gamma=0.1$ and ϵ_k given by Eq. (19) with $\delta\epsilon/\Gamma=0.1$.

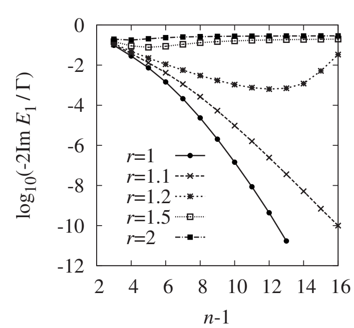

FIG. 5. Width of the E_1 resonance for a model (iii) system with $n-1$ bound states calculated with $A_2/\Gamma=0.1$ and ϵ_k given by Eq. (19) with $\delta\epsilon/\Gamma=0.1$.

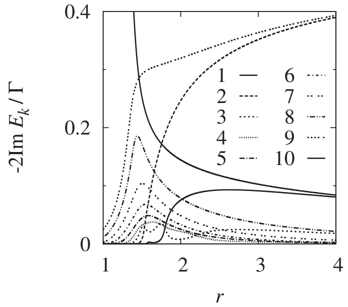

FIG. 6. Width of resonances versus r for a model (iii) system with $n-1=10$ bound states, $A_2/\Gamma=0.001$ and ϵ_k given by Eq. (19) with $\delta\epsilon/\Gamma=0.1$. Numbers indicate the index k of the root E_k.

resonances with this order of the bound state energies display qualitatively the same dependence on r as in Fig. 4. However, the resonance $k=7$ is now dominant in the limit of large r. The resonances $k=10$ and $k=4$ become wide and the resonances $k=1$, $k=2$, and $k=3$ become very narrow. We conclude that the width of a resonance depends both on the index k (i.e., the coupling scheme) and the position of the bound state energy ϵ_k relative to energies of other bound states.

Model (iv). The couplings $A_2=A_3=\cdots=A_{n-2}=A \ll A_{n-1}$ and the energies ϵ_k are regularly spaced in the sequence order. Figure 7 demonstrates the dependence of the resonance widths on the coupling strength A for the different resonances. It is interesting to note that the width of resonance $k=1$ decreases with increasing A_{n-1}, once n reaches a certain value. Generally, there are two wide resonances and the

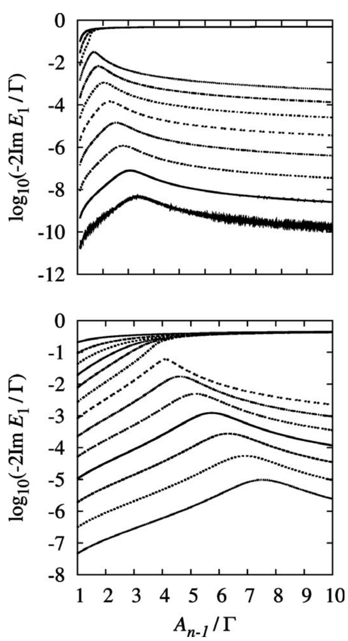

FIG. 7. Logarithmic plot of width of the E_1 resonance versus A_{n-1}/Γ for a model (iv) system with $A_{k\neq n-1}/\Gamma=0.1$ (upper plot) and 1 (lower plot), and ϵ_k given by Eq. (19) with $\delta\epsilon/\Gamma=0.1$. From top to bottom, the curves correspond to systems with $n-1=3$ to 13 (upper plot) and $n-1=3$ to 15 (lower plot).

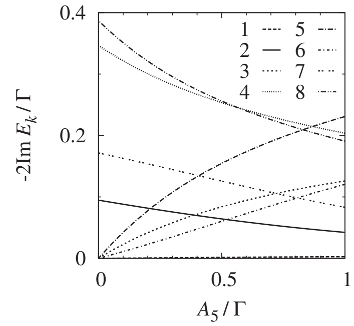

FIG. 8. Widths of resonances in a model (v) system with $A_{k\neq 5}/\Gamma=1$, $k\neq n-1$, and ϵ_k given by Eq. (19) with $\delta\epsilon/\Gamma=0.1$. Numbers indicate the index k of the root E_k.

widths of the other resonances decrease as A_{n-1} becomes large. This behavior is qualitatively the same for different orders of the bound state positions.

Model (v). The couplings $A_5 \ll A_{k\neq 5}=A$. This model represents a system with a bottleneck coupling A_5, i.e., the coupling between the bound states with the energies ϵ_4 and ϵ_5. As A_5 vanishes, some resonances become very narrow, while others remain wide (see Fig. 8). These calculations were performed with the energies ϵ_k in the sequential order, and with them randomly ordered. The A_k couplings lead to mixing of the bound states $|\phi_k\rangle$, and hence the resonances that vanish are not necessarily E_1 through E_4. For small A, resonances with lower k are more likely to vanish.

IV. CONCLUSION

We have presented a formal analysis of multichannel Feshbach resonances mediated by second- and higher-order couplings, i.e., scattering resonances induced by the interaction with a bound state that is not directly coupled to the initial scattering state. Our analysis focuses on a system with one open scattering state and multiple closed channels. Only one of the closed channels is coupled directly to the scattering state. Multiple Feshbach resonances arise due to the couplings between the closed channels. Such resonances may occur in collisions involving complex molecules with multiple degrees of freedom, chemical reaction dynamics, and electron-molecule and electron-positron scattering. Polyatomic molecules can potentially be cooled to ultracold temperatures by elastic collisions in a reservoir of ultracold atoms [22–24]. The experimental realization of this method may be complicated by naturally occurring Feshbach resonances. The energy spectrum of polyatomic molecules is usually dense and collisions of large molecules with ultracold atoms may lead to long-lived Feshbach resonances that would complicate translational energy exchange and result in sticking of atoms to molecules and the formation of clusters. It is therefore very important to understand the mechanisms of Feshbach resonances in collisions of polyatomic molecules with atoms.

If the molecule is sufficiently large, the atom-molecule scattering state of interest may not be directly coupled to all molecular states in a collision. The atom-molecule interaction potential, however, induces couplings between different states of the molecule, and the entire spectrum of molecular states may be coupled to the atom-molecule scattering state through a sequence of one continuum-bound and several bound-bound couplings. The simplest example of this coupling mechanism is a collision system of a structureless atom and a diatomic molecule interacting through the long-range dispersion interaction. The dispersion interaction couples the ground rotational state $N=0$ of the molecule only with the first and second rotationally excited states $N=1$ and $N=2$; however, the bound states of the atom-molecule complex corresponding asymptotically to $N>2$ may give rise to Feshbach resonances in collisions of ground-state molecules through a sequence of $N>2-N=2$ and $N=2-N=0$ couplings.

We have shown that the resonant variation of the T matrix element can be represented by a general form given by Eq. (14). This equation can be used to fit the experimental data on multistate Feshbach resonances involving one continuum state and several bound states. The polynomials P_k in Eq. (14) depend on the structure of the molecule and the atom-molecule interaction potentials. They can be evaluated using the recursive procedure described by Eqs. (9). We have presented a numerical analysis of the polynomial roots for several model systems that represent atomic or molecular collisions with anisotropic interactions. The calculations are performed for five different models: (i) a system with similar couplings; (ii) a system with slowly increasing couplings; (iii) a system with rapidly increasing couplings; (iv) a system with one coupling much larger than all other couplings; and (v) a system with a bottleneck coupling. Our discussion focuses on generic properties that are independent of the order of the bound states, the number of bound states and the regularity, and the magnitude of their spacing.

Our results demonstrate that Feshbach resonances may occur even if the scattering state is separated from the resonant bound state by a sequence of several indirectly coupled bound states. The ladder character of the couplings ensures that the scattering amplitude exhibits a pole near the energy of the bound state. In the limit of strong couplings, some of the intermediate Feshbach resonances may however vanish, which is reminiscent of dark states in atomic spectroscopy. This implies that the number of Feshbach resonances with significant widths in a system with strong indirect couplings should be much smaller than the density of the bound states as certain bound states will not give rise to Feshbach resonances. Our results suggest mechanisms for controlling interactions of ultracold atoms or molecules with external fields. For example, the "dark" states in Fig. 4 can be used to induce or suppress photoassociation of ultracold atoms by applying laser light in resonance with a high-order bound-bound transition. The results and discussion of Fig. 5 suggest that shifting rotationally excited states of ultracold molecules with electric or magnetic fields may dramatically modify the scattering dynamics of ultracold molecules in the ground state, even if the ground state is not affected by external fields. This mechanism of external field control should be particularly useful for quantum computation applications based on ultracold molecules. Understanding mechanisms of multistate Feshbach resonances is particularly important for the analysis of energy transfer mechanisms in ultracold collisions involving large complex molecules. Tuning multistate Feshbach resonances such as described in this work may be an approach to elucidating the role of ergodicity and multiple encounters in reactions of complex molecules [31].

ACKNOWLEDGMENT

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