Perfect state transfer over distance-regular spin networks

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Christandl *et al.* have noted that the d-dimensional hypercube can be projected to a linear chain with $d+1$ sites so that, by considering fixed but different couplings between the qubits assigned to the sites, the perfect state transfer (PST) can be achieved over arbitrarily long distances in the chain [Phys. Rev. Lett. **92**, 187902 (2004); Phys. Rev. A **71**, 032312 (2005)]. In this work we consider distance-regular graphs as spin networks and note that any such network (not just the hypercube) can be projected to a linear chain and so can allow PST over long distances. We consider some particular spin Hamiltonians which are the extended version of those of Christandl *et al.* Then, by using techniques such as stratification of distance-regular graphs and spectral analysis methods, we give a procedure for finding a set of coupling constants in the Hamiltonians so that a particular state initially encoded on one site will evolve freely to the opposite site without any dynamical control, i.e., we show how to derive the parameters of the system so that PST can be achieved. It is seen that PST is only allowed in distance-regular spin networks for which, starting from an arbitrary vertex as reference vertex (prepared in the initial state which we wish to transfer), the last stratum of the networks with respect to the reference state contains only one vertex; i.e., stratification of these networks plays an important role which determines in which kinds of networks and between which vertices of them, PST can be allowed. As examples, the cycle network with even number of vertices and d-dimensional hypercube are considered in details and the method is applied for some important distance-regular networks.

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I. INTRODUCTION

The transfer of a quantum state from one part of a physical unit, e.g., a qubit, to another part is a crucial ingredient for many quantum information processing protocols [1]. Currently, there are several ways of moving data around in a quantum computer. In a quantum-communication scenario, the transfer of quantum states from one location A to another location B, is rather explicit, since the goal is the communication between distant parties A and B (e.g., by means of photon transmission). Equally, in the interior of quantum computers good communication between different parts of the system is essential. The need is thus to transfer quantum states and generate entanglement between different regions contained within the system. There are various physical systems that can serve as quantum channels, one of them being a quantum spin system. This can be generally defined as a collection of interacting qubits (spin-1/2 particles) on a graph, whose dynamics is governed by a suitable Hamiltonian, e.g., the Heisenberg or XY Hamiltonian. Quantum communication over short distances through a spin chain, in which adjacent qubits are coupled by equal strength has been studied in detail, and an expression for the fidelity of quantum state transfer has been obtained [2,3]. Similarly, in Ref. [4], near perfect state transfer was achieved for uniform couplings provided a spatially varying magnetic field was introduced. The propagation of quantum information in rings has been also investigated in [5]. After the seminal paper by Bose [2], in which the potentialities of the so-called spin chains have been shown, several strategies were proposed to increase the transmission fidelity [5] and even to achieve, under appropriate conditions, perfect state transfer [6–11]. All of these proposals refer to ideal spin chains in which only nearest-neighbor couplings are present. In Refs. [6,7], the d-dimensional hypercube with 2^d vertices has been projected to a linear chain with $d+1$ sites so that, by considering fixed but different couplings between the qubits assigned to the sites, the PST can be achieved over arbitrarily long distances in the chain. In fact this is possible since the hypercube is a graph which is contained in an important category of graphs called distance-regular graphs. These graphs possess very useful properties, for instance, for these graphs one can classify the vertices in terms of a distance called shortest path distance so that the graph is stratified into distinct strata (these strata have been called in Refs. [6,7] as columns for the hypercube). The other preference of distance-regular graphs is that, these graphs are underlying graphs of association schemes [12] and so possess some useful algebraic properties. In fact, the theory of association schemes [12] has its origin in the design of statistical experiments. The connection of association schemes to spin models [13,14], algebraic codes, strongly regular graphs, distance regular graphs, design theory, etc., further intensified their study. In this paper we use the algebraic properties of distance-regular networks to note that any distance-regular network (not just the hypercube) with diameter $d=N-1$ (which contains $d+1=N$ strata), can be projected to a linear chain with N sites and so can allow PST over long distances. This is due to the fact that, the adjacency matrix corresponding to a given distance-regular network takes a tridiagonal form in the so-called "stratification space" spanned by stratification basis (this

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space has been also called "column space" by Christandl *et al.* in [6,7] and "walk space" by Jafarizadeh *et al.* in Ref. [15]). The more realistic case of long-range couplings, in particular magnetic-dipole-like couplings, has been studied [16,17]. In Ref. [16], it has been demonstrated that one can incorporate additional terms in the Hamiltonian by solving an inverse eigenvalue problem. Also in [16], Kay has shown that PST or, at least, high transmission fidelity can be obtained by appropriately choosing the system parameters, such as local magnetic fields and interspin distances.

In this work we focus on the situation in which state transference is perfect, i.e., the fidelity is unity. We will consider distance-regular graphs as spin networks in the sense that with each vertex of a distance-regular graph a qubit or a spin is associated (although qubits represent generic two state systems, for convenience of exposition we will use the term spin as it provides a simple physical picture of the network). Then, we use some techniques such as stratification and spectral analysis methods, in order to find suitable coupling constants in some particular spin Hamiltonians so that perfect transference of a quantum state between antipodes of the networks can be achieved. More clearly, for a given distance-regular spin network first we stratify the network with respect to an arbitrary chosen vertex of the network called reference vertex (for details about stratification of graphs, see [15,18,19]). Then, we consider coupling constants so that vertices belonging to the same stratum with respect to the reference vertex possess the same coupling strength with the reference vertex whereas vertices belonging to distinct strata possess different coupling strengths. Then we give a method for finding a suitable set of coupling constants so that PST over antipodes of the networks is possible. It is seen that PST is only allowed in distance-regular spin networks for which, starting from an arbitrary vertex as reference vertex (prepared in the initial state which we wish to transfer), the last stratum of the networks with respect to the reference state contains only one vertex (node); i.e., stratification of these networks plays an important role which determines in which kinds of networks and between which vertices of them, PST can be allowed. As examples we will consider the cycle networks with even number of vertices and d-dimensional hypercube networks in details and some important distance-regular networks in Appendix C.

The organization of the paper is as follows: In Sec. II, we review some preliminary facts about distance-regular graphs, their stratifications and spectral analysis techniques. Section III is devoted to PST over antipodes of distance-regular spin networks, where a method for finding suitable coupling constants in particular spin Hamiltonians so that PST is possible, is given. The paper is ended with a brief conclusion and three appendixes.

II. PRELIMINARIES

A. Distance-regular graphs and their stratifications

Distance-regular graphs lie in an important category of graphs which possess some useful properties. In these graphs, the adjacency matrices A_i are defined based on shortest path distance denoted by ∂. More clearly, for a given finite graph $\Gamma=(V,E)$ where V denotes the finite set of its vertices and E the set of its edges [two vertices $\alpha,\beta\in V$ are adjacent if $(\alpha,\beta)\in E$], if $\partial(\alpha,\beta)$ (distance between the vertices $\alpha,\beta\in V$) be the length of the shortest walk connecting α and β (recall that a finite sequence $\alpha_0,\alpha_1,\ldots,\alpha_n\in V$ is called a walk of length n if $\alpha_{k-1}\sim\alpha_k$ for all $k=1,2,\ldots,n$, where $\alpha_{k-1}\sim\alpha_k$ means that α_{k-1} is adjacent with α_k), then the adjacency matrices A_i for $i=0,1,\ldots,d$ in distance-regular graphs are defined as $(A_i)_{\alpha,\beta}=1$ if and only if $\partial(\alpha,\beta)=i$ and $(A_i)_{\alpha,\beta}=0$ otherwise, where $d:=\max\{\partial(\alpha,\beta):\alpha,\beta\in V\}$ is called the diameter of the graph.

Definition. An undirected connected graph $\Gamma=(V,E)$ is called distance-regular graph with diameter d if it satisfies the following distance-regularity condition.

For all $i,j,k\in\{0,1,\ldots,d\}$, and α,β with $\partial(\alpha,\beta)=k$, the number

$$p_{ij}^k=|\{\gamma\in V:\partial(\alpha,\gamma)=i \text{ and } \partial(\gamma,\beta)=j\}| \quad (2.1)$$

is constant in that it depends only on k,i,j but does not depend on the choice of α and β. This number is called the intersection number.

In Appendix B 1 some properties of distance-regular graphs have been given, where it has been noted that for the corresponding adjacency matrices A_i we have

$$A_i=P_i(A), \quad i=0,1,\ldots,d, \quad (2.2)$$

where, P_i is a polynomial of degree i. As an immediate consequence of (2.2), one can show that the eigenvalues of the adjacency matrix A_i denoted by P_{ji} for $j=0,1,\ldots,d$ are polynomials of eigenvalues $P_{j1}=\lambda_j$ (eigenvalues of the adjacency matrix $A\equiv A_1$), i.e., one can write $P_{ji}=P_i(\lambda_j)$, where P is the eigenvalue matrix associated with the graph which is defined as a $d\times d$ matrix such that its entry at jth row and ith column is the jth eigenvalue of the ith adjacency matrix of the graph. As it has been illustrated in Appendix B 2, in distance-regular graphs polynomials P_i are easily obtained via three-term recursion relations so that we need only to know the intersection array (defined in Appendix B 1) of the graphs.

In the rest of this section, we recall the notion of stratification in distance-regular graphs which will play an essential role in our investigation of PST over these graphs.

For a given vertex $\alpha\in V$, let $\Gamma_i(\alpha):=\{\beta\in V:\partial(\alpha,\beta)=i\}$ denote the set of all vertices being at distance i from α. Then, the vertex set V can be written as disjoint union of $\Gamma_i(\alpha)$ for $i=0,1,2,\ldots,d$, i.e.,

$$V=\bigcup_{i=0}^{d}\Gamma_i(\alpha), \quad (2.3)$$

In fact, by fixing a point $o\in V$ as an origin of the graph, hereafter referred to as reference vertex, the relation (2.3) stratifies the graph into a disjoint union of associate classes $\Gamma_i(o)$ (called the ith stratum or ith column with respect to o). Let $l^2(V)$ denote the Hilbert space of C-valued square summable functions on V. With each associate class $\Gamma_i(o)$ we associate a unit vector in $l^2(V)$ defined by

$$|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in \Gamma_i(o)} |\alpha\rangle, \tag{2.4}$$

where $|\alpha\rangle$ denotes the eigenket of αth vertex at the associate class $\Gamma_i(o)$ and $\kappa_i = |\Gamma_i(o)|$ is called the ith valence of the graph.

We will refer to the space spanned by unit vectors $|\phi_i\rangle$, $i=0,1,\ldots,d$ as "stratification space." This space has been called "column space" in [7] and "walk space" in [15]. Childs *et al.* [20] note that the evolution with the adjacency matrix $A \equiv A_1$ (which is considered in continuous-time quantum walks on graphs [19–24]) for the class of networks with this stratification, starting in $|\phi_0\rangle$, always remains in the stratification space because every vertex in stratum (column) i is connected to the same number of vertices in stratum $i+1$ and every vertex in stratum $i+1$ is connected to the same number of vertices in stratum i.

Now, let A_i be the ith adjacency matrix of the graph $\Gamma=(V,E)$. Then, for the reference state $|\phi_0\rangle$ ($|\phi_0\rangle = |o\rangle$, with $o \in V$ as reference vertex), one can write

$$A_i|\phi_0\rangle = \sum_{\beta \in \Gamma_i(o)} |\beta\rangle. \tag{2.5}$$

Then, by using (2.4) and (2.5), we obtain

$$A_i|\phi_0\rangle = \sqrt{\kappa_i}|\phi_i\rangle. \tag{2.6}$$

One should notice that, the above introduced stratification is reference state independent, namely one can choose any arbitrary vertex as reference vertex (site). In this paper, we will deal with PST over particular distance-regular graphs (as spin networks) for which starting from an arbitrary vertex as reference vertex (prepared in the initial state which we wish to transfer), the last stratum of the networks with respect to the reference state contains only one vertex of the networks. Therefore, stratification of these networks plays an important role which determines in which kinds of distance-regular graphs and between which vertices of them, PST can be allowed. In Appendix B 1, it has been also noted that the adjacency matrix $A \equiv A_1$ of distance-regular graphs takes a tridiagonal form in the stratification space, so that, for this type of network, we can restrict our attention to the stratification space for the purpose of PST from $|\phi_0\rangle$ (state associated with reference vertex) to $|\phi_d\rangle$ (state associated with the last stratum of the graph); for more details, see Sec. III A.

It has been shown in Ref. [15] that for any arbitrary graph (not just for distance-regular graphs), one can employ a modified version of the classical Gram-Schmidt orthogonalization process called Lanczos algorithm in order to give a tridiagonal form to the adjacency matrix of the graph. One of the preferences of this structure is the ability of determining the spectral distribution associated with the graph by using the so-called Stieltjes-Hilbert transform introduced in Appendix B 2.

B. Spectral techniques

In this section, we recall some preliminary facts about spectral techniques used in the paper, where more details have been given in Appendix B 2 and Refs. [19,25,26].

One of the most important applications of spectral analysis method is to analyze a set of two-state diffusion equations, which was first used by Zusman [27] to treat solvent effects on three-electron transfer in the nonadiabatic limit. In [28], the spectral analysis approach developed in [29] has been employed to study the electron transfer dynamics in mixed-valence systems. Also, since the advent of random matrix theory (RMT), there has been considerable interest in the statistical analysis of spectra [30–32]. RMT can be viewed as a generalization of the classical probability calculus, where the concept of probability density distribution for a one-dimensional random variable is generalized onto an averaged spectral distribution of the ensemble of large, noncommuting random matrices. Such a structure exhibits several phenomena known in classical probability theory, including central limit theorems [33].

Actually the spectral analysis of operators is an important issue in quantum mechanics, operator theory and mathematical physics [34,35]. As an example $\mu(dx) = |\psi(x)|^2 dx$ $[\mu(dp) = |\tilde{\psi}(p)|^2 dp]$ is a spectral distribution which is assigned to the position (momentum) operator $\hat{X}(\hat{P})$. Moreover, in general quasidistributions are the assigned spectral distributions of two Hermitian noncommuting operators with a prescribed ordering. For example, the Wigner distribution in phase space is the assigned spectral distribution for two noncommuting operators $\hat{X}$ (shift operator) and $\hat{P}$ (momentum operator) with Wyle ordering among them [36,37].

It is well known that, for any pair $(A,|\phi_0\rangle)$ of a matrix A and a vector $|\phi_0\rangle$, one can assign a measure μ as follows:

$$\mu(x) = \langle \phi_0|E(x)|\phi_0\rangle, \tag{2.7}$$

where $E(x) = \sum_i |u_i\rangle\langle u_i|$ is the operator of projection onto the eigenspace of A corresponding to eigenvalue x, i.e.,

$$A = \int xE(x)dx. \tag{2.8}$$

Then, for any polynomial $P(A)$ we have

$$P(A) = \int P(x)E(x)dx, \tag{2.9}$$

where for discrete spectrum the above integrals are replaced by summation. Therefore, using the relations (2.7) and (2.9), the expectation value of powers of adjacency matrix A over reference vector $|\phi_0\rangle$ can be written as

$$\langle \phi_0|A^m|\phi_0\rangle = \int_R x^m \mu(dx), \quad m = 0,1,2,\ldots. \tag{2.10}$$

Obviously, the relation (2.10) implies an isomorphism from the Hilbert space of the stratification onto the closed linear span of the orthogonal polynomials with respect to the measure μ.

III. PERFECT STATE TRANSFER OVER ANTIPODES OF DISTANCE-REGULAR SPIN NETWORKS

A. State transfer in quantum spin systems

The perfect state transfer algorithm was proposed by Christandl *et al.* [6,7], and it can be implemented in the XY chain. The algorithm can transfer an arbitrary quantum state between the two ends of the chain in a fixed period time, only using XY interactions. For one-dimensional fermionic chains, the model of a system consisting of spinless fermions (or bosons) hopping freely in a network of N lattice sites can be mapped to spin chains in which spins are coupled through the XY Hamiltonian

$$H=\frac{1}{2}\sum_{j=1}^{N-1} J_j(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y)+\frac{1}{2}\sum_{j=1}^{N}\lambda_j(\sigma_j^z+1), \quad (3.1)$$

by the Jordan-Wigner transformation, where J_j is the time-independent coupling constant between nearest-neighbor sites j and $j+1$, and λ_j represents the strength of the external static potential at site j.

A quantum spin system associated with a simple, connected, finite graph $G=(V,E)$ as a spin network is defined by attaching a spin-1/2 particle to each vertex of the graph so that with each vertex $i\in V$ one can associate a Hilbert space $\mathcal{H}_i\simeq\mathcal{C}^2$. The Hilbert space associated with G is then given by

$$\mathcal{H}_G=\underset{i\in V}{\otimes}\mathcal{H}_i=(\mathcal{C}^2)^{\otimes N}, \quad (3.2)$$

where $N:=|V|$ denotes the total number of vertices in G. On the other hand, quantum state transfer over a network is similar to the quantum random walk problem, where a variety of networks are equivalent to one-dimensional chains [6,22]. Therefore, it can be focused on a chain of N sites. For $j=1,2,\ldots,N$, let $|j\rangle$ be the state where a single fermion (or boson) is at the site j but is in the empty state $|0\rangle$ for all other sites and let $|\underline{0}\rangle$ be the vacuum state where all sites are empty. For spin chains, $|\underline{0}\rangle$ corresponds to the state where all the spins are in the spin-down state $|\downarrow\rangle$ and $|j\rangle$ corresponds to a spin-up state $|\uparrow\rangle$ for the jth spin and spin-down for all other spins. The Hamiltonian in this single-particle subspace can be written in a tridiagonal form, which is real and symmetric,

$$H=\begin{pmatrix} \lambda_1 & J_1 & 0 & \cdots & 0 \\ J_1 & \lambda_2 & J_2 & \cdots & 0 \\ 0 & J_2 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & J_{N-1} \\ 0 & 0 & 0 & J_{N-1} & \lambda_N \end{pmatrix}. \quad (3.3)$$

As regards the arguments of Sec. II A, the evolution with the adjacency matrix $H=A\equiv A_1$ for distance-regular networks starting in $|\phi_0\rangle$, always remains in the stratification space. If we consider particular distance-regular networks for which the last stratum, i.e., $|\phi_{N-1}\rangle$ contains only one site, then PST between the antipodes $|\phi_0\rangle$ and $|\phi_{N-1}\rangle$ is allowed [the $(N-1)$-dimensional hypercube with $2^{(N-1)}$ vertices is a well-known example of such networks where $|\phi_0\rangle=|\underbrace{00\ldots0}_{N-1}\rangle$ and $|\phi_{N-1}\rangle=|\underbrace{11\ldots1}_{N-1}\rangle$). Thus, we can restrict our attention to the stratification space for the purpose of PST from $|\phi_0\rangle$ to $|\phi_{N-1}\rangle$. As Eq. (B7) of Appendix B indicates, the matrix elements of the adjacency matrix restricted to the stratification space are given by

$$\langle\phi_i|H|\phi_i\rangle=\alpha_i, \quad i=0,1,\ldots,N-1,$$

$$\langle\phi_i|H|\phi_{i+1}\rangle=\beta_{i+1}, \quad i=0,1,\ldots,N-2,$$

where the parameters α_i and β_{i+1} for $i=0,1,\ldots,d-1$ depend on the characteristics of the network such as its size (the number of vertices) and its intersection array, as they have been defined in Eq. (B6). Hence, the above graph exhibits the same behavior as the XY chain with "engineered" parameters $\lambda_i=\alpha_{i-1}$, $J_i=\beta_i$,

$$H=\frac{1}{2}\sum_{j=1}^{N-1}\beta_j(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y)+\frac{1}{2}\sum_{j=1}^{N}\alpha_{j-1}(\sigma_j^z+\mathbf{1}), \quad (3.4)$$

The above argument notes that, any distance-regular network with diameter $d=N-1$ (which contains $d+1=N$ strata), can be projected to a chain with N sites.

The quantum state transfer protocol involves two steps: initialization and evolution. First, a quantum state $|\psi\rangle_A=\alpha|0\rangle_A+\beta|1\rangle_A\in\mathcal{H}_A$ (with $\alpha,\beta\in\mathcal{C}$ and $|\alpha|^2+|\beta|^2=1$) to be transmitted is created. The state of the entire spin system after this step is given by

$$|\psi(t=0)\rangle=|\psi_A0\cdots00_B\rangle=\alpha|0_A0\cdots00_B\rangle+\beta|1_A0\cdots00_B\rangle$$
$$=\alpha|\underline{0}\rangle+\beta|A\rangle, \quad (3.5)$$

with $|\underline{0}\rangle:=|0_A0\cdots00_B\rangle$. Then, the network couplings are switched on and the whole system is allowed to evolve under $U(t)=e^{-iHt}$ for a fixed time interval, say t_0. The final state becomes

$$|\psi(t_0)\rangle=\alpha|\underline{0}\rangle+\beta\sum_{j=1}^{N}f_{jA}(t_0)|j\rangle, \quad (3.6)$$

where $f_{jA}(t_0):=\langle j|e^{-iHt_0}|A\rangle$. Any site B is in a mixed state if $|f_{AB}(t_0)|<1$, which also implies that the state transfer from site A to B is imperfect. In this paper, we will focus only on PST. This means that we consider the condition

$$|f_{AB}(t_0)|=1 \quad \text{for some } 0<t_0<\infty \quad (3.7)$$

which can be interpreted as the signature of perfect communication (or PST) between A and B in time t_0. The effect of the modulus in (3.7) is that the state at B, after transmission, will no longer be $|\psi\rangle$, but will be of the form

$$\alpha|0\rangle + e^{i\phi}\beta|1\rangle. \tag{3.8}$$

The phase factor $e^{i\phi}$ is not a problem because ϕ is independent of α and β and will thus be a known quantity for the graph, which we can correct for with an appropriate phase gate (for more details see, for example, [6,7,10,11]).

The model we will consider is a distance-regular network consisting of N sites labeled by $\{1,2,\ldots,N\}$ and diameter d. Then we stratify the network with respect to a chosen reference site, say 1, and assume that the network contains only the output site N in its last stratum (i.e., $|\phi_d\rangle=|N\rangle$). At time $t=0$, the qubit in the first (input) site of the network is prepared in the state $|\psi_{\text{in}}\rangle$. We wish to transfer the state to the Nth (output) site of the network with unit efficiency after a well-defined period of time. Although our qubits represent generic two state systems, for the convenience of exposition we will use the term spin as it provides a simple physical picture of the network. The standard basis for an individual qubit is chosen to be $\{|0\rangle=|\downarrow\rangle,\ |1\rangle=|\uparrow\rangle\}$, and we shall assume that initially all spins point "down" along a prescribed z axis; i.e., the network is in the state $|\underline{0}\rangle=|0_A00\cdots00_B\rangle$. Then, we consider the dynamics of the system to be governed by the quantum-mechanical Hamiltonian

$$H_G = \frac{1}{2}\sum_{m=0}^{d} J_m \sum_{(i,j)\in R_m} H_{ij}, \tag{3.9}$$

with H_{ij} as

$$H_{ij} = \sigma_i \cdot \sigma_j, \tag{3.10}$$

where, σ_i is a vector with familiar Pauli matrices σ_i^x, σ_i^y, and σ_i^z as its components acting on the one-site Hilbert space $\mathcal{H}_i$, and J_m is the coupling strength between the reference site 1 and all of the sites belonging to the mth stratum with respect to 1.

The total spin of a quantum-mechanical system consisting of N elementary spins $\vec{\sigma}_i$ on a one-dimensional lattice or better-called chain is given by

$$\vec{\sigma} = \sum_{i=1}^{N} \vec{\sigma}_i. \tag{3.11}$$

One can easily see that, the Hamiltonian (3.9) commutes with the total spin operator (conservation). That is, since the total z component of the spin given by $\sigma^z_{\text{tot}}=\Sigma_{i\in V}\sigma_i^z$ is conserved, i.e., $[\sigma^z_{\text{tot}},H_G]=0$, hence the Hilbert space $\mathcal{H}_G$ decomposes into invariant subspaces, each of which is a distinct eigenspace of the operator σ^z_{tot} (this property would be important to use its symmetry to diagonalize the Hamiltonian in the well-known Bethe ansatz approach).

For the purpose of perfect quantum state transfer, we write the Hamiltonian (3.9) in terms of the adjacency matrices A_i, $i=0,1,\ldots,d$ of the underlying graph in order to use the techniques introduced in Sec. II such as stratification and spectral distribution associated with the graph. To do so, we recall that the kets $|i_1,i_2,\ldots,i_N\rangle$ with $i_1,\ldots,i_N\in\{\uparrow,\downarrow\}$ form an orthonormal basis for Hilbert space $\mathcal{H}_G$. Then, one can easily obtain

$$H_{ij}|\cdots \underbrace{\uparrow}_{i}\cdots \underbrace{\uparrow}_{j}\cdots\rangle = |\cdots \underbrace{\uparrow}_{i}\cdots \underbrace{\uparrow}_{j}\cdots\rangle,$$

$$H_{ij}|\cdots \underbrace{\uparrow}_{i}\cdots \underbrace{\downarrow}_{j}\cdots\rangle = -|\cdots \underbrace{\uparrow}_{i}\cdots \underbrace{\downarrow}_{j}\cdots\rangle + 2|\cdots \underbrace{\downarrow}_{i}\cdots \underbrace{\uparrow}_{j}\cdots\rangle, \tag{3.12}$$

where we have used the facts that $\sigma_z|\uparrow\rangle=|\uparrow\rangle$, $\sigma_z|\downarrow\rangle=-|\downarrow\rangle$, $\sigma_x|\uparrow\rangle=|\downarrow\rangle$, $\sigma_x|\downarrow\rangle=|\uparrow\rangle$, and $\sigma_y|\uparrow\rangle=i|\downarrow\rangle$, $\sigma_y|\downarrow\rangle=-i|\uparrow\rangle$. Equation (3.12) implies that the action of H_{ij} on the basis vectors is equivalent to the action of the operator $2P_{ij}-I_N$, i.e., we have

$$H_{ij} = 2P_{ij} - I_N, \tag{3.13}$$

where, P_{ij} denotes the permutation operator which permutes ith and jth sites and I_N is $N\times N$ identity matrix, where N is the number of vertices ($N:=|V|$). Now, let $|l\rangle$ denote the vector state where all components are $\uparrow$ except for l, i.e., $|l\rangle=|\uparrow\cdots\uparrow\underbrace{\downarrow}_{l}\uparrow\cdots\uparrow\rangle$. Then, we have

$$\sum_{(i,j)\in R_m} P_{ij}|l\rangle = \frac{1}{2}\Big(\sum_{i\in\Gamma_m(j);i,j\neq l} P_{ij} + 2\sum_{i\in\Gamma_m(l)} P_{il}\Big)|l\rangle = \left(\frac{N\kappa_m}{2} - \kappa_m\right)|l\rangle + \sum_{j\in\Gamma_m(l)}|j\rangle,$$

which implies that

$$\sum_{(i,j)\in R_m} P_{ij} = \left(\frac{\kappa_m(N-2)}{2}I + A_m\right). \tag{3.14}$$

Then, by using (3.13) and (3.14), the Hamiltonian in (3.9) can be written in terms of the adjacency matrices A_i, $i=0,1,\ldots,d$ as follows:

$$H = \sum_{m=0}^{d} J_m \sum_{(i,j)\in R_m}(2P_{ij}-I_N) = 2\sum_{m=0}^{d} J_m A_m + \frac{N-4}{2}\sum_{m=0}^{d} J_m\kappa_m I. \tag{3.15}$$

Actually the PST suitable Hamiltonians which are associated with permutations [38], can be considered within the present theoretical framework. For the purpose of the perfect transfer of state, we consider distance-regular graphs with $\kappa_d=|\Gamma_d(o)|=1$, i.e., the last stratum of the graph contains only one site. Then, we impose the constraints that the amplitudes $\langle\phi_i|e^{-iHt}|\phi_0\rangle$ be zero for all $i=0,1,\ldots,d-1$ and $\langle\phi_d|e^{-iHt}|\phi_0\rangle=e^{i\theta}$, where θ is an arbitrary phase. Therefore, these amplitudes must be evaluated. To do so, we use the stratification and spectral distribution associated with distance-regular graphs [Eqs. (2.6) and (B4)] to write

$$\langle\phi_i|e^{-iHt}|\phi_0\rangle = e^{-i(N-4)t/2\Sigma_{m=0}^{d}J_m\kappa_m}\langle\phi_i|e^{-2it\Sigma_{m=0}^{d}J_mA_m}|\phi_0\rangle$$
$$= \frac{1}{\sqrt{\kappa_i}}e^{-i(N-4)t/2\Sigma_{m=0}^{d}J_m\kappa_m}\langle\phi_0|A_ie^{-2it\Sigma_{m=0}^{d}J_mP_m(A)}|\phi_0\rangle.$$

Let the spectral distribution of the graph be $\mu(x)=\Sigma_{k=0}^{d}\gamma_k\delta(x-x_k)$ [see Eq. (B15) in Appendix B 2]. Then, $\langle\phi_i|e^{-iHt}|\phi_0\rangle=0$ implies that

$$\sum_{k=0}^{d}\gamma_kP_i(x_k)e^{-2it\Sigma_{m=0}^{d}J_mP_m(x_k)}=0,\quad i=0,1,\ldots,d-1.$$

Denoting $e^{-2it\Sigma_{m=0}^{d}J_mP_m(x_k)}$ by η_k, the above constraints are rewritten as follows:

$$\sum_{k=0}^{d}P_i(x_k)\,\eta_k\gamma_k=0,\quad i=0,1,\ldots,d-1,$$

$$\sum_{k=0}^{d}P_d(x_k)\,\eta_k\gamma_k=e^{i\theta}. \tag{3.16}$$

As it was discussed in Sec. II A, $P_i(x_k)$ are entries of the eigenvalue matrix P $[P_{ki}=P_i(x_k)]$ which is invertible [see Eq. (A3)], i.e., the Eq. (3.16) can be written as

$$\begin{pmatrix}\eta_0\gamma_0\\ \eta_1\gamma_1\\ \vdots\\ \eta_d\gamma_d\end{pmatrix}=(\mathcal{P}^t)^{-1}\begin{pmatrix}0\\ \vdots\\ 0\\ e^{i\theta}\end{pmatrix}. \tag{3.17}$$

The above equation implies that $\eta_k\gamma_k$ for $k=0,1,\ldots,d$ are the same as the entries in the last column of the matrix $(\mathcal{P}^t)^{-1}=\frac{1}{v}\mathcal{Q}^t$ multiplied with the phase $e^{i\theta}$, i.e., for the purpose of PST, the following equations must be satisfied:

$$\eta_k\gamma_k=\gamma_ke^{-2it_0\Sigma_{m=0}^{d}J_mP_m(x_k)}=\frac{e^{i\theta}}{v}(\mathcal{Q}^t)_{kd}\quad\text{for }\ k=0,1,\ldots,d. \tag{3.18}$$

In the following, we investigate PST between antipodes of some distance-regular networks such as cycle networks with even number of nodes and d-dimensional hypercube networks.

B. Examples

1. Cycle graph C_{2m}

A well-known example of distance-regular networks, is the cycle graph with N vertices denoted by C_N (see Fig. 1 for even $N=2m$). For the purpose of perfect transfer of state, we consider the cycle graph with even number of vertices, since as it can be seen from Fig. 1, in this case the last stratum contains a single state corresponding to the mth vertex. From Fig. 1 it can be seen that, for even number of vertices $N=2m$, the adjacency matrices are given by

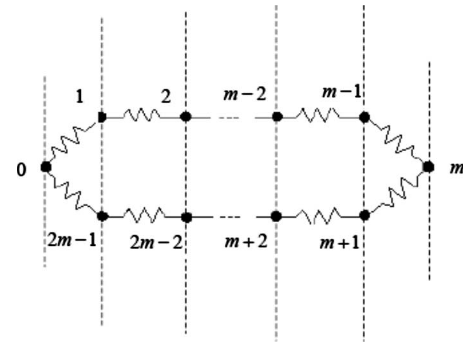

FIG. 1. Denotes the cycle network C_{2m}, where the $m+1$ vertical dashed lines show the $m+1$ strata of the network.

$$A_0=I_{2m},\quad A_i=S^i+S^{-i},\quad i=1,2,\ldots,m-1,\quad A_m=S^m, \tag{3.19}$$

where, S is the $N\times N$ circulant matrix with period N $(S^N=I_N)$ defined as follows:

$$S=\begin{pmatrix}0&1&0&\cdots&0\\ \vdots&\ddots&\ddots&\ddots&\vdots\\ 0&\cdots&0&1&0\\ 0&0&\cdots&0&1\\ 1&0&\cdots&0&0\end{pmatrix}. \tag{3.20}$$

By using (3.19), one can obtain the following recursion relations for C_{2m}:

$$A_1A_i=A_{i-1}+A_{i+1},\quad i=0,1,\ldots,m-1;\quad A_1A_m=A_{m-1} \tag{3.21}$$

(the graph C_{2m} consists of $m+1$ strata). By comparing (3.21) with three-term recursion relations (B3), we obtain the intersection arrays for C_{2m} as

$$\{b_0,\ldots,b_{m-1};c_1,\ldots,c_m\}=\{2,1,\ldots,1,1;1,\ldots,1,2\}. \tag{3.22}$$

Then, by using (B6), the QD parameters are given by

$$\alpha_i=0,\quad i=0,1,\ldots,m;\quad \omega_1=\omega_m=2,\quad \omega_i=1,$$
$$i=2,\ldots,m-1. \tag{3.23}$$

By using the recursion relations (B10), one can show that

$$Q_0(x)=P_0(x)=1,\quad Q_i(x)=P_i(x)=2T_i(x/2),$$
$$i=1,\ldots,m-1,\quad Q_m(x)=2P_m(x)=2T_m(x/2), \tag{3.24}$$

where T_i's are Tchebychev polynomials of the first kind.

Then, the eigenvalues of the adjacency matrix $A\equiv A_1$ [roots of $Q_{m+1}(x)=2T_{m+1}(x/2)$] are given by

$$x_i=\omega^i+\omega^{-i}=2\cos(2\pi i/N),\quad i=0,1,\ldots,m$$

with $\omega:=e^{2\pi i/N}$. Also, one can show that γ_i's (degeneracy's of eigenvalues x_i) are given by

$$\gamma_0 = \gamma_m = 1/2m, \quad \gamma_i = 1/m, \quad i = 1,2, \ldots, m-1. \tag{3.25}$$

Now, as regards Eq. (A5), the matrix P^t associated with cycle graph C_{2m} reads as

$$P^t = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2\cos(2\pi/N) & \cdots & 2\cos[2(m-1)\pi/N] & 2\omega^m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2\cos[2(m-1)\pi/N] & \cdots & 2\cos[(m-1)^2\pi/N] & 2\omega^{m(m-1)} \\ 1 & \omega^m & \cdots & \cdots & \omega^{m^2} \end{pmatrix}. \tag{3.26}$$

One can see that $(P^t)^2 = NI$, so the inverse of P^t is given by $(P^t)^{-1} = \frac{1}{N}P^t$. Therefore, by using (3.17) and (3.25), we obtain

$$\eta_i = e^{-it\sum_{l=0}^{m} 2J_l T_l[\cos(2\pi i/N)]} = (-1)^i e^{i\theta}, \quad i = 0,1, \ldots, m. \tag{3.27}$$

For instance, for $N=4$, we obtain

$$\eta_0 = e^{-it_0(J_0+2J_1+J_2)} = e^{i\theta},$$

$$\eta_1 = e^{-it_0(J_0-J_2)} = -e^{i\theta},$$

$$\eta_2 = e^{-it_0(J_0-2J_1+J_2)} = e^{i\theta}, \tag{3.28}$$

which gives us the following equations:

$$-t(J_0 + 2J_1 + J_2) = \theta + 2l\pi,$$

$$-t(J_0 - J_2) = \theta + (2l' + 1)\pi,$$

$$-t(J_0 - 2J_1 + J_2) = \theta + 2l''\pi. \tag{3.29}$$

For $l = l' = l'' = 0$, one can obtain

$$J_0 = -\frac{2\theta + \pi}{4t_0}, \quad J_1 = 0, \quad J_2 = \frac{\pi}{4t_0}, \tag{3.30}$$

whereas by choosing $l = l' = 0, \ l'' = 1$, the solution to (3.29) is given by

$$J_0 = -\frac{\theta + \pi}{2t_0}, \quad J_1 = \frac{\pi}{4t_0}, \quad J_2 = 0. \tag{3.31}$$

In the first case, the time t_0 at which the state $|\phi_0\rangle = |0\rangle = |1000\rangle$ is perfectly transferred to the vertex $|\phi_2\rangle = |2\rangle = |0010\rangle$ is given by

$$t_0 = -\frac{2\theta + \pi}{4J_0} = \frac{\pi}{4J_2}, \tag{3.32}$$

whereas in the latter case t_0 is given by

$$t_0 = -\frac{\theta + \pi}{2J_0} = \frac{\pi}{4J_1}. \tag{3.33}$$

2. Hypercube network

The hypercube of dimension d [known also as binary Hamming scheme $H(d,2)$] is a network with $N = 2^d$ nodes, each of which can be labeled by a d-bit binary string. Two nodes on the hypercube described by bit strings $\vec{x}$ and $\vec{y}$ are connected by an edge if $|\vec{x} - \vec{y}| = 1$, where $|\vec{x}|$ is the Hamming weight of $\vec{x}$. In other words, if $\vec{x}$ and $\vec{y}$ differ by only a single bit flip, then the two corresponding nodes on the network are connected. Thus, each of the 2^d nodes on the hypercube has degree d. For the hypercube network with dimension d we have $d+1$ strata with

$$\kappa_i = \frac{d!}{i!\,(d-i)!}, \quad 0 \le i \le d-1. \tag{3.34}$$

The intersection numbers are given by

$$b_i = d - i, \quad 0 \le i \le d-1; \quad c_i = i, \quad 1 \le i \le d. \tag{3.35}$$

Furthermore, the adjacency matrices of this network are given by

$$A_i = \sum_{\text{perm}} \underbrace{\sigma_x \otimes \sigma_x \cdots \otimes \sigma_x}_{i} \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{n-i} \quad i = 0,1, \ldots, n, \tag{3.36}$$

where, the summation is taken over all possible nontrivial permutations.

It has been shown that the eigenmatrices P and Q for the Hamming scheme $H(d,2)$ are the same, i.e., this scheme is self-dual [39]. Also, Delsarte [40] showed that the entries of the eigenmatrix $P = Q$ for the Hamming scheme $H(d,2)$ can be found using the Krawtchouk polynomials as follows:

$$P_{il} = Q_{il} = K_l(i), \tag{3.37}$$

where $K_l(x)$ are the Krawtchouk polynomials defined as

$$K_l(x) = \sum_{i=0}^{l} \binom{x}{i}\binom{d-x}{l-i}(-1)^i. \tag{3.38}$$

Therefore, we have $((P^t)^{-1})_{il} = \frac{1}{2^d}Q_{li} = \frac{1}{2^d}K_i(l)$.

The eigenvalues x_l of the adjacency matrix $A \equiv A_1$ and corresponding degeneracy's γ_l are given by

$$x_l = 2l - d,$$

$$\gamma_l = \frac{d!}{2^d l!\,(d-l)!}, \quad l = 0,1, \ldots, d. \tag{3.39}$$

By using (3.37), we have

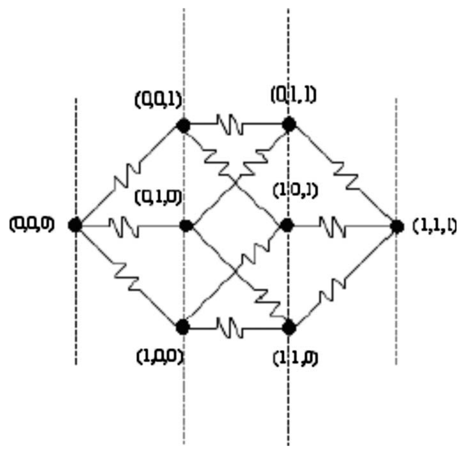

FIG. 2. Shows the cube or Hamming scheme $H(3,2)$ with vertex set $V=\{(ijk):\ i,j,k=0,1\}$, where the vertical dashed lines denote the four strata of the cube.

$$\eta_l = e^{-2it\sum_{m=0}^{d} J_m K_m(l)}, \quad l=0,1,\ldots,d. \tag{3.40}$$

Now, in order to evaluate the time t_0 at which PST takes place, the following equations must be satisfied:

$$\eta_l \gamma_l = \frac{e^{i\theta}}{2^d} Q_{dl} = \frac{e^{i\theta}}{2^d} K_l(d), \quad \forall \quad l=0,1,\ldots,d,$$

which are equivalent to

$$\frac{d!}{l!\,(d-l)!} e^{-2it_0\sum_{m=0}^{d} J_m K_m(l)} = e^{i\theta} K_l(d), \quad \forall \quad l=0,1,\ldots,d. \tag{3.41}$$

For instance, in the case of $d=3$ (see Fig. 2), we must solve the following equations:

$$e^{-2it_0(J_0+3J_1+3J_2+J_3)} = e^{i\theta},$$

$$e^{-2it_0(J_0+J_1-J_2-J_3)} = -e^{i\theta},$$

$$e^{-2it_0(J_0-J_1-J_2+J_3)} = e^{i\theta},$$

$$e^{-2it_0(J_0-3J_1+3J_2-J_3)} = -e^{i\theta}. \tag{3.42}$$

By solving Eqs. (3.42) one can obtain the following solution:

$$J_0 = -\frac{2\theta+3\pi}{4t_0}, \quad J_1 = -\frac{\pi J_0}{2\theta+3\pi} = \frac{\pi}{4t_0}, \quad J_2 = J_3 = 0;$$
$$\theta \neq -3\pi/2 \tag{3.43}$$

that is the time t_0 at which PST takes place is given by

$$t_0 = -\frac{2\theta+3\pi}{4J_0} = \frac{\pi}{4J_1}. \tag{3.44}$$

In Appendix C we consider PST over antipodes of some important finite distance-regular networks [46–49].

IV. CONCLUSION

It was noted that any distance-regular network (not just the hypercube which was discussed by Christandl *et al.* in [6,7]) as a spin network can be projected to a linear chain and so can allow PST over arbitrarily long distances. By using spectral analysis techniques and algebraic combinatoric structures of distance-regular graphs such as stratification and Bose-Mesner algebra, a method for finding a suitable set of coupling constants in some particular spin Hamiltonians associated with spin networks of distance-regular type was given so that PST between antipodes of the networks can be achieved. As examples, the cycle network with even number of vertices and d-dimensional hypercube were considered in details and some other important distance-regular networks are discussed in Appendix C.

APPENDIX A: SOME USEFUL ALGEBRAIC PROPERTIES OF DISTANCE-REGULAR GRAPHS

In this appendix we recall some more algebraic properties of distance-regular graphs which are necessary for obtaining the essential results of the paper.

First, we note that the algebraic structure of distance-regular graphs has its origin in the fact that these graphs are underlying graphs of so-called association schemes [12]. As a consequence of this fact, the adjacency matrices A_i of a given distance-regular graph with diameter d fulfill the relation

$$A_i A_j = A_j A_i = \sum_{k=0}^{d} p_{ij}^k A_k, \tag{A1}$$

which indicates that A_i for $i=0,1,\ldots,d$ form a commutative algebra called Bose-Mesner algebra (this algebra is defined not only for any distance-regular graph but also for any underlying graph of association schemes). The Bose-Mesner algebra has a second basis $E_0,\ldots,E_d$ such that, $E_iE_j=\delta_{ij}E_i$ and $\sum_{i=0}^{d} E_i = I$ with $E_0=1/vJ_v$ $(v:=|V|$ is the size of the graph). The matrices E_i for $0\le i\le d$ are known as primitive idempotents of the graph. Furthermore, there are matrices $\mathcal{P}$ and $\mathcal{Q}$ such that the two bases of the Bose-Mesner algebra can be related to each other as follows:

$$A_i = \sum_{j=0}^{d} \mathcal{P}_{ji} E_j, \quad 0 \le j \le d,$$

$$E_i = \frac{1}{v}\sum_{j=0}^{d} \mathcal{Q}_{ji} A_j, \quad 0 \le j \le d. \tag{A2}$$

Then, clearly we have

$$\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = vI. \tag{A3}$$

It also follows that

$$A_j E_i = \mathcal{P}_{ij} E_i, \tag{A4}$$

which indicates that $\mathcal{P}_{ij}$ is the ith eigenvalue of A_j and that the columns of E_i are corresponding eigenvectors (some times the matrix $\mathcal{P}$ is called eigenvalue matrix).

As it will be seen in Appendix B 1, in the case of distance-regular graphs, the adjacency matrices A_j are polynomials of the adjacency matrix $A\equiv A_1$, i.e., $A_j=P_j(A)$, where P_j is a polynomial of degree j, then the eigenvalues $\mathcal{P}_{ij}$ in (A4) are polynomials of eigenvalues $\mathcal{P}_{i1} \equiv \lambda_i$ (eigenvalues of the adjacency matrix A). This indicates that in distance-regular graphs, the matrix $\mathcal{P}^t$ is a polynomial transformation [41] as

$$\mathcal{P}^t = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ P_1(\lambda_0) & P_1(\lambda_1) & \cdots & P_1(\lambda_d) \\ P_2(\lambda_0) & P_2(\lambda_1) & \cdots & P_2(\lambda_d) \\ \vdots & \vdots & \cdots & \vdots \\ P_d(\lambda_0) & P_d(\lambda_1) & \cdots & P_d(\lambda_d) \end{pmatrix} \quad \text{(A5)}$$

or $\mathcal{P}_{ji} = P_i(\lambda_j)$.

APPENDIX B: THREE-TERM RECURSION STRUCTURE OF DISTANCE-REGULAR GRAPHS AND SOME SPECTRAL TECHNIQUES

1. Three-term recursion structure of distance-regular graphs

In this section, we note that distance-regular graphs possess a three-term recursion structure, in the sense that their adjacency matrices take tridiagonal form in the stratification basis. To do so, first we note that, for these graphs, the nonzero intersection numbers [defined in Eq. (2.1)] are given by

$$a_i = p^i_{i1}, \quad b_i = p^i_{i+1,1}, \quad c_i = p^i_{i-1,1}, \quad \text{(B1)}$$

respectively. The intersection numbers (B1) and the valences κ_i with $\kappa_1 \equiv \kappa$ [=deg(α), for each vertex α] satisfy the following obvious conditions:

$$a_i + b_i + c_i = \kappa, \quad \kappa_{i-1}b_{i-1} = \kappa_i c_i, \quad i = 1, \ldots, d,$$

$$\kappa_0 = c_1 = 1, \quad b_0 = \kappa_1 = \kappa \quad (c_0 = b_d = 0). \quad \text{(B2)}$$

Thus all parameters of a distance-regular graph can be obtained from its intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$. Then, it can be shown that the following recursion relations are satisfied:

$$A_1 A_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1}, \quad i = 1,2, \ldots, d-1,$$

$$A_1 A_d = b_{d-1}A_{d-1} + (\kappa - c_d)A_d. \quad \text{(B3)}$$

The recursion relations (B3) imply that

$$A_i = P_i(A), \quad i = 0,1, \ldots, d, \quad \text{(B4)}$$

where, P_i is a polynomial of degree i. Then, one can easily obtain the following three-term recursion relations for the unit vectors $|\phi_i\rangle$, $i=0,1,\ldots,d$:

$$A|\phi_i\rangle = \beta_{i+1}|\phi_{i+1}\rangle + \alpha_i|\phi_i\rangle + \beta_i|\phi_{i-1}\rangle, \quad \text{(B5)}$$

where, the coefficients α_i and β_i are defined as

$$\alpha_0 = 0, \quad \alpha_k \equiv a_k = \kappa - b_k - c_k, \quad \omega_k \equiv \beta_k^2 = b_{k-1}c_k, \quad \text{(B6)}$$

$$k = 1, \ldots, d.$$

Following Ref. [18], we will refer to the parameters α_k and ω_k as QD (quantum decomposition) parameters (see Refs. [19,25,26] for more details).

The three-term recursion relations (B5) indicate that the adjacency matrix $A \equiv A_1$ is represented as a tridiagonal matrix in the stratification space, i.e., we have

$$A = \begin{pmatrix} \alpha_0 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_1 & \beta_2 & \cdots & 0 \\ 0 & \beta_2 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \beta_d \\ 0 & 0 & 0 & \beta_d & \alpha_d \end{pmatrix}. \quad \text{(B7)}$$

2. Spectral techniques

In this section we recall some facts about spectral techniques used in the paper and define the Stieltjes-Hilbert transform associated with a distance-regular graph.

From orthonormality of the unit vectors $|\phi_i\rangle$ given in Eq. (2.4) (with $|\phi_0\rangle$ as unit vector assigned to reference vertex) we have

$$\delta_{ij} = \langle\phi_i|\phi_j\rangle = \frac{1}{\sqrt{\kappa_i\kappa_j}}\langle\phi_0|A_iA_j|\phi_0\rangle = \int_R P'_i(x)P'_j(x)\mu(dx), \quad \text{(B8)}$$

with $P'_i(A) := \frac{1}{\sqrt{\kappa_i}}P_i(A)$ where, we have used the Eqs. (2.6) and (B4) to write

$$|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}}A_i|\phi_0\rangle = \frac{1}{\sqrt{\kappa_i}}P_i(A)|\phi_0\rangle \equiv P'_i(A)|\phi_0\rangle. \quad \text{(B9)}$$

Now, by substituting (B9) in (B5) and rescaling P'_k as $Q_k = \beta_1 \cdots \beta_k P'_k$, the spectral distribution μ under question will be characterized by the property of orthonormal polynomials $\{Q_k\}$ defined recurrently by

$$Q_0(x) = 1, \quad Q_1(x) = x,$$

$$xQ_k(x) = Q_{k+1}(x) + \alpha_k Q_k(x) + \beta_k^2 Q_{k-1}(x), \quad k \geq 1. \quad \text{(B10)}$$

If such a spectral distribution is unique, the spectral distribution μ is determined by the identity

$$G_\mu(x) = \int_R \frac{\mu(dy)}{x-y} = \cfrac{1}{x - \alpha_0 - \cfrac{\beta_1^2}{x - \alpha_1 - \cfrac{\beta_2^2}{x - \alpha_2 - \cfrac{\beta_3^2}{x - \alpha_3 - \cdots}}}}$$

$$= \frac{Q_d^{(1)}(x)}{Q_{d+1}(x)} = \sum_{l=0}^{d} \frac{\gamma_l}{x - x_l}, \quad \text{(B11)}$$

where, x_l are the roots of polynomial $Q_{d+1}(x)$. $G_\mu(x)$ is called the Stieltjes-Hilbert transform of spectral distribution μ or Stieltjes function and polynomials $\{Q_k^{(1)}\}$ are defined recurrently as

$$Q_0^{(1)}(x) = 1, \quad Q_1^{(1)}(x) = x - \alpha_1,$$

$$xQ_k^{(1)}(x) = Q_{k+1}^{(1)}(x) + \alpha_{k+1}Q_k^{(1)}(x) + \beta_{k+1}^2 Q_{k-1}^{(1)}(x), \quad k \geq 1, \tag{B12}$$

respectively. The coefficients γ_l appearing in (B11) are calculated as

$$\gamma_l := \lim_{x\to x_l}[(x-x_l)G_\mu(x)]. \tag{B13}$$

Now let $G_\mu(z)$ be known, then the spectral distribution μ can be recovered from $G_\mu(z)$ by means of the Stieltjes-Hilbert inversion formula as

$$\mu(y)-\mu(x) = -\frac{1}{\pi}\lim_{v\to 0^+}\int_x^y \mathrm{Im}[G_\mu(u+iv)]du. \tag{B14}$$

Substituting the right-hand side of (B11) in (B14), the spectral distribution can be determined in terms of $x_l, l=1,2,\ldots$ and Gauss quadrature constants $\gamma_l, l=1,2,\ldots$ as

$$\mu = \sum_{l=0}^{d} \gamma_l \delta(x-x_l) \tag{B15}$$

(for more details see Refs. [42–45]).

APPENDIX C: PST OVER SOME IMPORTANT FINITE DISTANCE-REGULAR NETWORKS

In this appendix we consider some important finite distance-regular networks such that their last stratum contains only one node. Then by using the prescription of Sec. III, we investigate PST over antipodes of these networks.

1. Icosahedron [46]

The intersection array is

$$\{b_0,b_1,b_2;c_1,c_2,c_3\} = \{5,2,1;1,2,5\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 5, \quad \kappa_2 = 5, \quad \kappa_3 = 1,$$

$$\alpha_0 = 0, \quad \alpha_1 = \alpha_2 = 2, \quad \alpha_3 = 0; \quad \omega_1 = 5, \quad \omega_2 = 4, \quad \omega_3 = 5.$$

The polynomials $P_i(x)$ are

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(x^2 - 2x - 5),$$

$$P_3(x) = \frac{1}{10}(x^3 - 4x^2 - 5x + 10).$$

The Stieltjes function is

$$G_\mu(x) = \frac{x^3 - 4x^2 - 5x + 10}{x^4 - 4x^3 - 10x^2 + 20x + 25}.$$

The spectral distribution $[\mu(x)=\sum_{l=0}^{d}\gamma_l\delta(x-x_l)]$ is

$$\mu(x) = \frac{1}{12}[5\delta(x+1) + \delta(x-5) + 3\delta(x-\sqrt{5}) + 3\delta(x+\sqrt{5})].$$

Now, one can obtain the matrix P^t and its inverse. Then by solving Eq. (3.18), the solution is obtained as follows:

$$J_0 = -\frac{6\theta + 7\pi}{12t_0}, \quad J_1 = -\frac{(5-3\sqrt{5})\pi}{60t_0},$$

$$J_2 = -\frac{(5+3\sqrt{5})\pi}{60t_0}, \quad J_3 = \frac{5\pi}{12t_0}. \tag{C1}$$

Then, the time t_0 at which PST takes place is given by

$$t_0 = -\frac{2\theta + \pi}{4J_0} = \frac{\pi}{4J_3}. \tag{C2}$$

2. Desargues graph [46]

The intersection array is

$$\{b_0,b_1,b_2,b_3,b_4;c_1,c_2,c_3,c_4,c_5\} = \{3,2,2,1,1;1,1,2,2,3\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 3, \quad \kappa_2 = 6, \quad \kappa_3 = 6, \quad \kappa_4 = 3, \quad \kappa_5 = 1,$$

$$\alpha_i = 0, \quad i = 0,1,\ldots,5; \quad \omega_1 = 3, \quad \omega_2 = 2, \quad \omega_3 = 4,$$
$$\omega_4 = 2, \quad \omega_5 = 3.$$

The polynomials $P_i(x)$ are

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - 3,$$

$$P_3(x) = \frac{1}{2}(x^3 - 5x), \quad P_4(x) = \frac{1}{4}(x^4 - 9x^2 + 12),$$

$$P_5(x) = \frac{1}{12}(x^5 - 11x^3 + 22x).$$

The Stieltjes function is

$$G_\mu(x) = \frac{x^5 - 11x^3 + 22x}{x^6 - 14x^4 + 49x^2 - 36}.$$

The spectral distribution is

$$\mu(x) = \frac{1}{20}[5\delta(x+1) + 5\delta(x-1) + 4\delta(x+2) + 4\delta(x-2) + \delta(x+3) + \delta(x-3)].$$

The solution to Eq. (3.18) is given by

$$J_0 = -\frac{30\theta + 51\pi}{60t_0}, \quad J_1 = \frac{\pi}{10t_0}, \quad J_2 = -\frac{4\pi}{15t_0},$$

$$J_3 = 0, \quad J_4 = \frac{\pi}{15t_0}, \quad J_5 = \frac{\pi}{4t_0}.$$

3. Dodecahedron [46]

The intersection array is

$$\{b_0,b_1,b_2,b_3,b_4;c_1,c_2,c_3,c_4,c_5\} = \{3,2,1,1,1;1,1,1,2,3\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 3, \quad \kappa_2 = 6, \quad \kappa_3 = 6, \quad \kappa_4 = 3, \quad \kappa_5 = 1,$$

$$\alpha_0=\alpha_1=0, \quad \alpha_2=\alpha_3=1, \quad \alpha_4=\alpha_5=0; \quad \omega_1=3,$$

$$\omega_2=2, \quad \omega_3=1, \quad \omega_4=2, \quad \omega_5=3.$$

The polynomials $P_i(x)$ are

$$P_0=1, \quad P_1(x)=x, \quad P_2(x)=x^2-3,$$

$$P_3(x)=x^3-5x-x^2+3, \quad P_4(x)=\frac{1}{2}(x^4-5x^2-2x^3+8x),$$

$$P_5(x)=\frac{1}{6}(x^5-7x^3-2x^4+10x^2+10x-6).$$

The Stieltjes function is

$$G_\mu(x)=\frac{x^5-7x^3-2x^4+10x^2+10x-6}{x^6-10x^4-2x^5+16x^3+25x^2-30x}.$$

The spectral distribution is

$$\mu(x)=\frac{1}{20}[4\delta(x)+5\delta(x-1)+4\delta(x+2)+\delta(x-3)+3\delta(x-\sqrt{5})+3\delta(x+\sqrt{5})].$$

The solution to Eq. (3.18) is given by

$$J_0=-\frac{\theta+2\pi}{2t_0}, \quad J_1=\frac{(2+3\sqrt{5})\pi}{60t_0}, \quad J_2=-\frac{17\pi}{60t_0}, \quad J_3=\frac{\pi}{60t_0}, \quad J_4=\frac{(2-3\sqrt{5})\pi}{60t_0}.$$

4. Taylor graph over the Paley graph [$P(13)$] [47]

The intersection array is

$$\{b_0,b_1,b_2;c_1,c_2,c_3\}=\{13,6,1;1,6,13\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0=1, \quad \kappa\equiv\kappa_1=13, \quad \kappa_2=13, \quad \kappa_3=1,$$

$$\alpha_0=0, \quad \alpha_1=\alpha_2=6, \quad \alpha_3=0; \quad \omega_1=13, \quad \omega_2=36, \quad \omega_3=13.$$

The polynomials $P_i(x)$ are

$$P_0=1, \quad P_1(x)=x, \quad P_2(x)=\frac{1}{6}(x^2-6x-13),$$

$$P_3(x)=\frac{1}{78}(x^3-12x^2-13x+78).$$

The Stieltjes function is

$$G_\mu(x)=\frac{x^3-12x^2-13x+78}{x^4-12x^3-26x^2+156x+169}.$$

The spectral distribution is

$$\mu(x)=\frac{1}{28}[13\delta(x+1)+\delta(x-13)+7\delta(x-\sqrt{13})+7\delta(x+\sqrt{13})].$$

The solution to Eq. (3.18) is given by

$$J_0=-\frac{14\theta+15\pi}{28t_0}, \quad J_1=-\frac{(13-7\sqrt{13})\pi}{364t_0}, \quad J_2=-\frac{(13+7\sqrt{13})\pi}{364t_0}, \quad J_3=\frac{13\pi}{28t_0}.$$

5. Taylor graph [47]

The intersection array is

$$\{b_0,b_1,b_2;c_1,c_2,c_3\}=\{15,8,1;1,8,15\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0=1, \quad \kappa\equiv\kappa_1=15, \quad \kappa_2=15, \quad \kappa_3=1,$$

$$\alpha_0=0, \quad \alpha_1=\alpha_2=6, \quad \alpha_3=0; \quad \omega_1=15, \quad \omega_2=64, \quad \omega_3=15.$$

The polynomials $P_i(x)$ are

$$P_0=1, \quad P_1(x)=x, \quad P_2(x)=\frac{1}{8}(x^2-6x-15),$$

$$P_3(x)=\frac{1}{120}(x^3-12x^2-43x+90).$$

The Stieltjes function is

$$G_\mu(x)=\frac{x^3-12x^2-43x+90}{x^4-12x^3-58x^2+180x+225}.$$

The spectral distribution is

$$\mu(x)=\frac{1}{32}[15\delta(x+1)+10\delta(x-3)+6\delta(x+5)+\delta(x-15)].$$

The solution to Eq. (3.18) is given by

$$J_0=-\frac{16\theta+15\pi}{32t_0}, \quad J_1=\frac{\pi}{32t_0}, \quad J_2=-\frac{3\pi}{32t_0}, \quad J_3=\frac{13\pi}{32t_0}.$$

6. Taylor graph over [$T(6)$] [47]

The intersection array is

$$\{b_0,b_1,b_2;c_1,c_2,c_3\}=\{15,6,1;1,6,15\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0=1, \quad \kappa\equiv\kappa_1=15, \quad \kappa_2=15, \quad \kappa_3=1,$$

$$\alpha_0=0, \quad \alpha_1=\alpha_2=8, \quad \alpha_3=0; \quad \omega_1=15, \quad \omega_2=36, \quad \omega_3=15.$$

The polynomials $P_i(x)$ are

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{6}(x^2 - 8x - 15),$$

$$P_3(x) = \frac{1}{90}(x^3 - 16x^2 + 13x + 120).$$

The Stieltjes function is

$$G_\mu(x) = \frac{x^3 - 16x^2 + 13x + 120}{x^4 - 16x^3 - 2x^2 + 240x + 225}.$$

The spectral distribution is

$$\mu(x) = \frac{1}{32}[15\delta(x+1) + 10\delta(x+3) + 6\delta(x-5) + \delta(x-15)].$$

The solution to Eq. (3.18) is given by

$$J_0 = -\frac{16\theta + 15\pi}{32t_0}, \quad J_1 = -\frac{3\pi}{32t_0},$$

$$J_2 = \frac{\pi}{32t_0}, \quad J_3 = \frac{13\pi}{32t_0}.$$

7. Wells [48]

The intersection array is

$$\{b_0,b_1,b_2,b_3;c_1,c_2,c_3,c_4\} = \{5,4,1,1;1,1,4,5\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 5, \quad \kappa_2 = 20, \quad \kappa_3 = 5, \quad \kappa_4 = 1,$$

$$\alpha_0 = \alpha_1 = 0, \quad \alpha_2 = 3, \quad \alpha_3 = \alpha_4 = 0; \quad \omega_1 = 5,$$
$$\omega_2 = \omega_3 = 4, \quad \omega_4 = 5.$$

The polynomials $P_i(x)$ are

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - 5,$$

$$P_3(x) = \frac{1}{4}(x^3 - 9x - 3x^2 + 15),$$

$$P_4(x) = \frac{1}{20}(x^4 - 13x^2 - 3x^3 + 15x + 20).$$

The Stieltjes function is

$$G_\mu(x) = \frac{x^4 - 13x^2 - 3x^3 + 15x + 20}{x^5 - 18x^3 - 3x^4 + 30x^2 + 65x - 75}.$$

The spectral distribution is

$$\mu(x) = \frac{1}{32}[10\delta(x-1) + 5\delta(x+3) + \delta(x-5) + 8\delta(x+\sqrt{5})$$
$$+ 8\delta(x-\sqrt{5})].$$

The solution to Eq. (3.18) is given by

$$J_0 = -\frac{16\theta + 23\pi}{32t_0}, \quad J_1 = \frac{(5-8\sqrt{5})\pi}{160t_0},$$

$$J_2 = -\frac{3\pi}{32t_0}, \quad J_3 = \frac{(5+8\sqrt{5})\pi}{160t_0}, \quad J_4 = \frac{9\pi}{32t_0}.$$

8. Hadamard network [49]

The intersection array is

$$\{b_0,b_1,b_2,b_3;c_1,c_2,c_3,c_4\} = \{8,7,4,1;1,4,7,8\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 8, \quad \kappa_2 = 14, \quad \kappa_3 = 8, \quad \kappa_4 = 1,$$

$$\alpha_i = 0, \quad i = 0,1,\ldots,4; \quad \omega_1 = 8, \quad \omega_2 = 28, \quad \omega_3 = 28,$$
$$\omega_4 = 8.$$

The polynomials $P_i(x)$ are

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{4}(x^2 - 8),$$

$$P_3(x) = \frac{1}{28}(x^3 - 36x), \quad P_4(x) = \frac{1}{224}(x^4 - 64x^2 + 224).$$

The Stieltjes function is

$$G_\mu(x) = \frac{x^4 - 64x^2 + 224}{x^5 - 72x^3 + 512x}.$$

The spectral distribution is

$$\mu(x) = \frac{1}{32}[14\delta(x) + \delta(x-8) + \delta(x+8) + 8\delta(x-2\sqrt{2})$$
$$+ 8\delta(x+2\sqrt{2})].$$

The solution to Eq. (3.18) is given by

$$J_0 = -\frac{16\theta + 19\pi}{32t_0}, \quad J_1 = \frac{(1+2\sqrt{2})\pi}{32t_0}, \quad J_2 = -\frac{3\pi}{32t_0},$$

$$J_3 = \frac{(1-2\sqrt{2})\pi}{32t_0}, \quad J_4 = \frac{13\pi}{32t_0}.$$

9. Taylor graph over the Paley graph [$P(17)$] [47]

The intersection array is

$$\{b_0,b_1,b_2;c_1,c_2,c_3\} = \{17,8,1;1,8,17\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 17, \quad \kappa_2 = 17, \quad \kappa_3 = 1,$$

$$\alpha_0 = 0, \quad \alpha_1 = \alpha_2 = 8, \quad \alpha_3 = 0; \quad \omega_1 = 17, \quad \omega_2 = 64,$$
$$\omega_3 = 17.$$

The polynomials $P_i(x)$ are

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{8}(x^2 - 8x - 17),$$

$$P_3(x) = \frac{1}{136}(x^3 - 16x^2 - 17x + 136).$$

The Stieltjes function is

$$G_\mu(x) = \frac{x^3 - 16x^2 - 17x + 136}{x^4 - 16x^3 - 34x^2 + 272x + 289}.$$

The spectral distribution is

$$\mu(x) = \frac{1}{36}[17\delta(x+1) + \delta(x-17) + 9\delta(x-\sqrt{17}) + 9\delta(x+\sqrt{17})].$$

The solution to Eq. (3.18) is given by

$$J_0 = -\frac{18\theta + 19\pi}{36t_0}, \quad J_1 = -\frac{(17-9\sqrt{17})\pi}{612t_0}, \quad J_2 = -\frac{(17+9\sqrt{17})\pi}{612t_0}, \quad J_3 = \frac{17\pi}{36t_0}.$$

10. Hadamard network [47]

The intersection array is

$$\{b_0,b_1,b_2,b_3;c_1,c_2,c_3,c_4\} = \{12,11,6,1;1,6,11,12\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 12, \quad \kappa_2 = 22, \quad \kappa_3 = 12, \quad \kappa_4 = 1,$$

$$\alpha_i = 0, \quad i = 0,1,\ldots,4; \quad \omega_1 = 12, \quad \omega_2 = \omega_3 = 66, \quad \omega_4 = 12.$$

The polynomials $P_i(x)$ are

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{6}(x^2 - 12),$$

$$P_3(x) = \frac{1}{66}(x^3 - 78x),$$

$$P_4(x) = \frac{1}{792}(x^4 - 144x^2 + 792).$$

The Stieltjes function is

$$G_\mu(x) = \frac{x^4 - 144x^2 + 792}{x^5 - 156x^3 + 1728x}.$$

The spectral distribution is

$$\mu(x) = \frac{1}{48}[22\delta(x) + \delta(x+12) + \delta(x-12) + 12\delta(x-2\sqrt{3}) + 12\delta(x+2\sqrt{3})].$$

The solution to Eq. (3.18) is given by

$$J_0 = -\frac{24\theta + 27\pi}{48t_0}, \quad J_1 = -\frac{(1-2\sqrt{3})\pi}{48t_0}, \quad J_2 = -\frac{\pi}{16t_0}, \quad J_3 = -\frac{(1+2\sqrt{3})\pi}{48t_0}, \quad J_4 = \frac{21\pi}{48t_0}.$$

11. Taylor graph over the strongly regular graph [*SRG*(25,12)] [47]

The intersection array is

$$\{b_0,b_1,b_2;c_1,c_2,c_3\} = \{25,12,1;1,12,25\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 25, \quad \kappa_2 = 25, \quad \kappa_3 = 1,$$

$$\alpha_0 = 0, \quad \alpha_1 = \alpha_2 = 12, \quad \alpha_3 = 0; \quad \omega_1 = 25, \quad \omega_2 = 144, \quad \omega_3 = 25.$$

The polynomials $P_i(x)$ are

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{12}(x^2 - 12x - 25),$$

$$P_3(x) = \frac{1}{300}(x^3 - 24x^2 - 25x + 300).$$

The Stieltjes function is

$$G_\mu(x) = \frac{x^3 - 24x^2 - 25x + 300}{x^4 - 24x^3 - 50x^2 + 600x + 625}.$$

The spectral distribution is

$$\mu(x) = \frac{1}{52}[25\delta(x+1) + 13\delta(x-5) + 13\delta(x+5) + \delta(x-25)].$$

The solution to Eq. (3.18) is given by

$$J_0 = -\frac{26\theta + 27\pi}{52t_0}, \quad J_1 = \frac{2\pi}{65t_0}, \quad J_2 = -\frac{9\pi}{130t_0}, \quad J_3 = \frac{25\pi}{52t_0}.$$

12. The Gosset graph corresponding to the Taylor graph over the Schläfli graph [47]

The intersection array is

$$\{b_0,b_1,b_2;c_1,c_2,c_3\} = \{27,10,1;1,10,27\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0 = 1, \quad \kappa \equiv \kappa_1 = 27, \quad \kappa_2 = 27, \quad \kappa_3 = 1,$$

$$\alpha_0 = 0, \quad \alpha_1 = \alpha_2 = 16, \quad \alpha_3 = 0; \quad \omega_1 = 27, \quad \omega_2 = 100, \quad \omega_3 = 27.$$

The polynomials $P_i(x)$ are

$$P_0 = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{10}(x^2 - 16x - 27),$$

$$P_3(x) = \frac{1}{270}(x^3 - 32x^2 + 129x + 432).$$

The Stieltjes function is

$$G_\mu(x) = \frac{x^3 - 32x^2 + 129x + 432}{x^4 - 32x^3 + 102x^2 + 864x + 729}.$$

The spectral distribution is

$$\mu(x) = \frac{1}{56}[27\delta(x+1) + 21\delta(x+3) + 7\delta(x-9) + \delta(x-27)].$$

The solution to Eq. (3.18) is given by

$$J_0=-\frac{14\theta+11\pi}{28t_0},\quad J_1=-\frac{5\pi}{84t_0},\quad J_2=\frac{\pi}{42t_0},\quad J_3=\frac{5\pi}{14t_0}.$$

13. Taylor(Co-Schläfli) [47]

The intersection array is

$$\{b_0,b_1,b_2;c_1,c_2,c_3\}=\{27,16,1;1,16,27\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0=1,\quad \kappa\equiv\kappa_1=27,\quad \kappa_2=27,\quad \kappa_3=1,$$

$$\alpha_0=0,\quad \alpha_1=\alpha_2=10,\quad \alpha_3=0;\quad \omega_1=27,\quad \omega_2=256,\quad \omega_3=27.$$

The polynomials $P_i(x)$ are

$$P_0=1,\quad P_1(x)=x,\quad P_2(x)=\frac{1}{16}(x^2-10x-27),$$

$$P_3(x)=\frac{1}{432}(x^3-20x^2-183x+270).$$

The Stieltjes function is

$$G_\mu(x)=\frac{x^3-20x^2-183x+270}{x^4-20x^3-210x^2+540x+729}.$$

The spectral distribution is

$$\mu(x)=\frac{1}{56}[27\delta(x+1)+21\delta(x-3)+7\delta(x+9)+\delta(x-27)].$$

The solution to Eq. (3.18) is given by

$$J_0=-\frac{14\theta+11\pi}{28t_0},\quad J_1=\frac{\pi}{42t_0},\quad J_2=-\frac{5\pi}{84t_0},\quad J_3=\frac{5\pi}{14t_0}.$$

14. Taylor [$SRG(29,14)$] [47]

The intersection array is

$$\{b_0,b_1,b_2;c_1,c_2,c_3\}=\{29,14,1;1,14,29\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0=1,\quad \kappa\equiv\kappa_1=29,\quad \kappa_2=29,\quad \kappa_3=1,$$

$$\alpha_0=0,\quad \alpha_1=\alpha_2=14,\quad \alpha_3=0;\quad \omega_1=29,\quad \omega_2=196,\quad \omega_3=29.$$

The polynomials $P_i(x)$ are

$$P_0=1,\quad P_1(x)=x,\quad P_2(x)=\frac{1}{14}(x^2-14x-29),$$

$$P_3(x)=\frac{1}{406}(x^3-28x^2-29x+406).$$

The Stieltjes function is

$$G_\mu(x)=\frac{x^3-28x^2-29x+406}{x^4-28x^3-58x^2+812x+841}.$$

The spectral distribution is

$$\mu(x)=\frac{1}{60}[29\delta(x+1)+\delta(x-29)+15\delta(x-\sqrt{29})+15\delta(x+\sqrt{29})].$$

The solution to Eq. (3.18) is given by

$$J_0=-\frac{30\theta+31\pi}{60t_0},\quad J_1=-\frac{(29-15\sqrt{29})\pi}{1740t_0},\quad J_2=-\frac{(29+15\sqrt{29})\pi}{1740t_0},\quad J_3=\frac{29\pi}{60t_0}.$$

15. Doubled Odd graph DO (4) [47]

The intersection array is

$$\{b_0,b_1,b_2,b_3,b_4,b_5,b_6;c_1,c_2,c_3,c_4,c_5,c_6,c_7\}=\{4,3,3,2,2,1,1;1,1,2,2,3,3,4\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0=1,\quad \kappa\equiv\kappa_1=4,\quad \kappa_2=12,\quad \kappa_3=18,\quad \kappa_4=18,\quad \kappa_5=12,\quad \kappa_6=4,\quad \kappa_7=1,$$

$$\alpha_i=0,\quad i=0,1,\ldots,7;\quad \omega_1=4,\quad \omega_2=3,\quad \omega_3=6,\quad \omega_4=4,\quad \omega_5=6,\quad \omega_6=3,\quad \omega_7=4.$$

The polynomials $P_i(x)$ are

$$P_0=1,\quad P_1(x)=x,\quad P_2(x)=x^2-4,$$

$$P_3(x)=\frac{1}{2}(x^3-7x),\quad P_4(x)=\frac{1}{4}(x^4-13x^2+24),$$

$$P_5(x)=\frac{1}{12}(x^5-17x^3+52x),\quad P_6(x)=\frac{1}{36}(x^6-23x^4+130x^2-144),\quad P_7(x)=\frac{1}{144}(x^7-26x^5+181x^3-300x).$$

The Stieltjes function is

$$G_\mu(x)=\frac{x^7-26x^5+181x^3-300x}{x^8-30x^6+273x^4-820x^2+576}.$$

The spectral distribution is

$$\mu(x)=\frac{1}{70}\{14[\delta(x-1)+\delta(x+1)+\delta(x-2)+\delta(x+2)]+6[\delta(x-3)+\delta(x+3)]+\delta(x-4)+\delta(x+4)\}.$$

The solution to Eq. (3.18) is given by

$$J_0=-\frac{70\theta+151\pi}{140t_0},\quad J_1=0,\quad J_2=-\frac{56\pi}{245t_0},$$

$$J_3=0,\quad J_4=\frac{4\pi}{105t_0},\quad J_5=-\frac{\pi}{70t_0},\quad J_6=-\frac{\pi}{35t_0},$$

$$J_7=\frac{\pi}{4t_0}.$$

16. The Johnson graph J(8,4) [47]

The intersection array is

$$\{b_0,b_1,b_2,b_3;c_1,c_2,c_3,c_4\}=\{16,9,4,1;1,4,9,16\}.$$

The sizes of the strata and the QD parameters are

$$\kappa_0=1,\quad \kappa\equiv\kappa_1=16,\quad \kappa_2=36,\quad \kappa_3=16,\quad \kappa_4=1,$$

$$\alpha_0=0,\quad \alpha_1=6,\quad \alpha_2=8,\quad \alpha_3=6,\quad \alpha_4=0;$$
$$\omega_1=16,\quad \omega_2=36,\quad \omega_3=36,\quad \omega_4=16.$$

The polynomials $P_i(x)$ are

$$P_0=1,\quad P_1(x)=x,\quad P_2(x)=\frac{1}{4}(x^2-6x-16),$$

$$P_3(x)=\frac{1}{36}(x^3-14x^2-4x+128),$$

$$P_4(x)=\frac{1}{576}(x^4-20x^3+44x^2+368x-192).$$

The Stieltjes function is

$$G_\mu(x)=\frac{x^4-20x^3+44x^2+368x-192}{x^5-20x^4+28x^3+592x^2-128x-2048}.$$

The spectral distribution is

$$\mu(x)=\frac{1}{70}[\delta(x-16)+7\delta(x-8)+14\delta(x+4)+28\delta(x+2)+20\delta(x-2)].$$

The solution to Eq. (3.18) is given by

$$J_0=-\frac{70\theta+199\pi}{140t_0},\quad J_1=\frac{\pi}{35t_0},\quad J_2=\frac{13\pi}{210t_0},$$
$$J_3=-\frac{\pi}{14t_0},\quad J_4=-\frac{17\pi}{140t_0}.$$

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