Decoherence and dissipation of a quantum harmonic oscillator coupled to two-level systems

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We derive and analyze the Born-Markov master equation for a quantum harmonic oscillator interacting with a bath of independent two-level systems. This hitherto virtually unexplored model plays a fundamental role as one of the four "canonical" system-environment models for decoherence and dissipation. To investigate the influence of further couplings of the environmental spins to a dissipative bath, we also derive the master equation for a harmonic oscillator interacting with a single spin coupled to a bosonic bath. Our models are experimentally motivated by quantum-electromechanical systems and micron-scale ion traps. Decoherence and dissipation rates are found to exhibit temperature dependencies significantly different from those in quantum Brownian motion. In particular, the systematic dissipation rate for the central oscillator decreases with increasing temperature and goes to zero at zero temperature, but there also exists a temperature-independent momentum-diffusion (heating) rate.

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I. INTRODUCTION

Theoretical studies of decoherence and dissipation in quantum systems have hitherto focused on three *canonical* system-environment models: (i) A harmonic oscillator (or, more generally, a particle moving in phase space) coupled to a bath of other harmonic oscillators (quantum Brownian motion) [1–4]; (ii) a quantum two-level system (TLS), represented by a spin-$\frac{1}{2}$ particle, interacting with a bath of harmonic oscillators (spin-boson model) [5]; and (iii) a spin-$\frac{1}{2}$ particle coupled to a bath of other spins (spin-spin model) [6]. Surprisingly, however, the fourth possible canonical combination, namely, a single harmonic oscillator interacting with a bath of spin-$\frac{1}{2}$ particles—which, in obvious nomenclature, shall henceforth be referred to as the oscillator-spin model [Fig. 1(a)]—has not yet been studied in any detail in the literature in terms of a Markovian master equation.

It is the purpose of this paper to close this gap by giving a microscopic treatment of the oscillator-spin model. We will derive the Born-Markov master equation and compare the resulting dynamics to those induced by an oscillator bath (quantum Brownian motion). Apart from its relevance in completing the set of canonical models, the oscillator-spin model is also motivated by recent experiments on quantum-electromechanical systems (QEMS) [7–9] and micron-scale ion traps. In both systems, a central quantum-mechanical vibrational degree of freedom interacts with two-level defects causing dissipation and decoherence of the oscillator. We may represent this situation by a harmonic oscillator coupled to a collection of TLS, i.e., by a model of the oscillator-spin type.

QEMS are nanometer-to-micrometer-sized crystalline mechanical resonators coupled to nanoscale electronic transducers that detect the high-frequency (MHz-GHz) vibrational motion of the resonator. Since only the lowest, fundamental flexural mode of the resonator turns out to be relevant [7], the resonator can be modeled as a single quantum-mechanical harmonic oscillator. Recent experimental evidence [10,11] (see also the molecular-dynamics simulation of [12] and earlier results in [13,14]) strongly suggests that the dominant source of decoherence and dissipation in QEMS is the interaction with two-level defects intrinsically present in the resonator itself.

In ion traps a single ion can be trapped by a time-dependent potential and cooled to very low energies [15]. Under the right conditions the motion of the ion is equivalent to that of a quasi-one-dimensional harmonically bound particle. A major source of decoherence in ion traps is thought to arise from fluctuating patch potentials on the trap electrodes [16]. Roughly speaking, this causes a fluctuating linear potential that results in random forces acting on the ion. The net effect is a slow heating of the ions. This problem is particularly acute in small traps [17,18], where anomalous heating has been experimentally observed [19]. More recent experiments have cooled the traps to a few K and show a dramatic reduction in heating [20]. We anticipate that with further cooling the heating will ultimately be attributable to charge fluctuations in two-level traps, especially for oxide barriers in semiconductor substrates.

In many cases, it is reasonable to assume that each of the two-level defects will be also coupled to its own environment, which we may model as a bosonic bath. We are thus led to a more complicated model in which the central oscillator couples to a collection of independent spin-boson models [Fig. 1(b)]. In this paper we will consider the special case

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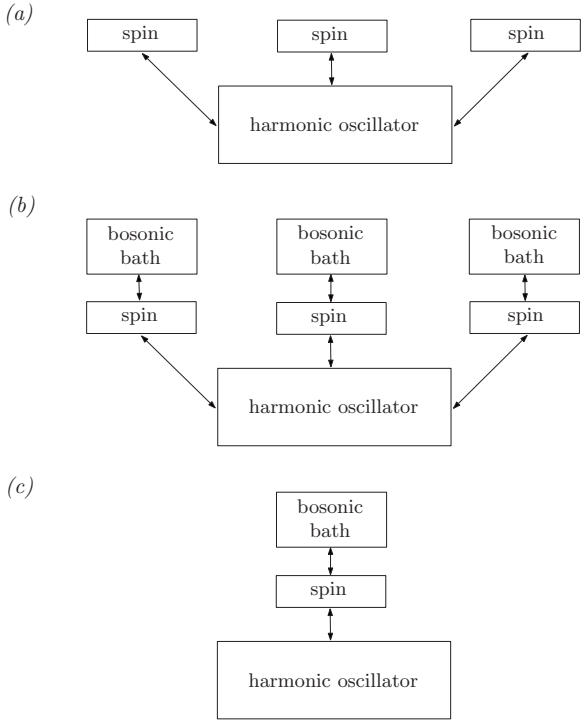

FIG. 1. (a) Oscillator-spin model. (b) Oscillator-spin model with each spin coupled to an additional bosonic bath. (c) Limiting case of (b) in which the oscillator interacts with a single environmental spin coupled to a bosonic bath.

of only a single TLS interacting with a bosonic bath [Fig. 1(c)]. This simplification allows us to analytically derive the master equation for the central oscillator in the limit in which the oscillator is strongly coupled to a steady-state TLS. It is also experimentally motivated by the fact that in GHz QEMS and micron scale, cryogenic ion traps, the number of defects that participate in the dynamics is thought to be quite small.

We emphasize that the focus of this paper is a study of the general dynamics and properties of the oscillator-spin model in the context of the canonical models, and it is not our aim to present detailed models for ion traps and QEMS. However, these systems lend urgent experimental relevance to the oscillator-spin model, and our model may serve as a starting point for the development of models tailored to specific experimental situations. In existing models of ion traps, the fluctuating forces have thus far been treated classically [21–25]. For QEMS, a realistic and quantitatively accurate modeling of the influence of the various defects on the resonator is rather involved. First theoretical studies (see, e.g., [11,26]) were recently followed by detailed work by Seoánez, Guinea, and Castro Neto [27,28]. In order to be able to use a spectral-function treatment for the environmental TLS, these authors focused on the limit $k_BT \gg \hbar\Omega_0$, where Ω_0 is the natural frequency of the resonator. However, attaining the quantum regime of QEMS requires the opposite limit $k_BT \ll \hbar\Omega_0$ (GHz QEMS). In this case, it is likely that only very few TLS will be relevant to the dynamics of the resonator and we suggest that the correct description may be closer to the model discussed in Sec. III below.

This paper is organized as follows: Section II presents the derivation of the Born-Markov master equation for the central oscillator coupled to a spin bath. In Sec. III we derive the master equation for a harmonic oscillator interacting with a single spin coupled to a bosonic bath. We summarize our results in Sec. IV.

II. MASTER EQUATION FOR A HARMONIC OSCILLATOR COUPLED TO A SPIN BATH

A. Model

We consider a single quantum harmonic oscillator (the system $\mathcal{S}$) with self-Hamiltonian

$$\hat{H}_{\mathcal{S}} = \frac{\hat{P}^2}{2M} + \frac{M\Omega_0}{2}\hat{X}^2. \tag{1}$$

The oscillator interacts with an environment $\mathcal{E}$ of N independent spin-$\frac{1}{2}$ particles (quantum TLS). The environment is described by the self-Hamiltonian (setting $\hbar \equiv 1$)

$$\hat{H}_{\mathcal{E}} \equiv \sum_{i=1}^{N} \hat{H}_{\mathcal{E}}^{(i)} = \sum_{i=1}^{N} \frac{\omega_i}{2}\hat{\sigma}_z^{(i)} + \sum_{i=1}^{N} \frac{\Delta_i}{2}\hat{\sigma}_x^{(i)}, \tag{2}$$

where ω_i and Δ_i are, respectively, the asymmetry energy and tunneling matrix element of the ith bath spin. The environment couples linearly to the position coordinate of the oscillator via the interaction Hamiltonian

$$\hat{H}_{\text{int}} = \hat{X} \otimes \sum_{i=1}^{N} g_i \hat{\sigma}_z^{(i)} \equiv \hat{X} \otimes \hat{E}. \tag{3}$$

The total system-environment combination is then described by the Hamiltonian

$$\hat{H} = \hat{H}_{\mathcal{S}} + \hat{H}_{\mathcal{E}} + \hat{H}_{\text{int}}. \tag{4}$$

We assume the limit of weak system-environment couplings and no initial system-environment correlations, $\hat{\rho}(0) = \hat{\rho}_{\mathcal{S}}(0) \otimes \hat{\rho}_{\mathcal{E}}(0)$. We take the environment to be in thermal equilibrium at temperature T. Since the spins of this thermal bath are independent, we have

$$\hat{\rho}_{\mathcal{E}}(0) = \frac{1}{Z} e^{-\beta \hat{H}_{\mathcal{E}}} \equiv \frac{1}{Z} \prod_{i=1}^{N} e^{-\beta \hat{H}_{\mathcal{E}}^{(i)}}, \tag{5}$$

where $\beta \equiv 1/k_BT$ and $Z = \text{Tr}_{\mathcal{E}}\, e^{-\beta \hat{H}_{\mathcal{E}}}$. We would now like to derive the Born-Markov master equation for this spin-bath model. For the interaction Hamiltonian (3), the general form of the master equation is [29]

$$\frac{d}{dt}\hat{\rho}_{\mathcal{S}}(t) = -i[\hat{H}_{\mathcal{S}}, \hat{\rho}_{\mathcal{S}}(t)] - \int_0^{\infty} d\tau \{\mathcal{C}(\tau)[\hat{X}, \hat{X}(-\tau)\hat{\rho}_{\mathcal{S}}(t)] + \mathcal{C}(-\tau) \times [\hat{\rho}_{\mathcal{S}}(t)\hat{X}(-\tau), \hat{X}]\}. \tag{6}$$

Here

$$\hat{X}(\tau) = \hat{X}\cos(\Omega_0\tau) + \frac{1}{M\Omega_0}\hat{P}\sin(\Omega_0\tau) \tag{7}$$

denotes the system's position operator $\hat{X}$ in the interaction picture. The spin-environment self-correlation function $\mathcal{C}(\tau)$ appearing in Eq. (6) is given by

$$\mathcal{C}(\tau) \equiv \langle \hat{E}(\tau)\hat{E}\rangle_{\hat{\rho}_\mathcal{E}}, \tag{8}$$

where $\hat{E}(\tau)=e^{i\hat{H}_\mathcal{E}\tau}\hat{E}e^{-i\hat{H}_\mathcal{E}\tau}$ is the environment operator $\hat{E}$ in the interaction picture and the average is taken over the initial state $\hat{\rho}_\mathcal{E}\equiv\hat{\rho}_\mathcal{E}(0)$ of the environment [the Born approximation means that $\hat{\rho}(t)\approx\hat{\rho}_\mathcal{S}(t)\otimes\hat{\rho}_\mathcal{E}(0)$ for all t].

B. Calculation of the environment self-correlation function

First, we compute the environment self-correlation function (8), which we may write as

$$\mathcal{C}(\tau)=\sum_{ij} g_i g_j \langle e^{i\hat{H}_\mathcal{E}^{(i)}\tau}\hat{\sigma}_z^{(i)}e^{-i\hat{H}_\mathcal{E}^{(i)}\tau}\hat{\sigma}_z^{(j)}\rangle_{\hat{\rho}_\mathcal{E}} \equiv \sum_{ij} g_i g_j \langle \hat{\sigma}_z^{(i)}(\tau)\hat{\sigma}_z^{(j)}\rangle_{\hat{\rho}_\mathcal{E}}. \tag{9}$$

Because the environmental spins do not directly interact with each other, they are uncorrelated,

$$\langle \hat{\sigma}_z^{(i)}(\tau)\hat{\sigma}_z^{(j)}\rangle_{\hat{\rho}_\mathcal{E}} = \langle \hat{\sigma}_z^{(i)}(\tau)\rangle_{\hat{\rho}_\mathcal{E}}\langle \hat{\sigma}_z^{(j)}\rangle_{\hat{\rho}_\mathcal{E}} \quad \text{for } i\neq j, \tag{10}$$

and thus Eq. (9) can be decomposed as

$$\mathcal{C}(\tau)=\sum_i g_i\langle \hat{\sigma}_z^{(i)}(\tau)\rangle_{\hat{\rho}_\mathcal{E}}\sum_{j\neq i} g_j\langle \hat{\sigma}_z^{(j)}\rangle_{\hat{\rho}_\mathcal{E}} + \sum_i g_i^2\langle \hat{\sigma}_z^{(i)}(\tau)\hat{\sigma}_z^{(i)}\rangle_{\hat{\rho}_\mathcal{E}}. \tag{11}$$

Let us assume that at $t=0$ the average of the "quantum force" due to the collective action of all environmental spins is equal to zero,

$$\langle \hat{E}\rangle_{\hat{\rho}_\mathcal{E}} = \sum_i g_i\langle \hat{\sigma}_z^{(i)}\rangle_{\hat{\rho}_\mathcal{E}} = 0. \tag{12}$$

This is a nonrestrictive assumption, since Eq. (12) can always be fulfilled by simply adding a constant to the Hamiltonian. Then the term $\Sigma_{j\neq i}g_j\langle \hat{\sigma}_z^{(j)}\rangle_{\hat{\rho}_\mathcal{E}}$ appearing in Eq. (11) will also tend to zero, and Eq. (9) simplifies to

$$\begin{aligned}\mathcal{C}(\tau) &= \sum_i g_i^2\langle \hat{\sigma}_z^{(i)}(\tau)\hat{\sigma}_z^{(i)}\rangle_{\hat{\rho}_\mathcal{E}} \\ &= \sum_i g_i^2\,\mathrm{Tr}_\mathcal{E}\left[\left(\frac{1}{Z}\prod_i e^{-\hat{H}_\mathcal{E}^{(i)}/k_BT}\right)\hat{\sigma}_z^{(i)}(\tau)\hat{\sigma}_z^{(i)}\right] \\ &= \sum_i g_i^2\frac{1}{Z_i}\mathrm{Tr}_{\mathcal{E}_i}[e^{-\hat{H}_\mathcal{E}^{(i)}/k_BT}\hat{\sigma}_z^{(i)}(\tau)\hat{\sigma}_z^{(i)}],\end{aligned} \tag{13}$$

where $Z_i=\mathrm{Tr}_{\mathcal{E}_i}e^{-\hat{H}_{\mathcal{E}_i}/k_BT}$ and in the last line we have again used the fact that the bath spins are uncorrelated.

To calculate the interaction-picture operator $\hat{\sigma}_z^{(i)}(\tau)=e^{-i\hat{H}_\mathcal{E}^{(i)}\tau}\hat{\sigma}_z^{(i)}e^{-i\hat{H}_\mathcal{E}^{(i)}\tau}$, we write the environment Hamiltonian $\hat{H}_\mathcal{E}^{(i)}$ in matrix form in the eigenbasis $\{|0\rangle_i,|1\rangle_i\}$ of $\hat{\sigma}_z^{(i)}$,

$$\hat{H}_\mathcal{E}^{(i)}=\frac{1}{2}\begin{pmatrix}\omega_i & \Delta_i\\ \Delta_i & -\omega_i\end{pmatrix}. \tag{14}$$

The matrix eigenvalues are $E_\pm^{(i)}=\pm\frac{1}{2}\sqrt{\omega_i^2+\Delta_i^2}\equiv\pm\frac{1}{2}\tilde{\omega}_i$ with corresponding eigenvectors

$$|+\rangle_i=\cos\frac{\theta_i}{2}|0\rangle_i+\sin\frac{\theta_i}{2}|1\rangle_i, \tag{15a}$$

$$|-\rangle_i=-\sin\frac{\theta_i}{2}|0\rangle_i+\cos\frac{\theta_i}{2}|1\rangle_i, \tag{15b}$$

where $\theta_i=\arctan(\Delta_i/\omega_i)$. With respect to the basis $\{|+\rangle_i,|-\rangle_i\}$, the matrix representation of $\hat{\sigma}_z^{(i)}$ reads

$$\hat{\sigma}_z^{(i)}=\begin{pmatrix}\cos\theta_i & -\sin\theta_i\\ -\sin\theta_i & -\cos\theta_i\end{pmatrix}. \tag{16}$$

We can now evaluate Eq. (13) directly by carrying out the relevant matrix products and then taking the trace. The result is

$$\mathcal{C}(\tau)=\mathcal{C}_0+\sum_i\left(\frac{g_i\Delta_i}{\tilde{\omega}_i}\right)^2[\cos(\tilde{\omega}_i\tau)-i\tanh(\beta\tilde{\omega}_i/2)\sin(\tilde{\omega}_i\tau)], \tag{17}$$

where $\mathcal{C}_0=\Sigma_i\left(\frac{g_i\omega_i}{\tilde{\omega}_i}\right)^2$ is a time-independent constant.

C. Continuum limit

Let us introduce the spectral density function

$$J(\tilde{\omega})\equiv\sum_i\left(\frac{g_i\Delta_i}{\tilde{\omega}_i}\right)^2\delta(\tilde{\omega}-\tilde{\omega}_i) \tag{18}$$

and write Eq. (17) as

$$\begin{aligned}\mathcal{C}(\tau) &= \mathcal{C}_0+\int_0^\infty d\tilde{\omega}J(\tilde{\omega})[\cos(\tilde{\omega}\tau)-i\tanh(\beta\tilde{\omega}/2)\sin(\tilde{\omega}\tau)] \\ &\equiv \mathcal{C}_0+\nu(\tau)-i\eta(\tau).\end{aligned} \tag{19}$$

The functions $\nu(\tau)$ and $\eta(\tau)$ take the same functional form as the noise and dissipation kernels, respectively, in the case of an oscillator bath (quantum Brownian motion) with spectral density $J_{\mathrm{osc}}(\tilde{\omega})$,

$$\nu_{\mathrm{osc}}(\tau)=\int_0^\infty d\tilde{\omega}J_{\mathrm{osc}}(\tilde{\omega})\coth\left(\frac{\tilde{\omega}}{2k_BT}\right)\cos(\tilde{\omega}\tau), \tag{20a}$$

$$\eta_{\mathrm{osc}}(\tau)=\int_0^\infty d\tilde{\omega}J_{\mathrm{osc}}(\tilde{\omega})\sin(\tilde{\omega}\tau), \tag{20b}$$

provided we choose

$$J(\tilde{\omega})=J_{\mathrm{osc}}(\tilde{\omega})\coth\left(\frac{\tilde{\omega}}{2k_BT}\right) \tag{21}$$

in Eq. (19). Conversely, ignoring the constant term $\mathcal{C}_0$, we can map the spin bath with spectral density $J(\tilde{\omega})$ onto an oscillator bath with "surrogate" spectral density

$$J_{\mathrm{osc}}(\tilde{\omega})=J(\tilde{\omega})\tanh\left(\frac{\tilde{\omega}}{2k_BT}\right). \tag{22}$$

This is an example of the general result, first derived by Feynman and Vernon [30], that in the limit of sufficiently weak coupling any dissipative bath (including the spin bath) can be mapped onto a bath of oscillators. We note that in the limit $\Delta_i\gg\omega_i$ for all i (and thus $\mathcal{C}_0\to 0$), expression (19) co-

incides with a result previously obtained by Caldeira, Castro Neto, and de Carvalho [31] for a model of a general system interacting linearly and weakly with a spin bath, where the environmental self-Hamiltonian was assumed to take a more simple form than in our model.

D. Master equation

Inserting Eq. (19) into Eq. (6) and using Eq. (7) leads to the master equation

$$\frac{d}{dt}\hat{\rho}_S(t) = -i\left(\hat{H}_S + \frac{1}{2}M\tilde{\Omega}_0^2\hat{X}^2, \hat{\rho}_S(t)\right) - i\gamma[\hat{X},\{\hat{P},\hat{\rho}_S(t)\}] - D[\hat{X},[\hat{X},\hat{\rho}_S(t)]] - f[\hat{X},[\hat{P},\hat{\rho}_S(t)]]. \tag{23}$$

Here, the coefficients $\tilde{\Omega}_0^2$, γ, D, and f are defined as

$$\tilde{\Omega}_0^2 \equiv -\frac{2}{M}\int_0^\infty d\tau \eta(\tau)\cos(\Omega_0\tau), \tag{24a}$$

$$\gamma \equiv \frac{1}{M\Omega_0}\int_0^\infty d\tau \eta(\tau)\sin(\Omega_0\tau), \tag{24b}$$

$$D \equiv \int_0^\infty d\tau[\mathcal{C}_0 + \nu(\tau)]\cos(\Omega_0\tau) \equiv D_0 + \int_0^\infty d\tau\nu(\tau)\cos(\Omega_0\tau) \equiv D_0 + D_1, \tag{24c}$$

$$f \equiv -\frac{1}{M\Omega_0}\int_0^\infty d\tau[\mathcal{C}_0 + \nu(\tau)]\sin(\Omega_0\tau) \equiv f_0 - \frac{1}{M\Omega_0}\int_0^\infty d\tau\nu(\tau)\sin(\Omega_0\tau) \equiv f_0 + f_1. \tag{24d}$$

The interpretation of these coefficients is analogous to the case of quantum Brownian motion. The coefficient $\tilde{\Omega}_0^2$ describes a frequency shift ("Lamb-shift" renormalization of the natural frequency of the oscillator), γ is the momentum-damping (and thus dissipation) rate, and D and f are the normal-diffusion and anomalous-diffusion coefficients describing decoherence. We see that $\tilde{\Omega}_0^2$ and γ are explicitly temperature-dependent while D and f are not, which is exactly opposite as in quantum Brownian motion. Formally, this difference is easily understood from the fact that using the surrogate spectral density (22) in the expressions for the oscillator-bath noise and dissipation kernels (20a) and (20b) eliminates the temperature-dependent term in the integral (20a) while introducing the term $\tanh(\Omega_0/2k_BT)$ in the integral (20b).

E. Example: Ohmic spectral density

Let us consider an ohmic spectral density for the spin bath with a Lorentz-Drude high-frequency cutoff,

$$J(\tilde{\omega}) = \frac{2M\gamma_0}{\pi}\tilde{\omega}\frac{\Lambda^2}{\Lambda^2 + \tilde{\omega}^2}. \tag{25}$$

The coefficient γ is given by a double Fourier sine transform of the function $J(\tilde{\omega})\tanh(\beta\tilde{\omega}/2)$, which returns the original function up to a prefactor of $\pi/2$,

$$\gamma = \frac{1}{M\Omega_0}\int_0^\infty d\tau \sin(\Omega_0\tau)\int_0^\infty d\tilde{\omega}J(\tilde{\omega})\tanh(\beta\tilde{\omega}/2)\sin(\tilde{\omega}\tau) = \gamma_0\frac{\Lambda^2}{\Lambda^2 + \Omega_0^2}\tanh\left(\frac{\Omega_0}{2k_BT}\right). \tag{26}$$

Similarly, the coefficient D_1 is given by a double Fourier cosine transform of $J(\tilde{\omega})$, which leads to

$$D = D_0 + M\gamma_0\Omega_0\frac{\Lambda^2}{\Lambda^2 + \Omega_0^2}. \tag{27}$$

For quantum Brownian motion these coefficients read [29] as

$$\gamma_{\mathrm{QBM}} = \gamma_0\frac{\Lambda^2}{\Lambda^2 + \Omega_0^2}, \tag{28a}$$

$$D_{\mathrm{QBM}} = M\gamma_0\Omega_0\frac{\Lambda^2}{\Lambda^2 + \Omega_0^2}\coth\left(\frac{\Omega_0}{2k_BT}\right). \tag{28b}$$

Disregarding the constant term D_0, we see that the spin-bath coefficients are given by the oscillator-bath coefficients multiplied by the term $\tanh(\Omega_0/2k_BT)$, which is simply a direct consequence of the use of the surrogate spectral density (22).

F. Comparison between spin and oscillator baths

In Fig. 2 we have plotted the temperature dependencies of the coefficients γ and D for the spin bath [see Eqs. (26) and (27)] and for an oscillator bath (quantum Brownian motion) [see Eqs. (28a) and (28b)], with both baths described by the spectral density (25). Figure 2(a) shows that the spin-bath dissipation rate γ, Eq. (26), decreases with temperature. This initially surprising result is easily explained. While a harmonic oscillator can absorb an infinite amount of energy, there are only two energy levels for a spin-$\frac{1}{2}$ particle. It follows that, as the bath temperature is raised, the spin bath saturates quickly and the dissipative influence on the central system must decrease when compared with that of the oscillator bath, whose dissipation rate is temperature independent (for linear quantum Brownian motion). Indeed, the $\tanh(\Omega_0/2k_BT)$ temperature dependence of the spin-bath dissipation rate (26) has been explicitly observed in systems such as glasses [32] where dissipation is mainly caused by interactions between phonon modes and TLS [33].

A similar argument also allows us to understand the absence of any temperature dependence of the normal-diffusion coefficient D [see Eq. (27)] for the spin bath and thus of the rate of spatial decoherence [Fig. 2(b)]. Equation (28b) shows that for quantum Brownian motion this rate increases with temperature as $\coth(\Omega_0/2k_BT)$ (and linearly with T in the high-temperature limit of the Caldeira-Leggett model [3]). This increase is due to the fact that, as the temperature is

raised, increasingly excited energy levels will be occupied in each harmonic oscillator, and thus the characteristic wavelengths present in the bath will decrease. Shorter environmental wavelengths mean that the bath will be able to better resolve the position of the central system, leading to stronger decoherence (localization) of superpositions of well-separated positions. The quick saturation of the spin bath with increasing temperature implies that the characteristic wavelengths cannot become significantly shorter, resulting for our model in a temperature-independent expression for D.

The fact that the spin-bath decoherence rate D has a constant component implies that there is a heating source in the oscillator-spin model. This term in the master equation drives a diffusion process in the momentum and thus causes the average kinetic energy of the oscillator to increase linearly in time at a rate of at least D_0. This is likely to be a problematic source of heating in micron-scale ion traps. As this is independent of temperature, it will be apparent even if no systematic dissipation is observable ($\gamma \approx 0$).

III. HARMONIC OSCILLATOR COUPLED TO A SINGLE TWO-LEVEL SYSTEM INTERACTING WITH A BOSONIC BATH

The discussion in the preceding section indicates that the temperature dependence of the damping and dissipation rates arise from the fact that the bath seen by the central oscillator cannot absorb an arbitrarily large amount of energy. We are thus led to consider an extreme case in which the oscillator is coupled to only a single TLS which is itself weakly coupled to a bosonic bath. As mentioned in the introduction, both in micron-scale, cryogenic ion traps and in GHz QEMS (the frequency regime relevant to the observation of quantum effects) it is likely that only a few TLS take part in the dynamics, rather than a TLS environment with continuous spectrum as used in [27,28]. This lends particular experimental relevance to the single-TLS model considered here. For example, our model could represent a single charge trap electrostatically coupled to a nanomechanical resonator, with the charge trap itself coupled to Johnson-Nyquist electrical noise in the surrounding circuit.

We begin with a single vibrational degree of freedom (the central harmonic oscillator $\mathcal{S}$) coupled to a single spin-$\frac{1}{2}$ particle (TLS),

$$\hat{H} = \Omega_0 \hat{a}^\dagger \hat{a} + g\hat{X} \otimes \hat{\sigma}_z + \frac{\Delta}{2}\hat{\sigma}_x, \tag{29}$$

where $\hat{X} = (\hat{a} + \hat{a}^\dagger)$ is a dimensionless displacement operator and we have again set $\hbar \equiv 1$. In the case of a realization of the model as a charge trap, $\hat{\sigma}_z$ refers to two distinct charge configurations of a single microscopic trap in one of the electrodes near a nanomechanical resonator or trapped ion. We could think of this as some kind of double well, in which case the eigenstates of $\hat{\sigma}_z$ refer to states localized on one side of the barrier or the other. The tunnel-split ground states under the barrier are eigenstates of $\hat{\sigma}_x$ and the tunnel splitting is Δ. If we define the eigenstates of $\hat{\sigma}_x$ by $\hat{\sigma}_x|\pm\rangle = \pm|\pm\rangle$, then we can define the "bit-flip" operator as $\hat{\sigma}_z = |+\rangle\langle -| + |-\rangle\langle +|$.

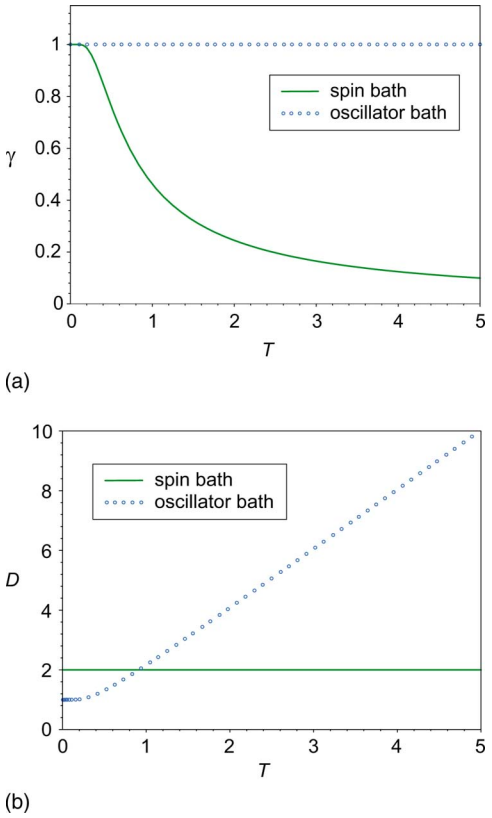

FIG. 2. (Color online) Dissipation and normal-diffusion coefficients (a) γ and (b) D for the oscillator-spin model (solid line) and for quantum Brownian motion (circles), assuming the ohmic spectral density (25). The vertical axis is normalized in units of the zero-temperature values $\gamma(T=0)$ and $D_{\mathrm{QBM}}(T=0)$, respectively. The horizontal (temperature) axis is displayed in units of Ω_0/k_B. We use $D_0 = D_{\mathrm{QBM}}(T=0)$.

We expect that the TLS remains close to thermodynamic equilibrium with a heat bath at temperature T even in the presence of the coupling to the central oscillator. Thus we assume that its state at all times can be approximated by the thermal state

$$\hat{\rho}_T = p_+|+\rangle\langle +| + p_-|-\rangle\langle -|, \tag{30}$$

where

$$\frac{p_+}{p_-} = e^{-\Delta/k_B T}. \tag{31}$$

The dynamical process that maintains the TLS in thermodynamic equilibrium could be quite obscure. However, a simple model can be given by weakly coupling the single TLS to a bosonic bath at temperature T. The coupling is capacitive,

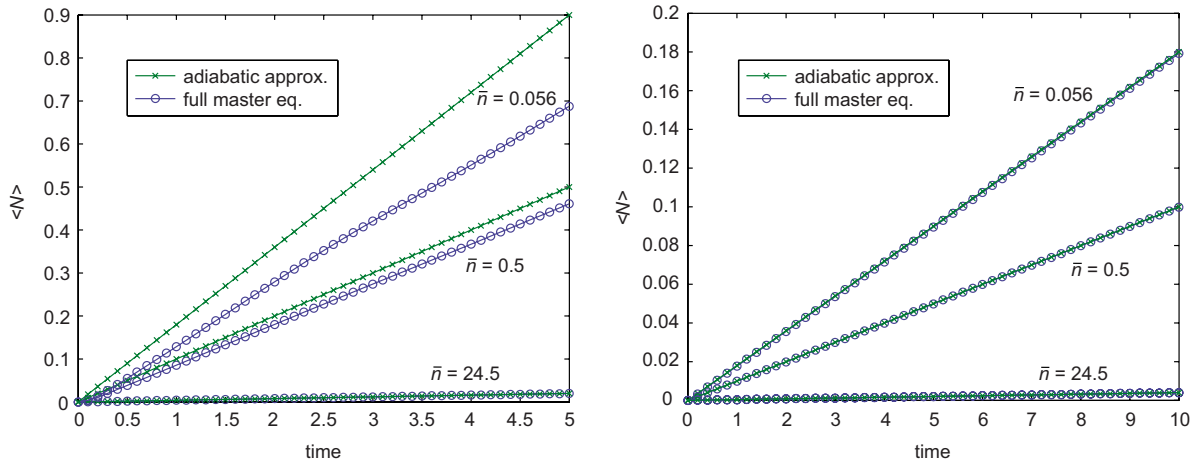

FIG. 3. (Color online) Comparison of the adiabatic approximation (crosses) and the full master equation (circles) for the simple model discussed in Sec. III. We show the mean occupation number $\langle N\rangle$ of the central harmonic oscillator as a function of time for three different temperatures [a larger value of $\bar{n}$, see Eq. (35), corresponds to higher temperature]. The time axis is in units of inverse Δ (with $\hbar \equiv 1$). We use $\Delta=1$, $\Omega_0=1$, $g=1$, and (a) $\gamma=10$, (b) $\gamma=100$.

$$\hat{H}_{\text{coupling}} = \hat{\sigma}_z \otimes \sum_k g_k \hat{q}_k(t), \tag{32}$$

while the free Hamiltonian for the bath is a sum over harmonic oscillators, each with canonical coordinates $\hat{q}_k, \hat{p}_k$ and frequency ω_k. If the coupling is weak so that $g_k \ll \Delta, \omega_k$, the corresponding Markov master equation for the density operator $\hat{\rho}$ of the joint oscillator-TLS system is [34]

$$\frac{d\hat{\rho}}{dt} = -i\Omega_0[\hat{a}^\dagger\hat{a},\hat{\rho}] - i\Delta[\hat{\sigma}_x,\hat{\rho}] - ig[\hat{X}\hat{\sigma}_z,\hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{\sigma}_-]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{\sigma}_+]\hat{\rho}, \tag{33}$$

where Ω_0 is the vibrational frequency of the central oscillator and γ determines the heating rate. The super-operator $\mathcal{D}[\hat{A}]$ is defined by

$$\mathcal{D}[\hat{A}]\hat{\rho} \equiv \hat{A}\hat{\rho}\hat{A}^\dagger - \frac{1}{2}(\hat{A}^\dagger\hat{A}\hat{\rho} + \hat{\rho}\hat{A}^\dagger\hat{A}), \tag{34}$$

and $\hat{\sigma}_+ = \hat{\sigma}_-^\dagger = |+\rangle\langle -|$ are raising and lowering operators in the eigenstates of $\hat{\sigma}_x$, i.e., the energy eigenstates. Finally, the parameter $\bar{n}$ is defined by

$$\bar{n} \equiv (e^{\beta\Delta/2} - 1)^{-1}. \tag{35}$$

It is easy to see that in the absence of the coupling to the oscillator, the steady state for the TLS implied by Eq. (33) is simply given by Eq. (31).

We can now calculate a heating rate for the oscillator. We do this by adiabatic elimination [35] of the TLS, i.e., we assume that $\gamma \gg \Omega_0, g, \Delta$ so that the TLS remains in steady state slaved to the motion of the oscillator. We define the operators acting on the vibrational degree of freedom by $\hat{\rho}_{++} \equiv \langle +|\hat{\rho}|+\rangle$, $\hat{\rho}_{--} \equiv \langle -|\hat{\rho}|-\rangle$, and $\hat{\rho}_{+-} \equiv \langle +|\hat{\rho}|-\rangle$. Note that the reduced density operator for the vibrational degree of freedom is just given by $\hat{\rho}_S = \hat{\rho}_{++} + \hat{\rho}_{--}$. The master equation (33) then implies that

$$\frac{d\hat{\rho}_{++}}{dt} = -ig(\hat{X}\hat{\rho}_{-+} - \hat{\rho}_{+-}\hat{X}) - i\Omega_0[\hat{a}^\dagger\hat{a},\hat{\rho}_{++}] - \gamma(\bar{n}+1)\hat{\rho}_{++} + \gamma\bar{n}\hat{\rho}_{--}, \tag{36a}$$

$$\frac{d\hat{\rho}_{--}}{dt} = -ig(\hat{X}\hat{\rho}_{+-} - \hat{\rho}_{-+}\hat{X}) - i\Omega_0[\hat{a}^\dagger\hat{a},\hat{\rho}_{--}] + \gamma(\bar{n}+1)\hat{\rho}_{++} - \gamma\bar{n}\hat{\rho}_{--}, \tag{36b}$$

$$\frac{d\hat{\rho}_{+-}}{dt} = -ig(\hat{X}\hat{\rho}_{--} - \hat{\rho}_{++}\hat{X}) - i\Omega_0[\hat{a}^\dagger\hat{a},\hat{\rho}_{+-}] - \frac{\gamma}{2}(2\bar{n}+1)\hat{\rho}_{+-} - 2i\Delta\hat{\rho}_{+-}. \tag{36c}$$

Assuming that the off-diagonal operators $\hat{\rho}_{+-}$ reach a steady state, we find that

$$\hat{\rho}_{+-} \approx -\frac{2ig}{\gamma(2\bar{n}+1)}(\hat{X}\hat{\rho}_{--} - \hat{\rho}_{++}\hat{X}). \tag{37}$$

Substituting this into the equation of motion for the diagonal components leads to

$$\frac{d\hat{\rho}_S}{dt} = -i\Omega_0[\hat{a}^\dagger\hat{a},\hat{\rho}_S] - \Gamma[\hat{X},[\hat{X},\hat{\rho}_S]], \tag{38}$$

where the last term implies diffusive heating and the rate is given by

$$\Gamma = \frac{2g^2}{\gamma(2\bar{n}+1)}. \tag{39}$$

A full numerical simulation (see Fig. 3) confirms that this master equation is a good description of the dynamics in the limit of large γ. However, even when the adiabatic approximation is not the same as the full master equation, the full model still shows diffusive heating.

At zero temperature the TLS is in the ground state $|-\rangle$. As this is a superpositon of the two eigenstates of $\hat{\sigma}_z$, the vibra-

tional degree of freedom suffers momentum kicks of equal magnitude but random sign. As the temperature goes to infinity, the TLS state is the identity operator and the vibrational degree of freedom suffers no kicks at all. For this reason the heating rate goes to zero at high temperature—a reappearance of the damping-rate feature of the oscillator-spin model discussed in Sec. II F. Also note that at zero temperature there is a fixed momentum-diffusion rate that causes the oscillator to heat, which again is the case for the full spin-bath model of Sec. II.

IV. SUMMARY AND CONCLUSIONS

We have derived the master equations for a single harmonic oscillator coupled (i) to a bath of two-level systems (the oscillator-spin model) and (ii) to a single two-level system interacting with a bosonic bath. These models and the derivation of the relevant master equations not only close an important gap in the set of canonical system-environment models for decoherence and dissipation, but are also motivated by and relevant to current experiments such as quantum-electromechanical systems and micron-scale ion traps.

For both models the key features that arise are the following: (i) The systematic dissipation rate for the oscillator decreases with increasing temperature; (ii) at zero temperature the systematic dissipation rate goes to zero; but (iii) there is a temperature-independent momentum-diffusion rate (a heating rate). Interestingly, this behavior is very different from the model for quantum Brownian motion, and we have explained how it can be understood as arising from a rapid saturation of the spin environment.

In future presentations we will report results of an application of the models considered in this paper to a number of physically relevant experimental systems, including nanomechanical systems and ion traps.

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