Metal-insulator transition in the spectrum of a frequency-modulation mode-locked laser

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It is theoretically shown that the spectrum of a frequency mode-locked laser with an intracavity dielectric plate, which modulates the cavity axial mode resonances, can undergo a kind of metal-insulator transition analogous to that encountered in the incommensurate Harper model of solid-state physics. The transition should be observable as a rather abrupt spectral broadening of the laser emission, accompanied by a transition from exponential to Gaussian spectral localization, as the modulation depth is increased above a threshold value.

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Mode locking is a well-established operational regime of a laser in which the locking of multiple cavity axial modes using either active or passive means leads to a pulsed output radiation (see, for instance, Refs. [1,2]). The history of laser mode locking, as described by Haus in one of his last review papers, is "a progression of new and better ways to generate shorter and shorter pulses, and of improvements in the understanding of the mode-locking process" [2]. As the advent in the past decade of mode-locking methods and systems has largely revolutionized their applications in different fields of physics and science, mode-locked lasers have also offered to physicists an experimentally accessible laboratory tool to investigate complex dynamical behaviors and rather universal physical phenomena that go beyond traditional laser physics. For instance, detuned mode-locked lasers can show instabilities and turbulent behavior similar to hydrodynamic systems [3,4], whereas excess noise due to non-normal mode dynamics has been predicted and observed in frequency-modulation (FM) mode-locked lasers at the transition region between the locking and FM operational regimes [5,6]. Temporal oscillations in the spectrum of a FM mode-locked laser, which mimic the celebrated Bloch oscillations of an electron in an ordered crystal driven by a dc electric field, have been predicted and observed in Refs. [7,8]. Recently, elegant termodynamic approaches have been proposed to explain laser pulsation threshold in passively mode-locked lasers as a phase transition phenomenon [11,12]. Finally, FM mode-locked lasers with strong intracavity dispersion have been shown to exhibit a kind of spectral localization which is analogous to dynamic localization of the quantum kicked rotator and related to Anderson localization [9,10]. In this work it is shown that a FM mode-locked laser can undergo a kind of metal-insulator transition in its spectrum, i.e., a transition from extended to localized states, which is analogous to the metal-insulator transition encountered in disordered-free tight-binding lattice models with an incommensurate potential (see, for instance, [13]). In particular, we show that a FM mode-locked laser in a cavity with modulated axial resonances realizes an optical analogue of the celebrated Harper model [13,14], which originally appeared in the description of Bloch electrons in a magnetic field. In contrast to localization in other tight-binding lattice models with disorder (Anderson model) or in the periodically kicked quantum rotator model [15] mimicked by a FM mode-locked laser with strong dispersion in previous References [9,10], in the incommensurate Harper model both localized and extended states may exist with the interesting possibility of a metal-insulator transition as a function of the strength of the potential [13]. Realizations of the Harper model have been previously presented for microwave transmission in waveguides with scatterers [16], in dual-periodic dielectric multilayer structures [17], and in Bose-Einstein condensates in aperiodic optical lattices [18].

Let us consider an optical cavity made of two flat end mirrors containing a homogeneously broadened slow-gain medium [such as Nd: yttrium aluminum garnet (YAG)] with a resonance frequency ω_0 and a phase modulator, placed close to one of the two end mirrors. The other mirror, which we assume to be a perfect reflector (similar to an ideal metallic mirror), is placed in contact with a transparent dielectric medium (e.g., a thick glass plate) with flat facets of thickness d and refractive index n_1, as shown in Fig. 1(a). As discussed below, the role of the thick dielectric medium is to slightly modulate the spacing of the cavity axial modes from the uniform value $\Delta\omega_{ax}=\pi c_0/L_e$ imposed by the optical length L_e of the resonator. To study the mode-locking process, we adopt a standard frequency-domain description [1]

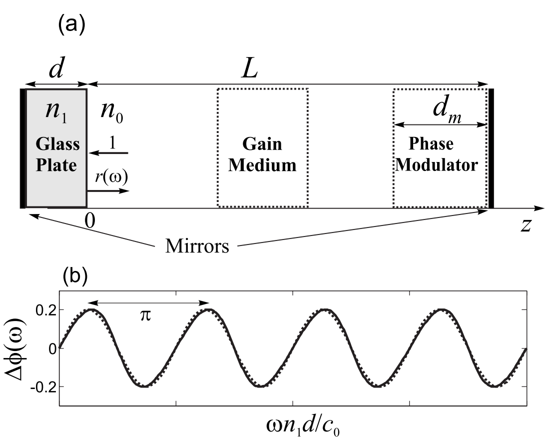

FIG. 1. (a) Schematic of the laser cavity with a thick glass plate placed in contact with an end metallic mirror. (b) Behavior of the phase term contribution $\Delta\phi(\omega)$, accumulated by a reflected monochromatic wave after incidence at plane $z=0$ on the thick glass, for $n_1=1.5$ and $n_0=1$. The solid curve is the exact behavior given by Eq. (2), whereas the dotted curve (almost overlapped with the solid one) is its approximation given by Eq. (3).

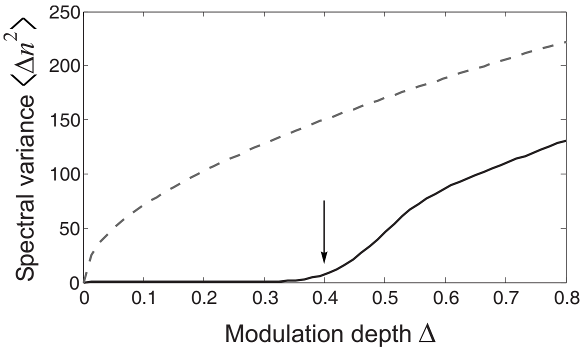

FIG. 2. Behavior of spectral variance $\langle \Delta n^2 \rangle$ versus modulation depth for $V=0.4/(2\pi)$ (solid curve) and for $V=0$ (dashed curve). The arrow in the figure indicates the metal-insulator transition point $2\pi V=\Delta$.

in which the electric field $E(z,t)$ in the cavity is decomposed as a superposition of normal (axial) modes $U_n(z)$ of the cold cavity according to $E(z,t)=\Sigma_n E_n(t) U_n(z) \exp[i(\omega_0+in\omega_m)t]$ +c.c., where ω_m is the modulation frequency and the complex mode amplitudes $E_n(t)$ vary slowly in time over a modulation cycle. For the sake of simplicity, we assume that the resonance frequency of one normal mode of the cavity, corresponding to the index $n=0$, coincides with the resonance frequency ω_0 of the homogeneously broadened gain medium. Indicating by $\omega_n=\omega_0+n\Delta\omega_{ax}+\delta\omega_n$ the resonance frequency of the nth cavity axial mode, where $\delta\omega_n$ is a small correction to $\Delta\omega_{ax}$ that accounts for the presence of the dielectric plate, for a synchronous modulation $\omega_m=\Delta\omega_{ax}$ the coupled mode equations for the mode amplitudes E_n read (see, for instance, Ref. [1])

$$T_R \frac{dE_n}{dt} = (g_n - l)E_n + 2\pi i \left(\frac{\delta\omega_n}{\Delta\omega_{ax}} \right) E_n + i\frac{\Delta}{2}(E_{n+1} + E_{n-1}), \tag{1}$$

where $T_R=2\pi/\Delta\omega_{ax}$ is the cavity round-trip time, g_n and l are round-trip saturated gain and loss parameters, and Δ is the modulation depth. For a homogeneously broadened gain medium with a slow relaxation time, one can assume for g_n the Lorentzian profile $g_n=g/[1+(n\omega_m/\Delta\omega_g)^2]$, where $\Delta\omega_g$ is the gain linewidth and g the saturated gain parameter. Atomic frequency pulling effects can be accounted for, if needed, by shifting the modulation frequency ω_m to match the effective cavity length and by adding a contribution to the frequency shift terms $\delta\omega_n$. For a low gain g, however, such a contribution is typically smaller than that introduced by the dielectric plate, and it will be thus neglected in the following analysis. In writing Eq. (1), we also assumed that the presence of the dielectric plate does not appreciably modulate the coupling strength Δ between neighboring spectral modes [19]. To calculate the shift $\delta\omega_n$ of cavity resonances due to the dielectric plate, let us first notice that the effective field reflectivity $r(\omega)$ at the plane $z=0$ of the metallic mirror with the dielectric plate [see Fig. 1(a)] can be written as $r(\omega)=\exp[-2in_1 d(\omega/c_0)+i\pi+2i\Delta\phi(\omega)]$, with

$$\Delta\phi(\omega) \equiv \frac{\omega n_1 d}{c_0} - \arctan\left[\frac{n_0}{n_1} \tan\left(\frac{\omega n_1 d}{c_0} \right) \right], \tag{2}$$

where $n_0=1$ is the refractive index of air. For a typical value $n_1 \simeq 1.5$ of a transparent glass, the phase term $\Delta\phi(\omega)$ entering in Eq. (2) can be very well approximated by the relation [see Fig. 1(b)]

$$\Delta\phi(\omega) \simeq \frac{f}{2}\left(1 - \frac{n_0}{n_1}\right) \sin\left(\frac{2\omega n_1 d}{c_0} \right), \tag{3}$$

where f is a numerical factor, close to 1, which depends on the ratio n_1/n_0 (for instance, $f \simeq 1.2$ for $n_1/n_0=1.5$). The resonance frequencies of cavity axial modes are then found by imposing that the accumulated phase in one round-trip be an integer multiple of 2π, i.e., ω_n are found as the roots of the transcendental equation

$$\frac{\omega}{c_0} L_e - \Delta\phi(\omega) + \phi_0 = l\pi, \tag{4}$$

where ϕ_0 is a constant phase shift ($\phi_0=0$ if both mirrors are metallic) and l is an integer number. Taking into account that $f(1-n_0/n_1) \ll 2\pi$, from Eqs. (3) and (4) it follows that the approximate solutions to Eq. (4) are given by

$$\omega_n = \omega_0 + n\Delta\omega_{ax} + \delta\omega_n \tag{5}$$

($n=0, \pm 1, \pm 2, \ldots$), where ω_0 is the cavity axial frequency in resonance with the atomic transition,

$$\frac{\delta\omega_n}{\Delta\omega_{ax}} = V \sin\left(\frac{2n_1 d\omega_0}{c_0} + 2\pi\alpha n \right) \tag{6}$$

is the normalized frequency shift of cavity resonances introduced by the dielectric plate,

$$\alpha \equiv \frac{\Delta\omega_{ax}}{\pi c_0/(n_1 d)} = \frac{n_1 d}{L_e} \tag{7}$$

is the ratio between the free spectral range of the full cavity and that of the dielectric plate ($\alpha < 1$), and

$$V \equiv \frac{f}{2\pi}\left(1 - \frac{n_0}{n_1}\right). \tag{8}$$

In the case where modal gain and loss were negligible, i.e., for $g_n=l=0$, the spectral eigenmodes of Eq. (1) satisfy the Harper equation

$$(\Delta/2)(E_{n+1} + E_{n-1}) + 2\pi V \sin(2\pi\alpha n + \varphi)E_n = \beta E_n, \tag{9}$$

where β is the eigenvalue and $\varphi=(2n_1 d\omega_0)/c_0$ an unimportant phase term. It is well known that, for an irrational value of α, there is a metal-insulator transition at $2\pi V=\Delta$, with all spectral eigenmodes extended for $\Delta > 2\pi V$ and exponentially localized for $\Delta < 2\pi V$, with an inverse localization length given by $\lambda=2\ln(2\pi V/\Delta)$ (see, for instance, [13]). In presence of gain and losses, the eigenvalues β become complex valued, the imaginary part of β being the decay rate of the corresponding spectral eigenmode. Moreover, the spectral mode extension is determined not only by the incommensurate modulation $\delta\omega_n$ of cavity resonances, but also on the gainline of the gain medium. Even though for $\Delta > 2\pi V$

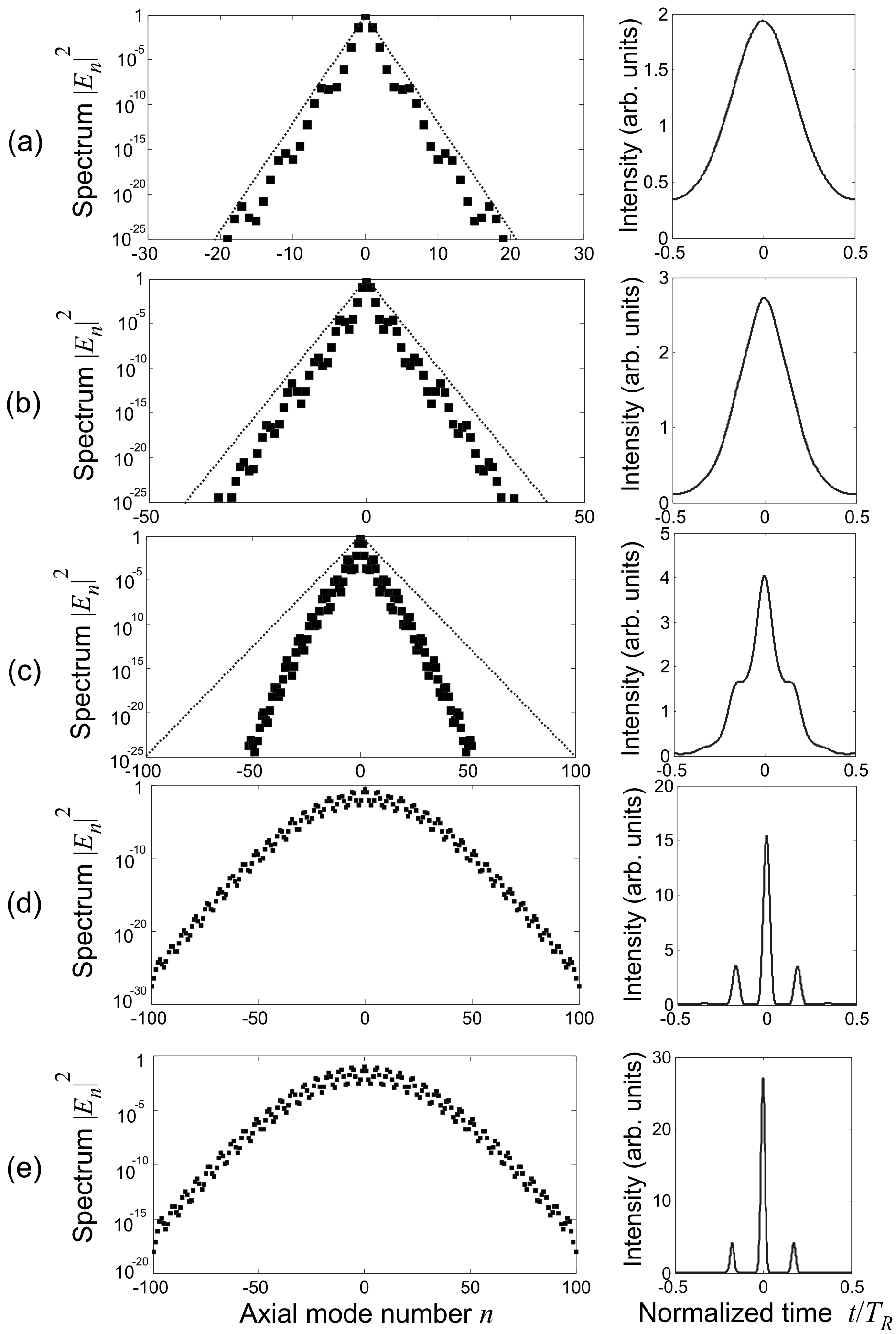

FIG. 3. Spectrum (in logarithmic scale, left columns) and corresponding pulse intensity profile (right columns) of the mode-locked laser for increasing values of the modulation depth Δ for $2\pi V=0.4$ and $\varphi=\pi/2$. (a) $\Delta=0.1$, (b) $\Delta=0.2$, (c) $\Delta=0.3$, (d) $\Delta=0.5$, and (e) $\Delta=0.7$. In the left plots of (a), (b), and (c) the dashed curves show the exponential localization behavior predicted by the Harper model in absence of gain and loss.

the spectral eigenmodes will not be fully extended but spectrally limited owing to the finite bandwidth of the atomic medium, a metal-insulator transition in the laser output spectrum is nevertheless expected to be observed. In this case, above the metal-insulator transition point $2\pi V=\Delta$ localization does not lead to an exponentially decaying spectrum, rather to a Gaussian spectral shape as in the usual FM mode-locking regime [1]. We checked the occurrence of such a kind of metal-insulator transition by numerically computing the spectral eigenmode of Eq. (1) corresponding to the lowest threshold lasing mode, which provides the shape of the output laser spectrum observed close to threshold [20]. The lasing threshold is attained for a gain parameter $g=g_{th}$ such that the most unstable eigenmode of Eq. (1) shows a vanishing imaginary part of its eigenvalue β. As the threshold of low-order modes is expected to slightly deviate from l, to compute the oscillating spectral eigenmode we may assume $g \simeq l$ in Eq. (1) for the saturated gain parameter. As an example, Fig. 2 shows the numerically computed spectral variance $\langle \Delta n^2 \rangle \equiv \Sigma (n-\langle n \rangle)^2 |E_n|^2 / \Sigma_n |E_n|^2$ of the laser spectrum versus the modulation depth Δ for parameter values $n_1=1.5$ (i.e., $2\pi V \simeq 0.4$), $\alpha = 1/(4\sqrt{2})$, $l=0.02$, $\Delta\omega_g/\omega_m=60$, and $\varphi = \pi/2 \pmod{2\pi}$. For comparison, in the figure it is also shown the behavior of the spectral variance $\langle \Delta n^2 \rangle$ versus Δ for a usual FM mode-locked laser, i.e., for $V=0$. Note that in this case, according to the standard Kuizenga-Siegman theory of FM mode locking [21], the spectral variance $\langle \Delta n^2 \rangle$ increases monotonously with Δ according to the square-root law $\langle \Delta n^2 \rangle \propto \sqrt{\Delta}$, without any abrupt transition. Conversely, for $V \neq 0$ a clear transition in the spectral extension, from a strongly localized spectrum to a broad spectrum, is observed at the transition point $\Delta \sim 2\pi V$, indicated by the arrow in the figure, which is the signature of a metal-insulator transition. For example, for a Nd:YAG laser $[\Delta\omega_g/\pi \simeq 120$ GHz$]$, the parameters used in the simulations correspond to a modulation frequency $\omega_m/(2\pi) \simeq 1$ GHz, an optical cavity length $L_e = 15$ cm, and a glass thickness $d \simeq 11.7$ mm with $n_1 = 1.5$. The behavior of the spectral variance $\langle \Delta n^2 \rangle$ versus modulation depth Δ and the onset of the metal-insulator transition turns out to be rather insensitive to a change of the phase φ. Figure 3 shows a few examples of spectral and temporal laser output at a few values of the modulation depth both below and above the transition point $\Delta = 2\pi V$. Note that for $\Delta < 2\pi V$ the spectral localization is exponential, with an inverse decay length which is well described, at least far from the transition, by the theoretical relation $\lambda = 2\ln(2\pi V/\Delta)$ [see Fig. 3(a)]. Conversely, for $\Delta > 2\pi V$ the oscillating spectrum, besides to be much broader, shows a typical Gaussian (rather than exponential) localization with a modulated spectrum at the free-spectral range of the dielectric plate which resembles the spectrum of a FM mode-locked laser in presence of étalon effects (see, for instance, Refs. [1,22]). However, it should be noted that the metal-insulator transition predicted in this work is related to a sinusoidal modulation of the cavity axial mode resonances, not to an effective spectral modulation of the gainline induced by étalon effects as in Ref. [22], which does not lead to a metal-insulator transition.

In conclusion, in this work it has been theoretically shown that the spectrum of a frequency mode-locked laser containing an intracavity dielectric plate can undergo a kind of metal-insulator transition analogous to that encountered in the Harper model describing the localization-delocalization transition of tight-binding incommensurate lattices. The predicted transition should be observable as an abrupt spectral broadening and corresponding temporal narrowing of the laser emission as the modulation depth is increased above a threshold value.

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