General relation between the transformation operator and an invariant under stochastic local operations and classical communication in quantum teleportation

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In this paper, we introduce a transformation operator expression to give a criterion for faithful teleportation of an arbitrary two-qubit state via a four-qubit entangled state. The general relation between the transformation operator and the SLOCC invariant of quantum channel is obtained and its applicability is illuminated by some known examples for the measurement in the Bell basis.

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Quantum teleportation is a prime example of a quantum-information processing task, where an unknown state can be perfectly transported from one place to another by using previously shared entanglement and classical communication between the sender and the receiver. Since the introduction of quantum teleportation protocol by Bennett [1], research on quantum teleportation has been attracting much attention both in theoretical and experimental aspects in recent years due to its significant applications in quantum calculation and quantum communication. A number of experimental implementations [2–4] of teleportation have been reported and some schemes of quantum teleportation have also been presented [5–8]. Up to now, most complete sets of measurements are of Bell state and the states of entangled channel are reducible to a pair of Bell state. In Ref. [7] a class of four-qubit entanglement channels (which are not reducible to a tensor product of two Bell states) was proposed to teleport a two-qubit state. In Ref. [8] the general form of the genuine multipartite entanglement channels was also proposed for faithful teleportation of an N-qubit state.

On the other hand, the entanglement is generally considered as a key resource in quantum information and computation such as teleportation, in which it is crucial to find ways to classify and quantify the entanglement properties of quantum states. Central to doing this is to know if the local invariant quantities can be employed to characterize the entanglement. Numerous researchers have investigated the equivalent classes of three-qubit states specified by SLOCC (stochastic local operations and classical communication) [9–14], but the relation of the SLOCC invariant and the quantum channel in teleportation has not been represented. One of the most important practical features of entanglement is the teleportation capability. We propose in this paper a criterion for faithful teleportation of an arbitrary two-qubit state via the transformation operator [15,16] and analyze the relation between the determinant of transformation operators and the SLOCC transformation invariant.

Suppose that the sender Alice has two particles 1 and 2 in an unknown state:

$$|\chi\rangle_{12} = (x_0|00\rangle + x_1|01\rangle + x_2|10\rangle + x_3|11\rangle)_{12}, \tag{1}$$

where x_0, x_1, x_2, and x_3 are arbitrary complex numbers, and it is assumed that the wave function satisfies the normalization condition $\Sigma_{i=0}^{3}|x_i|^2=1$. The entanglement channel between Alice and Bob is a four-qubit entangled state $|\varphi\rangle_{3456}$. The particle pairs (1, 2) and (3, 4) are in Alice's possession, and the other two particles (5, 6) are in Bob's possession. The system state of the six particles can be expressed as

$$|\psi\rangle_{123456} = |\chi\rangle_{12} \otimes |\varphi\rangle_{3456}. \tag{2}$$

It is well known that a quantum state can be transferred perfectly through a swap operator defined by [17]

$$P\langle ij| = \langle ji|, \tag{3}$$

where

$$P = \frac{1}{2}(I + \vec{\sigma}\cdot\vec{\sigma}). \tag{4}$$

$|\psi\rangle_{123456}$ can then be represented in the following form [17]:

$$\begin{aligned}|\psi\rangle_{123456} &= |\chi\rangle_{12} \otimes |\varphi\rangle_{3456} = P_{15}P_{26}|\chi\rangle_{56} \otimes |\varphi\rangle_{3412}\\ &= P_{15}P_{26}|\varphi\rangle_{3412} \otimes |\chi\rangle_{56}.\end{aligned} \tag{5}$$

On the other hand, we have

$$|\psi\rangle_{123456} = |\chi\rangle_{12} \otimes |\varphi\rangle_{3456} = \frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{4}\varphi_{13}^{i}\varphi_{24}^{j}\sigma_{56}^{ij}|\chi\rangle_{56}, \tag{6}$$

where φ_{13}^{i}, φ_{24}^{j} are Bell states, and

$$\varphi_{mn}^{1} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{mn},$$

$$\varphi_{mn}^{2} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)_{mn},$$

$$\varphi_{mn}^{3} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)_{mn}, \quad mn = 13,24,$$

$$\varphi_{mn}^{4} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{mn}, \tag{7}$$

$$|\chi\rangle_{56} = (x_0|00\rangle + x_1|01\rangle + x_2|10\rangle + x_3|11\rangle)_{56}. \tag{8}$$

From Eqs. (5) and (6), we obtain

$$\sigma_{56}^{ij} = 4\,{}_{13}\langle\varphi^i|_{24}\langle\varphi^j|P_{15}P_{26}|\varphi\rangle_{3412}. \tag{9}$$

The operator σ_{56}^{ij} is called the "transformation operator." The criterion for faithfully teleporting an arbitrary two-qubit state can be given in terms of the "transformation operator." If σ_{56}^{ij} is a unitary operator, Alice can inform Bob of two Bell state measurement outcomes via a classical channel. According to the outcomes received, Bob can determine the state of particles 5, 6 exactly by the inverse of the transformation operator $(\sigma_{56}^{ij})^{-1}$. Consequently, the unknown two-particle entangled state is teleported perfectly, and the successful possibilities and the fidelities both reach unity. Otherwise, if σ_{56}^{ij} is not a unitary operator and the transformation operator is reversible, Bob can introduce an auxiliary two-state particle a with the initial state $|0\rangle_a$ and perform a collective unitary transformation on particles 5, 6, and a as presented in Ref. [16]. Then Bob measures the state of particle a. If the measurement result is $|0\rangle_a$, the teleportation will be successfully realized. In contrast, the teleportation will fail if the measured result is $|1\rangle_a$. Hence the probability of successful teleportation is less than unity. If the transformation operator is not reversible, the unknown two-particle arbitrary entangled state cannot be teleported.

We assume that Alice and Bob share an arbitrary entangled channel:

$$\begin{aligned}|\varphi\rangle_{3456} = (&a_0|0000\rangle + a_1|0001\rangle + a_2|0010\rangle + a_3|0011\rangle\\ &+ a_4|0100\rangle + a_5|0101\rangle + a_6|0110\rangle + a_7|0111\rangle\\ &+ a_8|1000\rangle + a_9|1001\rangle + a_{10}|1010\rangle + a_{11}|1011\rangle\\ &+ a_{12}|1100\rangle + a_{13}|1101\rangle + a_{14}|1110\rangle\\ &+ a_{15}|1111\rangle)_{3456}.\end{aligned} \tag{10}$$

According to Eq. (10),

$$\begin{aligned}\sigma_{56}^{ij} &= 4\,{}_{13}\langle\varphi^i|_{24}\langle\varphi^j|P_{15}P_{26}|\varphi\rangle_{3412} = {}_{13}\langle\varphi^i|_{24}\langle\varphi^j|(I + \sigma_{1x}\sigma_{5x}\\ &\quad + \sigma_{1y}\sigma_{5y} + \sigma_{1z}\sigma_{5z})(I + \sigma_{2x}\sigma_{6x} + \sigma_{2y}\sigma_{6y} + \sigma_{2z}\sigma_{6z})|\varphi\rangle_{3412}.\end{aligned}$$

By using Eqs. (5) and (6), the transformation operator $\hat{\sigma}_{56}^{11}$ is obtained:

$$\begin{aligned}\sigma_{56}^{11} = \frac{1}{2}[&a_0(I_5 + \sigma_{5z})(I_6 + \sigma_{6z}) + a_1(I_5 + \sigma_{5z})(\sigma_{6x} + \sigma_{6y})\\ &+ a_2(\sigma_{5x} + \sigma_{5y})(I_6 + \sigma_{6z}) + a_3(\sigma_{5x} + \sigma_{5y})(\sigma_{6x} + \sigma_{6y})\\ &+ a_4(\sigma_{5x} - \sigma_{5y})(I_6 + \sigma_{6z}) + a_5(\sigma_{5x} - \sigma_{5y})(\sigma_{6x} + \sigma_{6y})\\ &+ a_6(I_5 - \sigma_{5z})(I_6 + \sigma_{6z}) + a_7(I_5 - \sigma_{5z})(\sigma_{6x} + \sigma_{6y})\\ &+ a_8(I_5 + \sigma_{5z})(\sigma_{6x} - \sigma_{6y}) + a_9(I_5 + \sigma_{5z})(I_6 - \sigma_{6z})\\ &+ a_{10}(\sigma_{5x} + \sigma_{5y})(\sigma_{6x} - \sigma_{6y}) + a_{11}(\sigma_{5x} + \sigma_{5y})(I_6 - \sigma_{6z})\\ &+ a_{12}(\sigma_{5x} - \sigma_{5y})(\sigma_{6x} - \sigma_{6y}) + a_{13}(\sigma_{5x} - \sigma_{5y})(I_6 - \sigma_{6z})\\ &+ a_{14}(I_5 - \sigma_{5z})(\sigma_{6x} - \sigma_{6y}) + a_{15}(I_5 - \sigma_{5z})(I_6 - \sigma_{6z})],\end{aligned} \tag{11}$$

which can be written in matrix form:

$$\sigma_{56}^{11} = 2\begin{pmatrix} a_0 & a_8 & a_4 & a_{12}\\ a_1 & a_9 & a_5 & a_{13}\\ a_2 & a_{10} & a_6 & a_{14}\\ a_3 & a_{11} & a_7 & a_{15}\end{pmatrix}. \tag{12}$$

Other transformation operators $\hat{\sigma}_{56}^{ij}$ are given by

$$\hat{\sigma}_{56}^{ij} = \hat{\sigma}_{56}^{11}(\sigma_5^i \otimes \sigma_6^j), \tag{13}$$

where $\hat{\sigma}_m^k = I_m, \sigma_{mz}, \sigma_{mx}, -i\sigma_{my}$, $m=5,6$, I_m is the two-dimensional identity, and $\sigma_{mz}, \sigma_{mx}, \sigma_{my}$ are the Pauli matrices. Apparently, if $\hat{\sigma}_{56}^{11}$ is a unitary operator, $\hat{\sigma}_{56}^{ij}$ are also unitary operators.

Moreover, the invariant L of SLOCC can be expressed as follows [12]:

$$L = \begin{vmatrix} a_0 & a_4 & a_8 & a_{12}\\ a_1 & a_5 & a_9 & a_{13}\\ a_2 & a_6 & a_{10} & a_{14}\\ a_3 & a_7 & a_{11} & a_{15}\end{vmatrix} \tag{14}$$

or

$$\begin{aligned}L = &(a_0a_5 - a_1a_4)(a_{10}a_{15} - a_{11}a_{14})\\ &- (a_0a_9 - a_1a_8)(a_6a_{15} - a_7a_{14})\\ &+ (a_0a_{13} - a_1a_{12})(a_6a_{11} - a_7a_{10})\\ &+ (a_4a_9 - a_5a_8)(a_2a_{15} - a_3a_{14})\\ &- (a_4a_{13} - a_5a_{12})(a_2a_{11} - a_3a_{10})\\ &+ (a_8a_{13} - a_9a_{12})(a_2a_7 - a_3a_6).\end{aligned}$$

From Eq. (11) or (12) and Eq. (14), the relation between the determinant of transformation operator σ_{56}^{11} and the SLOCC invariant L could be acquired.

In reality, the determinant of transformation operator σ_{56}^{11} is found just equal to the SLOCC invariant $L' = 16L$. Therefore, if transformation operator $\hat{\sigma}_{56}^{ij}$ are unitary operators, the normal of SLOCC invariant $|L'| = 1$. If transformation operator $\hat{\sigma}_{56}^{ij}$ are not unitary operators. the normal of SLOCC invariant $|L'| < 1$.

To show the applicability of this criterion we examine the probability of transportation for some known examples. We take the first example of the quantum channel given in Ref. [7]:

$$\begin{aligned}|\varphi\rangle_{3456} = \frac{1}{2\sqrt{2}}(&|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle\\ &+ |1010\rangle + |1100\rangle + |1111\rangle)_{3456},\end{aligned} \tag{15}$$

i.e.,

$$a_0 = a_6 = a_9 = a_{10} = a_{12} = a_{15} = \frac{1}{2\sqrt{2}}, \quad a_3 = a_5 = -\frac{1}{2\sqrt{2}}.$$

The transformation operator can then be calculated easily by

$$\sigma_{56}^{11}=\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{16}$$

Then $L'=-1$ and $|L'|=1$. So, for this quantum channel, the teleportation can be perfectly realized.

A cluster state is used as a quantum channel between Alice and Bob, which is in the following state [18]:

$$|\varphi\rangle_{3456}=\frac{1}{2}(|0000\rangle+|0110\rangle+|1001\rangle-|1111\rangle)_{3456},$$

i.e.,

$$a_0=a_6=a_9=\frac{1}{2}, \quad a_{15}=-\frac{1}{2},$$

and we obtain

$$\sigma_{56}^{11}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then we easily find that $L'=1$. So, for this quantum channel, the teleportation can also be perfectly realized.

If Alice and Bob share a quantum channel in the form of the following partly pure entangled four-particle state [19],

$$\begin{aligned}|\varphi\rangle_{3456}&=(B_1|00\rangle+C_1|11\rangle)_{35}(B_2|00\rangle+C_2|11\rangle)_{46}\\ &=(B_1B_2|0000\rangle+B_1C_2|0101\rangle+C_1B_2|1010\rangle\\ &\quad+C_1C_2|1111\rangle)_{3456},\end{aligned}$$

then, without loss of generality, it is assumed that $|C_1|<|B_1|$, $|C_2|<|B_2|$, i.e.,

$$a_0=B_1B_2, \quad a_5=B_1C_2, \quad a_{10}=C_1B_2, \quad a_{15}=C_1C_2,$$

and we obtain

$$\sigma_{56}^{11}=2\begin{pmatrix} B_1B_2 & 0 & 0 & 0 \\ 0 & 0 & B_1C_2 & 0 \\ 0 & C_1B_2 & 0 & 0 \\ 0 & 0 & 0 & C_1C_2 \end{pmatrix}$$

and the SLOCC invariant $L'=16B_1^2B_2^2C_1^2C_2^2$. Therefore, the transformation operator is reversible and is not a unitary operator. Then, Bob needs to introduce an auxiliary particle a with initial state $|0\rangle_a$ and perform a collective unitary transformation on particles 5, 6, and a. The probability of successful teleportation is $4|C_1|^2|C_2|^2<1$.

For the known quantum channel Greenberger-Horne-Zeilinger (GHZ) state, i.e., $|\varphi\rangle_{3456}=(1/2)(|0000\rangle+|1111\rangle)_{3456}$, the transformation operator is

$$\sigma_{56}^{11}=\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \tag{17}$$

and the normal of SLOCC invariant $|L'|=0$. Therefore, the transportation cannot be realized.

We consider another quantum channel in W state described by

$$|\varphi\rangle_{3456}=\frac{1}{2}(|0001\rangle+|0010\rangle+|0100\rangle+|1000\rangle)_{3456},$$

whose transformation operator is then given by

$$\sigma_{56}^{11}=\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

We find $|L'|=0$. From Eqs. (17) and (18), all the transformation operators are not inversible. Accordingly, the GHZ state and the W state cannot be employed to deterministically teleport arbitrary two qubits.

In this paper we have introduced a transportation operator and shown that there exists a relation between the determinant of transformation operator and the SLOCC invariant for Bell basis measurement. It could be affirmed that if the SLOCC invariant L of quantum channel is not zero, the unknown two-particle entangled state can be teleported successfully. If it is equal to zero, the teleportation of the unknown two-particle entangled state will by no means be realized. In addition, if the normal of SLOCC invariant L' of quantum channel is equal to 1, the unknown two-particle entangled state can be teleported perfectly. We believe for Bell basis measurement the relation between the determinant of the transformation operator and the SLOCC invariant can be expanded to the multipartite entangled state.

However, we know that the property of the transformation operator depends on both the channel and the measuring basis entanglement, but only the case of the Bell basis measurement has been discussed in this paper. The general relation of transformation operator with quantum channel and measuring basis and the relation between the probability of success with the SLOCC invariant need further investigation. The optimal match of measuring basis and quantum channel is also to be studied.

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