Time of arrival of electrons in the double-slit experiment

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(Received 11 December 2006; revised manuscript received 19 November 2007; published 14 January 2008)

Using Nelson's stochastic mechanics, quantum motion of electrons in the double-slit experiment is studied numerically. It is found that not only the distribution of arrival positions but also that of arrival times at the screen forms an interference pattern. It is demonstrated that the presence time and arrival time are well interpreted in terms of stochastic mechanics.

DOI: 10.1103/PhysRevA.77.014102 PACS number(s): 03.65.Ta, 03.65.Xp

There is no doubt that position and time are the most fundamental quantities for describing the motion of an object. In classical mechanics, the equation of motion determines the position of an object as a function of time, $\mathbf{x}(t)$. Then it is straightforward to calculate the time of arrival of the object moving from a position $\mathbf{x}_i$ to another $\mathbf{x}_f$.

In quantum mechanics, however, the time of arrival cannot be obtained within the standard procedure of calculating an expectation value because, so far, a general time operator has not been established [1]. It is worthwhile to mention that Pauli pointed out that there is no self-adjoint time operator conjugate to a Hamiltonian bounded from below [1,2].

From an experimental point of view, it is evident that the time of arrival can be measured, in principle, by means of the time-of-flight technique. Therefore, a complete theory must be able to predict the time of arrival. In this sense, the lack of an established method of obtaining the time of arrival may be regarded as a point of imperfection in quantum mechanics. Therefore, despite the fundamental difficulties mentioned above, considerable efforts have been put into solving the problem of the time of arrival [1,3–8].

An alternative way of calculating the time of arrival is to employ Nelson's stochastic mechanics [9,10]. Since stochastic mechanics gives a set of trajectories (sample paths), the time of arrival for each trajectory is obtained in the same way of classical mechanics. A few recent works showed that stochastic mechanics is indeed a powerful tool of calculating the tunneling time [11,12].

In this Brief Report, we consider the time of arrival in the double-slit experiment. Since the establishment of quantum mechanics the double-slit experiment by electrons has been one of the most famous gedanken experiments for demonstrating the fundamental concept of quantum mechanics. The realization of such double-slit experiments has been reported in recent years [13,14]. The double-slit experiment performed by Tonomura *et al.* recorded arrival positions of electrons one by one and confirmed that the accumulated arrival positions on the screen reproduce the interference pattern [14].

So far, the formation of a double-slit interference pattern has been studied based on the wave function in the stationary states. However, as shown by Tonomura *et al.* [14], the interference pattern is composed of the arrival positions of individual electrons. In this case the quantum motion of each electron should be described by a time-dependent wave packet rather than a stationary wave. Then the time of arrival of electrons at the screen becomes a fundamental subject as well as the position of arrival. Experimentally, it is possible to obtain a time of arrival distribution by measuring the time of arrival for each electron. The purpose of our present report is to predict the time of arrival distribution numerically by using stochastic mechanics.

In stochastic mechanics, a possible trajectory of an electron is expressed as a sample path calculated by the Ito stochastic differential equation given by [9,10]

$$d\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t),t)dt + d\mathbf{w}(t), \tag{1}$$

where $d\mathbf{w}(t)$ represents the Brownian motion defined by

$$\langle dw_i(t)\rangle = 0, \langle dw_i(t)dw_j(t)\rangle = \frac{\hbar}{m}\delta_{ij}dt \quad (i = x,y), \tag{2}$$

$\hbar$ is the Planck constant divided by 2π, and m the mass of an electron. $\mathbf{b}(\mathbf{x},t)$ represents the drift velocity given by

$$\mathbf{b}(\mathbf{x},t) = \frac{\hbar}{m}(\mathrm{Re} + \mathrm{Im})\,\nabla\,\ln \psi(\mathbf{x},t), \tag{3}$$

where $\psi(\mathbf{x},t)$ is the solution of the Schrödinger equation.

Using Eqs. (1)–(3) we calculate trajectories (sample paths) of electrons for a simple double-slit experiment. We assume that electrons moving along the y axis pass through a double slit placed at $y=0$ and then reach a screen placed at $y=Y$. For simplicity, we further assume that the wave function passing through the double-slit is approximated by a couple of two-dimensional Gaussian wave packets [15,16]:

$$\psi(x,y,t) = \psi_1(x,y,t) + \psi_2(x,y,t), \tag{4}$$

where

$$\psi_j(x,y,t) = \frac{1}{2\sqrt{\pi a}(1+i\xi t)}\exp\left\{-\left[\frac{1}{2a}\frac{(x-s_j)^2+y^2}{1+i\xi t}\right] + i\left(\frac{k_0 y + \omega_0 t}{1+i\xi t}\right)\right\} \quad (j=1,2), \tag{5}$$

$2s$ the distance between two slits, $s_1=-s$, $s_2=s$, $(0,\pm s)$ the coordinates of the slits, $\xi=\hbar/(ma)$, a the dispersion at $t=0$, and $\omega_0=\hbar k_0^2/(2m)$. Setting $\psi_j=\exp(R_j+iS_j)$ $(j=1,2)$ with

$$\mathbf{u}_j = \frac{\hbar}{m}\nabla R_j, \quad \mathbf{v}_j = \frac{\hbar}{m}\nabla S_j \quad (j=1,2), \tag{6}$$

we obtain the osmotic velocity $\mathbf{u}$ and the current velocity $\mathbf{v}$ as [9]

$$\mathbf{u} = \frac{\mathbf{u}_1 + \mathbf{u}_2}{2} + \frac{\sinh(R_1 - R_2)(\mathbf{u}_1 - \mathbf{u}_2) - \sin(S_1 - S_2)(\mathbf{v}_1 - \mathbf{v}_2)}{2[\cosh(R_1 - R_2) + \cos(S_1 - S_2)]}, \tag{7}$$

$$\mathbf{v} = \frac{\mathbf{v}_1 + \mathbf{v}_2}{2} + \frac{\sinh(R_1 - R_2)(\mathbf{v}_1 - \mathbf{v}_2) + \sin(S_1 - S_2)(\mathbf{u}_1 - \mathbf{u}_2)}{2[\cosh(R_1 - R_2) + \cos(S_1 - S_2)]}. \tag{8}$$

From the relation $\mathbf{b}(\mathbf{x},t) = \mathbf{u} + \mathbf{v}$ we obtain the drift velocity as follows:

$$b_x(\mathbf{x},t) = \frac{\hbar}{m}\left[\frac{\hbar t/m - a}{a^2 + (\hbar t/m)^2}\right]x - \frac{\hbar}{m}\left[\frac{s(\hbar t/m - a)}{a^2 + (\hbar t/m)^2}\right]\frac{\sinh\{2asx/[a^2 + (\hbar t/m)^2]\} + \sin\{2(\hbar t/m)sx/[a^2 + (\hbar t/m)^2]\}}{\cosh\{2asx/[a^2 + (\hbar t/m)^2]\} + \cos\{2(\hbar t/m)sx/[a^2 + (\hbar t/m)^2]\}}, \tag{9}$$

$$b_y(\mathbf{x},t) = \frac{\hbar}{m}\left[\frac{\hbar t/m - a}{a^2 + (\hbar t/m)^2}\right]y + \frac{p_0}{m}\left[\frac{a(a + \hbar t/m)}{a^2 + (\hbar t/m)^2}\right]. \tag{10}$$

Using Eqs. (4)–(10) we have solved the stochastic differential equation (1) numerically. Examples of calculated sample paths are shown in Fig. 1. From the figure one can see that the interference pattern becomes clearer as the number of sample paths increases.

Now, we consider the time of arrival. Before showing our numerical results, let us introduce two expressions that have been used for discussion of quantum mechanical time of arrival [1].

One is *the presence time* defined by

$$\langle T\rangle^p_{X,Y} = \int_0^\infty T\rho^p_{X,Y}(T)dT \tag{11}$$

with the presence time distribution

$$\rho^p_{X,Y}(T)dT = \frac{p(X,Y,T)dT}{\int_0^\infty p(X,Y,T)dT}, \tag{12}$$

where $p(X,Y,T) = |\psi(X,Y,T)|^2$. $\rho^p_{X,Y}(T)dT$ is proportional to the probability density of finding an electron at a detector position (X,Y) at a time interval $T \sim T+dT$. Although the presence time seems to be a simple definition of quantum mechanical time of arrival, Eq. (12) has not been derived within the standard procedure of calculating an observable.

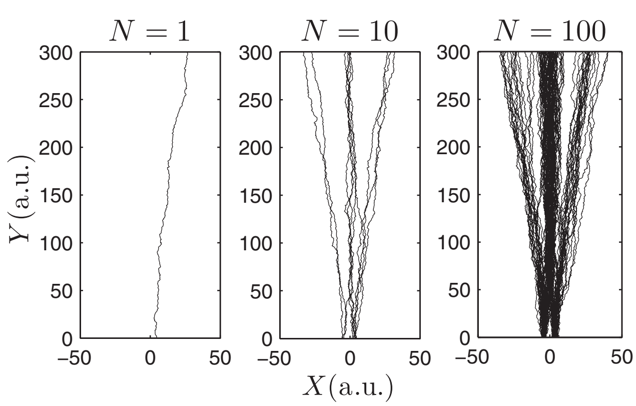

FIG. 1. Sample paths in the double-slit experiment. N denotes the number of sample paths. Two slits are placed on the X axis. Atomic units (a.u.) have been used.

The other one is *the arrival time* defined by

$$\langle T\rangle^a_{X,Y} = \int_0^\infty T\rho^a_{X,Y}(T)dT, \tag{13}$$

with the arrival time distribution

$$\rho^a_{X,Y}(T)dT = \frac{|\mathbf{j}(X,Y,T)\cdot d\mathbf{S}|dT}{\int_0^\infty |\mathbf{j}(X,Y,T)\cdot d\mathbf{S}|dT}, \tag{14}$$

where $\mathbf{j}(X,Y,T)$ represents the current density and $d\mathbf{S}$ is the surface element of a detector. The arrival time formula is derived within Bohmian mechanics [1,17].

In contrast to quantum mechanics, stochastic mechanics has no fundamental difficulty in defining a time of arrival: We can calculate classical mechanical-like time of arrival by regarding sample paths as possible classical trajectories. However, there are two schemes for defining the detection of a particle at a fixed point (see Fig. 2) [18,19]. One is the "first counting scheme." This scheme assumes that a particle is counted by a detector when its sample path traverses the surface of the detector for the first time. Therefore, the time of arrival in the first counting scheme for the sample path 1 in Fig. 2 is given by $(T_3+T_4)/2$, where $T_1, \ldots, T_5$ represent the arrival time at each point. Defining $N^f_{X,Y}(T)dT\,dX$ as the number of sample paths traversing the detector within the acceptance width $X - dX/2 \sim X + dX/2$ in the time interval $T \sim T+dT$, we introduce the "first counting arrival time distribution":

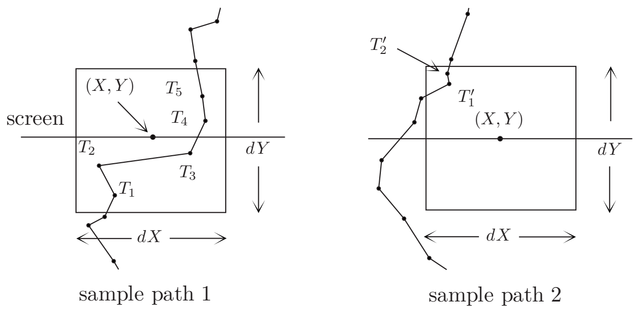

FIG. 2. First counting scheme and the multiple counting scheme. Detector's width and area for the point (X,Y) are dX and $dXdY$, respectively. The traversing time in the first counting scheme is $(T_3+T_4)/2$ for sample path 1 while the existing times in the multiple counting scheme are $(T_1+T_2+T_3+T_4+T_5)/5$ for the sample path 1 and $(T_1'+T_2')/2$ for the sample path 2.

$$\rho^f_{X,Y}dT = \frac{N^f_{X,Y}(T)dT}{\int_0^\infty N^f_{X,Y}(T)dT}. \quad (15)$$

Using $\rho^f_{X,Y}$ the time of arrival in the first counting scheme is given by

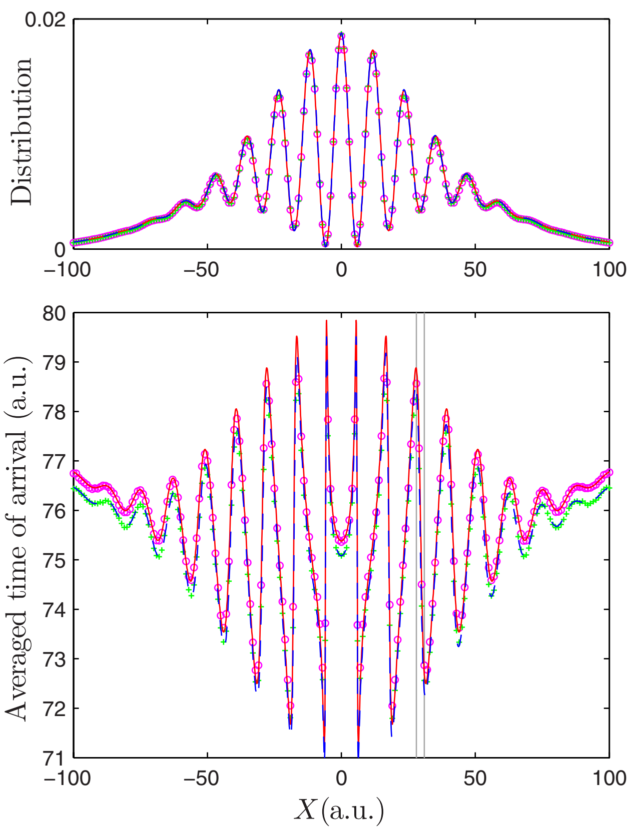

FIG. 3. (Color online) Number of electrons arrived at the screen (upper figure) and the averaged time of arrival (lower figure). The multiple counting scheme (magenta, ∘), first counting scheme (green, +), presence time (red solid line), and arrival time (blue broken line) are plotted. The window in the lower figure indicates the region shown in Fig. 4.

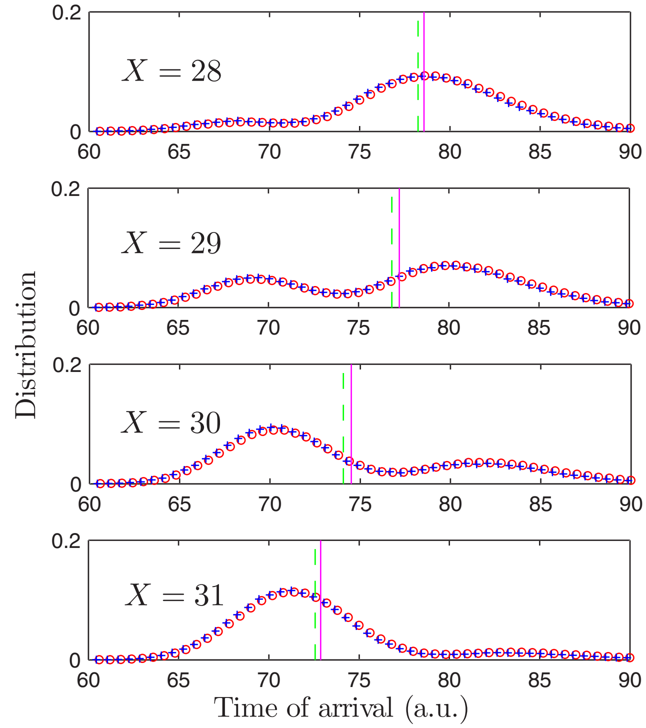

FIG. 4. (Color online) Time of arrival distributions at the screen positions $X=28$, 29, 30, 31 $(Y=600)$. Distributions are calculated by the multiple counting scheme (red, ○) and the first counting scheme (blue, +). The solid vertical lines and broken lines represent $\langle T\rangle^m_{X,Y}$ and $\langle T\rangle^f_{X,Y}$, respectively.

$$\langle T\rangle^f_{X,Y} = \int_0^\infty T\rho^f_{X,Y}(T)dT. \quad (16)$$

The other is the "multiple counting scheme." This scheme assumes that a particle is counted by a detector in a probabilistic manner when a sample path enters the detector. Therefore, the time of arrival in the multiple counting scheme for the sample path 1 in Fig. 2 is given by $(T_1+T_2+T_3+T_4+T_5)/5$. It should be mentioned that the sample path 2 contributes to the time of arrival in the multiple counting scheme but does not so in the first counting scheme. Defining $N^m_{X,Y}(T)dXdYdT$ as the number of the sample paths entering the detector area $(X-dX/2,Y-dY/2)\sim(X+dX/2,Y+dY/2)$ at the time $T\sim T+dT$, we introduce the "multiple counting arrival time distribution":

$$\rho^m_{X,Y}dT = \frac{N^m_{X,Y}(T)dT}{\int_0^\infty N^m_{X,Y}(T)dT}. \quad (17)$$

By its definition $\int N^m_{X,Y}(T)dT$ gives a greater number than the total number of the sample paths come into the detector. The time of arrival in the multiple counting scheme is represented by

$$\langle T\rangle^m_{X,Y} = \int_0^\infty T\rho^m_{X,Y}(T)dT. \quad (18)$$

In Fig. 3, we show typical numerical results. Parameters have been chosen as $k_0=8$, $Y=6\times 10^2$, $a=2$, $s=20$, $dt=0.3$, $dX/2=0.5$, $dY/2=3$ in atomic units. The total number of sample paths is $N=1.0\times 10^8$. The upper figure shows the distribution of electrons arrived at the screen, i.e., the interference pattern in the double-slit experiment.

It is found that the time of arrival in the first counting scheme $\langle T\rangle^f_{X,Y}$ coincides with the arrival time $\langle T\rangle^a_{X,Y}$ calculated by the probability current density in the quantum mechanics. On the other hand, the time of arrival in the multiple counting scheme $\langle T\rangle^m_{X,Y}$ coincides with the presence time $\langle T\rangle^p_{X,Y}$ calculated by the probability density. The latter result agrees with that of Aoki *et al.* [11]. It is also found that the time of arrival changes most rapidly about the local minima in the intensity distribution (upper figure).

In Fig. 4, we show the time of arrival distributions calculated by both schemes at the screen positions $X=28$, 29, 30, 31. These positions are located within the vertical lines in Fig. 3, i.e., one of the local minima in the intensity distribution. The time of arrival distributions about other minima behave similarly. From the figure we observe that about the local minima in the intensity distribution two tails of wave packets contribute to the intensity distribution.

In this Brief Report, we have shown that the arrival time of electrons in the double-slit experiment will have an oscillating behavior, which is, in principle, observable in an experiment. It is shown that the time of arrival is calculated clearly by using stochastic mechanics though there are two schemes for calculation: The first counting scheme and the multiple counting scheme. The former agrees with the arrival time based on Bohmian mechanics while the latter agrees with the presence time derived by a naive consideration in quantum mechanics. We have not succeeded in explaining these agreements analytically but leave this problem for further research.

It is worthwhile to mention that Szriftgiser *et al.* observed time of arrival distribution by using "temporal slits" [20]. Our result shows that even an ordinary double-slit in space will form an interference pattern in time, suggesting the possibility to observe the quantum features of the time of arrival in a simple manner.

This work is partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (C).

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