Bright-dark solitons and their collisions in mixed *N*-coupled nonlinear Schrödinger equations

M. Vijayajayanthi,[1,*] T. Kanna,[2,†] and M. Lakshmanan[1,‡]

[1]*Centre for Nonlinear Dynamics, School of Physics, Bharathidasan University, Tiruchirapalli-620 024, India*
[2]*Department of Physics, Bishop Heber College, Tiruchirapalli-620 017, India*

Mixed-type (bright-dark) soliton solutions of the integrable *N*-coupled nonlinear Schrödinger (CNLS) equations with mixed signs of focusing- and defocusing-type nonlinearity coefficients are obtained by using Hirota's bilinearization method. Generally, for the mixed *N*-CNLS equations the bright and dark solitons can be split up in $(N-1)$ ways. By analyzing the collision dynamics of these coupled bright and dark solitons systematically we point out that for $N>2$, if the bright solitons appear in at least two components, nontrivial effects, such as onset of intensity redistribution, amplitude-dependent phase shift, and change in relative separation distance take place in the bright solitons during collision. However their counterparts, the dark solitons, undergo elastic collision but experience the same amplitude-dependent phase shift as that of bright solitons. Thus, in the mixed CNLS system, there is a coexisting shape-changing collision of bright solitons and elastic collision of dark solitons with amplitude-dependent phase shift, thereby influencing each other mutually in an intricate way.

I. INTRODUCTION

Solitons in coupled nonlinear Schrödinger (CNLS) equations have been the subject of intense studies due to their intriguing collision properties and their robustness against external perturbations. The study of physical and mathematical aspects of CNLS equations is of considerable current interest as these equations arise in diverse areas of science, such as nonlinear optics, optical communication, biophysics, Bose-Einstein condensates (BECs), and plasma physics [1–6]. In the context of nonlinear optics, solitons arising as a result of competing focusing-type nonlinearity and anomalous dispersion (diffraction) of pulse (beam) are called bright solitons [3] as they are well localized structures of light. The dark solitons resulting due to compensation between defocusing-type nonlinearity and normal dispersion of pulse appear as localized intensity dips on a finite carrier wave background, and they are more robust than bright solitons [3]. The appearance of multicomponent CNLS-type equations as dynamical equations in various areas of physics [3,4,6] and the subsequent studies on these systems lead to the identification of bright-, dark-, bright-dark-, and dark-bright-type solitons [7–16]. Even though there are a number of works on bright and dark soliton propagation and collision separately, results are scarce for the study on bright-dark-type soliton propagation and their collision dynamics.

In the present paper we perform a study on bright-dark soliton solutions of mixed *N*-CNLS equations and examine the shape-changing collisions of multicomponent bright solitons in the presence of dark components. We consider the following set of integrable mixed *N*-CNLS equations (in dimensionless form):

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*jayanthi@cnld.bdu.ac.in
†tkans@rediffmail.com
‡Corresponding author; lakshman@cnld.bdu.ac.in

$$iq_{j,z} + q_{j,tt} + 2\left(\sum_{l=1}^{N} \sigma_l |q_l|^2\right) q_j = 0, \quad j = 1, 2, \ldots, N, \quad (1a)$$

where q_j, $j = 1, 2, \ldots, N$, is the complex amplitude of the *j*th component, the subscripts *z* and *t* denote the partial derivatives with respect to normalized distance and retarded time, respectively, and the coefficients, σ_l, define the sign of the nonlinearity. For convenience and without loss of generality, we define σ_l for this mixed case as

$$\sigma_l = \begin{cases} 1 & \text{for } l = 1, 2, \ldots, m, \\ -1 & \text{for } l = m+1, m+2, \ldots, N. \end{cases} \quad (1b)$$

Recent theoretical and experimental studies show that the bright solitons in the focusing case ($\sigma_l = 1$, $l = 1, 2, \ldots, N$) undergo fascinating shape-changing collisions characterized by intensity redistribution, amplitude-dependent phase shift, and relative separation distance [11–13], whereas the standard elastic collision of dark solitons occurs in the defocusing case ($\sigma_l = -1$, $l = 1, 2, \ldots, N$) [8,14]. Later, it has been observed in Ref. [16] that bright-bright solitons of mixed CNLS equation (1) also undergo shape-changing collision but of different nature. Very recently, it has also been shown that the bright solitons in the mixed CNLS system exhibit periodic energy switching during the shape-changing collision process in the presence of linear couplings [17]. Now it is of further interest to examine how the bright solitons are influenced by dark solitons and vice versa in this mixed CNLS system. The main aim of this paper is to investigate the nature of bright-dark soliton solutions of the mixed CNLS equation (1) and their collisions.

The pioneering works of Makhankov *et al.* [18] in the context of the Bose-Hubbard model and a few recent works [19,20] on left-handed materials (LHMs) in nonlinear optics, suggest that the mixed CNLS system studied in our paper could be of considerable physical significance. The mixed 2-CNLS system can be obtained as the modified Hubbard model (Lindner-Fedyanin system) in the long-wavelength

approximation by taking into account the electron-phonon interaction [18]. Here the two component Bose-condensate is a mixture of two "gases" with attractive and repulsive inter-boson fields and the bright-dark soliton solution corresponds to the so-called drop-bubble solution. A straightforward generalization of this mixed 2-CNLS system to an arbitrary number of fields is given in Ref. [18]. Exactly this is the system which we have considered in this paper [Eqs. (1)] where the first m components have positive sign of nonlinearity and the remaining components possess negative sign of nonlinearity. The mixed 2-CNLS equations also arise as governing equations for an electromagnetic field propagation in LHMs with Kerr-type nonlinearity [19]. In fact, this case arises when the fields in the medium experience different types of nonlinearity, leading to $\sigma_1 = -\sigma_2 = 1$ or $\sigma_2 = -\sigma_1 = 1$, corresponding to a medium of effective positive dielectric permittivity ϵ_{eff} and effective negative magnetic permeability μ_{eff} or vice versa. This suggests that mixed N-CNLS equations could be a possible generalization of multiple electromagnetic fields propagating in LHMs with suitably chosen effective permittivity and effective permeability.

Equation (1) with $N=2$ can also be viewed as the governing equation for two fields, q_1 and q_2^*, propagating in the anomalous and normal dispersion regimes, respectively (that is, the self-phase modulation coefficients are positive and cross-phase modulation coefficients are negative in both the components). A possible physical realization of such type of nonlinearities is multifield propagation in a quadratic medium with inefficient phase matching [21]. Another important physical realization of Eq. (1) arises in the context of boson-fermion gas mixtures. For example, the dynamics of two-species condensates is governed by mixed 2-CNLS equations for suitable choice of intraspecies (a_{11}, a_{22}) and interspecies (a_{12}, a_{21}) scattering lengths [22]. These two-species condensates offer a wider range of possibilities, the main one being the possibility of having a negative interspecies scattering length. This possibility has been theoretically explored in the context of Feshbach resonance management and realized experimentally for boson-fermion mixtures [23]. Thus, the interspecies interactions a_{12} and a_{21} can be tuned to be negative (attractive type) and positive (repulsive type) scattering lengths, respectively, by Feshbach resonance. It is also possible to choose the self-interactions as $a_{11} > 0$, $a_{22} < 0$ [24]. Therefore, studying N-CNLS equations (1) of mixed type will provide a better understanding on the dynamics of multispecies condensates with suitably tuned scattering lengths. In addition, mixed 2-CNLS equations arise in BECs involving two isotopes of the same element, for example, isotopes of rubidium (Rb^{87} and Rb^{85}) [25]. In fact, multicomponent BECs support nonlinear waves which do not exist in single component BECs such as domain-wall solitons, bright-dark solitons, etc. Thus, the mixed CNLS system which we have investigated could be of considerable physical relevance and significance in the context of nonlinear optics and matter waves.

The plan of the paper is as follows. In Sec. II, we briefly present the Hirota's bilinearization procedure for the mixed N-CNLS Eq. (1) to obtain exact soliton solutions. Section III is devoted to obtain exact one and two bright-dark soliton solutions for mixed 2-CNLS, 3-CNLS, and N-CNLS equations. The collision dynamics of bright-dark solitons in mixed CNLS equations is given in Sec. IV, where we have pointed out that if the bright solitons appear in more than one component then they undergo shape-changing collisions characterized by intensity redistribution, amplitude-dependent phase shift, and change in relative separation distance. We have also discussed the role of dark solitons on the shape-changing collisions of bright solitons and also the effect of bright solitons on the dark soliton collisions. In Sec. V, we summarize the results of our study. Asymptotic analysis of mixed 2-CNLS and mixed 3-CNLS equations are given in Appendixes A and B, respectively.

II. BILINEARIZATION METHOD FOR INTEGRABLE MIXED N-CNLS EQUATIONS

In this section, we briefly outline the procedure to obtain m-bright–n-dark soliton solution $(m+n=N)$ of the mixed N-CNLS equations using Hirota's bilinearization method [26]. We denote the soliton solution of Eq. (1) in which the bright and dark solitons are split up in the first m components and the remaining $(N-m) = n$ components, respectively, as "mixed soliton solutions," for brevity. To start with, let us apply the bilinearizing transformation

$$q_j = \frac{g^{(j)}}{f}, \quad j = 1, 2, \ldots, m, \tag{2a}$$

$$q_{l+m} = \frac{h^{(l)}}{f}, \quad l = 1, 2, \ldots, n, \tag{2b}$$

to Eq. (1), where $g^{(j)}$'s and $h^{(l)}$'s are arbitrary complex functions of z and t while f is a real function. Then, the set of mixed N-CNLS equations given by Eq. (1) reduces to the following set of bilinear equations:

$$D_1(g^{(j)} \cdot f) = 0, \quad j = 1, 2, \ldots, m, \tag{3a}$$

$$D_1(h^{(l)} \cdot f) = 0, \quad l = 1, 2, \ldots, n, \tag{3b}$$

$$D_2(f \cdot f) = 2 \left(\sum_{j=1}^{m} g^{(j)} g^{(j)*} - \sum_{l=1}^{n} h^{(l)} h^{(l)*} \right), \tag{3c}$$

where $D_1 = (iD_z + D_t^2 - \lambda)$, $D_2 = (D_t^2 - \lambda)$, $*$ denotes complex conjugate and λ is a constant to be determined. The Hirota's bilinear operators D_z and D_t are defined as

$$D_z^p D_t^q (a \cdot b) = \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right)^p \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^q$$
$$\times a(z,t) b(z',t') \big|_{(z=z', t=t')}. \tag{4}$$

Expanding $g^{(j)}$, $h^{(l)}$, and f formally as power series expansions in terms of a small arbitrary real parameter χ,

$$g^{(j)} = \chi g_1^{(j)} + \chi^3 g_3^{(j)} + \cdots, \quad j = 1, 2, \ldots, m, \tag{5a}$$

$$h^{(l)} = h_0^{(l)}(1 + \chi^2 h_2^{(l)} + \chi^4 h_4^{(l)} + \cdots), \quad l = 1, 2, \ldots, n, \tag{5b}$$

$$f = 1 + \chi^2 f_2 + \chi^4 f_4 + \cdots, \tag{5c}$$

and solving the resultant set of equations recursively, we can obtain the explicit forms of $g^{(j)}$, $h^{(l)}$, and f.

III. EXACT MIXED-TYPE (BRIGHT-DARK) SOLITON SOLUTIONS OF MULTICOMPONENT MIXED CNLS EQUATIONS

A significant feature of the system of integrable mixed CNLS equation (1) is that it admits a rich structure of soliton solutions, such as bright-, dark-, bright-dark-type soliton solutions depending upon the boundary conditions. Here, our focus is on bright-dark-type (or dark-bright-type) solutions. In order to understand the nature of such mixed soliton solutions, their propagation and collision dynamics, we first obtain the solution for the $N=2$ case. Then by extending the analysis to the arbitrary N-component case in the remaining sections, we point out that for $N>2$, if the bright solitons are split up in two or more components, there occurs shape-changing collision of bright solitons which are also influenced by the presence of a dark soliton.

A. Bright-dark soliton solutions of the mixed 2-CNLS equations

(a) *Mixed (bright-dark) one-soliton solution.* In this case the bright soliton appears in the q_1 component and the remaining component q_2 comprises of dark soliton (or vice versa). This case corresponds to the choice $\sigma_1 = 1$ and $\sigma_2 = -1$ (or $\sigma_1 = -\sigma_2 = -1$) for which Eq. (1) become

$$iq_{1,z} + q_{1,tt} + 2(|q_1|^2 - |q_2|^2)q_1 = 0, \tag{6a}$$

$$iq_{2,z} + q_{2,tt} + 2(|q_1|^2 - |q_2|^2)q_2 = 0. \tag{6b}$$

After restricting the power series expansion (5) as

$$g^{(1)} = \chi g_1^{(1)}, \tag{7a}$$

$$h^{(1)} = h_0^{(1)}(1 + \chi^2 h_2^{(1)}), \tag{7b}$$

$$f = 1 + \chi^2 f_2, \tag{7c}$$

and solving the resulting set of linear partial differential equations recursively, one can write down the mixed one-soliton solution explicitly as

$$q_1 = \frac{\alpha_1^{(1)} e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + R}} = A_1 k_{1R} e^{i\eta_{1I}} \operatorname{sech}\left(\eta_{1R} + \frac{R}{2}\right), \tag{8a}$$

$$q_2 = \frac{c_1 e^{i\zeta_1}\left[1 - \left(\dfrac{k_1 - ib_1}{k_1^* + ib_1}\right)e^{\eta_1 + \eta_1^* + R}\right]}{1 + e^{\eta_1 + \eta_1^* + R}}, \tag{8b}$$

where

$$e^R = \frac{|\alpha_1^{(1)}|^2}{(k_1 + k_1^*)^2}\left(1 - \frac{|c_1|^2}{|k_1 - ib_1|^2}\right)^{-1}, \quad A_1 = \left(\frac{\alpha_1^{(1)}}{2k_{1R}}\right)e^{-R/2}, \tag{8c}$$

$$\eta_1 = k_1 t + i(k_1^2 - \lambda)z, \quad k_1 = k_{1R} + ik_{1I}, \quad \alpha_1^{(1)} = \alpha_{1R}^{(1)} + i\alpha_{1I}^{(1)}, \tag{8d}$$

$$\zeta_1 = -(b_1^2 + \lambda)z + b_1 t, \quad \lambda = 2|c_1|^2, \quad c_1 = c_{1R} + ic_{1I}, \tag{8e}$$

with the condition

$$|c_1|^2 < |k_1 - ib_1|^2. \tag{8f}$$

Here $\alpha_1^{(1)}$, k_1, and c_1 are arbitrary complex parameters while b_1 is a real parameter. In the above equations and in the following sections, the suffixes R and I denote the real and imaginary parts, respectively. The above one-soliton solution is characterized by seven real parameters $\alpha_{1R}^{(1)}$, $\alpha_{1I}^{(1)}$, k_{1R}, k_{1I}, c_{1R}, c_{1I}, and b_1 along with the constraint (8f). In the context of nonlinear optics the quantity A_1 defined through the α parameter can be viewed as the polarization vector of the light pulse or beam and $A_1 k_{1R}$ as the amplitude of the bright soliton. By defining the quantities θ and $\phi_1^{(1)}$ as $\theta = \tan^{-1}(\alpha_{1I}^{(1)}/\alpha_{1R}^{(1)})$ and $\phi_1^{(1)} = \tan^{-1}[(k_{1I}-b_1)/k_{1R}]$, respectively, Eqs. (8a) and (8b) can be rewritten as

$$q_1 = \sqrt{k_{1R}^2 - |c_1|^2 \cos^2 \phi_1^{(1)}}\, e^{i\theta} \operatorname{sech}\left(k_{1R}(t - 2k_{1I}z)\right.$$
$$\left. + \frac{R}{2}\right) e^{ik_{1I}t + i(k_{1R}^2 - k_{1I}^2 - 2|c_1|^2)z}, \tag{9a}$$

$$q_2 = -c_1 e^{i\zeta_1}\left[\cos \phi_1^{(1)} \tanh\left(k_{1R}(t - 2k_{1I}z) + \frac{R}{2}\right) + i \sin \phi_1^{(1)}\right]. \tag{9b}$$

Now the condition (8f) becomes $k_{1R}^2 > |c_1|^2 \cos^2 \phi_1^{(1)}$. It can be inferred from Eq. (9a) that the intensity of the bright soliton increases with a decrease in the magnitude of the dark soliton parameter c_1 as shown in Fig. 1 (all of the quantities in Fig. 1 and the rest of the figures are dimensionless). This is a consequence of the particular type of cross-phase modulation coupling given in Eq. (6a) and (6b). Note that the α-parameter influences only the central position of the bright and dark solitons and not the intensities. However, it can be inferred from Eq. (9a) that the α parameters appear in the complex phase ($e^{i\theta}$) of the amplitude part.

(b) *Mixed two-soliton solution.* The mixed two-soliton solution can be obtained by terminating the power series expansion (5) as

$$g^{(1)} = \chi g_1^{(1)} + \chi^3 g_3^{(1)}, \tag{10a}$$

$$h^{(1)} = h_0^{(1)}(1 + \chi^2 h_2^{(1)} + \chi^4 h_4^{(1)}), \tag{10b}$$

$$f = 1 + \chi^2 f_2 + \chi^4 f_4. \tag{10c}$$

After solving the resulting bilinear equations recursively, the explicit two-soliton solution is obtained as

$$q_1 = \frac{1}{D}(\alpha_1^{(1)} e^{\eta_1} + \alpha_2^{(1)} e^{\eta_2} + e^{\eta_1 + \eta_1^* + \eta_2 + \delta_{11}} + e^{\eta_2 + \eta_2^* + \eta_1 + \delta_{21}}), \tag{11a}$$
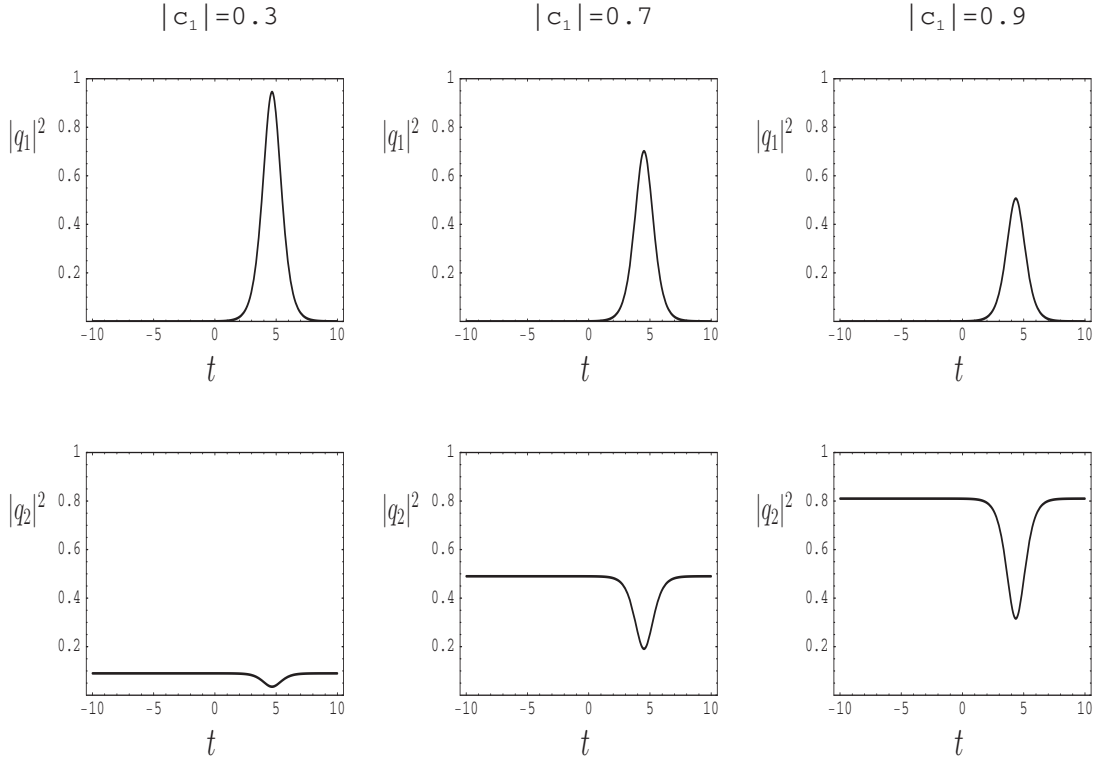
FIG. 1. Intensity plots of one-soliton of the mixed CNLS equations with $N=2$ for different values of the background parameter c_1 for a fixed value of z. Note that the intensity of the bright soliton increases as the depth of the dark soliton decreases. The other parameters are chosen as $k_1 = 1 + i$, $\alpha_1^{(1)} = 1$, $b_1 = 0.2$.

$$q_2 = \frac{1}{D}\left[c_1 e^{i\zeta_1}(1 + e^{\eta_1 + \eta_1^* + Q_{11}^{(1)}} + e^{\eta_1 + \eta_2^* + Q_{12}^{(1)}} + e^{\eta_2 + \eta_1^* + Q_{21}^{(1)}}\right.$$

$$\left. + e^{\eta_2 + \eta_2^* + Q_{22}^{(1)}} + e^{\eta_1 + \eta_1^* + \eta_2 + \eta_2^* + Q_3^{(1)}})\right], \qquad (11b)$$

where

$$D = 1 + e^{\eta_1 + \eta_1^* + R_1} + e^{\eta_1 + \eta_2^* + \delta_0} + e^{\eta_2 + \eta_1^* + \delta_0^*} + e^{\eta_2 + \eta_2^* + R_2}$$

$$+ e^{\eta_1 + \eta_1^* + \eta_2 + \eta_2^* + R_3} \qquad (11c)$$

and

$$\eta_j = k_j t + i(k_j^2 - 2|c_1|^2)z, \quad j = 1, 2. \qquad (11d)$$

In the above $\alpha_1^{(1)}$, $\alpha_2^{(1)}$, k_1, k_2, and c_1 are complex parameters and b_1 is a real parameter (see below). Introducing the quantity

$$\mu_{ij} = \frac{\alpha_i^{(1)} \alpha_j^{(1)*}}{(k_i + k_j^*)^2}\left(1 - \frac{|c_1|^2}{(k_i - ib_1)(k_j^* + ib_1)}\right)^{-1}, \quad i, j = 1, 2,$$

$$(12a)$$

the various other parameters in the expression (11) are defined as follows:

$$e^{R_1} = \mu_{11}, \quad e^{R_2} = \mu_{22}, \quad e^{\delta_0} = \mu_{12}, \quad e^{\delta_0^*} = \mu_{21}, \quad (12b)$$

$$e^{\delta_{11}} = \frac{(k_2 - k_1)^2}{\alpha_1^{(1)*}}\left(1 + \frac{|c_1|^2}{(k_1 - ib_1)(k_2 - ib_1)}\right)\mu_{11}\mu_{21},$$

$$(12c)$$

$$e^{\delta_{21}} = \frac{(k_2 - k_1)^2}{\alpha_2^{(1)*}}\left(1 + \frac{|c_1|^2}{(k_1 - ib_1)(k_2 - ib_1)}\right)\mu_{22}\mu_{12},$$

$$(12d)$$

$$e^{Q_{ij}^{(1)}} = -\frac{(k_i - ib_1)}{(k_j^* + ib_1)}\mu_{ij}, \quad i, j = 1, 2,$$

$$e^{Q_3^{(1)}} = \left(\frac{(k_1 - ib_1)(k_2 - ib_1)}{(k_1^* + ib_1)(k_2^* + ib_1)}\right)e^{R_3}, \qquad (12e)$$

and

$$e^{R_3} = \left(\frac{\mu_{11}\mu_{12}\mu_{21}\mu_{22}}{|\alpha_1^{(1)}\alpha_2^{(1)}|^2}\right)|k_1 - k_2|^2 \left|1 + \frac{|c_1|^2}{(k_1 - ib_1)(k_2 - ib_1)}\right|^2.$$

$$(12f)$$

The above two-soliton solution (11) is restricted by the conditions $k_{jR}^2 + (k_{jI} - b_1)^2 > |c_1|^2$, $j = 1, 2$, as in the case of one-soliton solution. The two-soliton solution (11) is characterized by 11 real parameters $\alpha_{1R}^{(1)}$, $\alpha_{2R}^{(1)}$, $\alpha_{1I}^{(1)}$, $\alpha_{2I}^{(1)}$, k_{1R}, k_{1I}, k_{2R}, k_{2I}, c_{1R}, c_{1I}, and b_1.

B. Bright-dark soliton solutions of the mixed 3-CNLS equations

Let us consider the set of mixed 3-CNLS equations which corresponds to Eq. (1) with $N=3$. In its explicit form the set of mixed 3-CNLS equations reads as

$$iq_{1,z} + q_{1,tt} + 2(\sigma_1|q_1|^2 + \sigma_2|q_2|^2 + \sigma_3|q_3|^2)q_1 = 0,$$
(13a)

$$iq_{2,z} + q_{2,tt} + 2(\sigma_1|q_1|^2 + \sigma_2|q_2|^2 + \sigma_3|q_3|^2)q_2 = 0,$$
(13b)

$$iq_{3,z} + q_{3,tt} + 2(\sigma_1|q_1|^2 + \sigma_2|q_2|^2 + \sigma_3|q_3|^2)q_3 = 0. \quad (13c)$$

In the above, the nonlinearity coefficients σ_j's, $j=1,2,3$ take the values either $\sigma_1=\sigma_2=+1$, $\sigma_3=-1$ or $\sigma_1=+1$, and $\sigma_2=\sigma_3=-1$. In Eq. (13), in the context of BECs the components q_1, q_2, and q_3 either denote the condensates of three isotopes of the same element (for example, Rb isotopes) [25] or the hyperfine spin states of spinor BECs [27]. This set of equations admits the following two distinct types of mixed soliton solutions.

(i) Two-bright–one-dark soliton solution: In this case the bright solitons are separated out into two of the three components and the dark soliton appears in the remaining component (with the choices $\sigma_1=\sigma_2=1$ and $\sigma_3=-1$).

(ii) One-bright–two-dark soliton solution: In this type of solution the dark solitons appear in two components and the remaining component comprises of the bright soliton (with the choices $\sigma_1=1$ and $\sigma_2=\sigma_3=-1$).

The procedure of obtaining these soliton solutions is similar to that of the mixed 2-CNLS equations.

1. Two-bright–one-dark soliton solution

We present below the explicit forms of the obtained one- and two-soliton solutions.

(a) *Mixed one-soliton solution.* The mixed one-soliton solution, in which two bright solitons appear in the first two components and the dark one appears in the third component, is obtained by Hirota's method as

$$q_j = \frac{\alpha_1^{(j)}e^{\eta_1}}{1 + e^{\eta_1+\eta_1^*+R}}, \quad j=1,2,$$
(14a)

that is

$$q_j = A_j k_{1R} e^{i\eta_{1I}} \operatorname{sech}\left(\eta_{1R} + \frac{R}{2}\right),$$
(14b)

$$q_3 = \frac{c_1 e^{i\zeta_1}\left[1 - \left(\dfrac{k_1-ib_1}{k_1^*+ib_1}\right)e^{\eta_1+\eta_1^*+R}\right]}{1 + e^{\eta_1+\eta_1^*+R}},$$
(14c)

where

$$e^R = \frac{\displaystyle\sum_{j=1}^{2}(\alpha_1^{(j)}\alpha_1^{(j)*})}{(k_1+k_1^*)^2}\left(1 - \frac{|c_1|^2}{|k_1-ib_1|^2}\right)^{-1},$$
(14d)

$$A_j = \left(\frac{\alpha_1^{(j)}}{2k_{1R}}\right)e^{-R/2}$$

$$\equiv \frac{\sqrt{k_{1R}^2 - |c_1|^2\cos^2\phi_1^{(1)}}}{k_{1R}}\left(\frac{\alpha_1^{(j)}}{\sqrt{(|\alpha_1^{(1)}|^2 + |\alpha_1^{(2)}|^2)}}\right), \quad j=1,2.$$
(14e)

Here the quantities η_1 and ζ_1 are as defined in Eqs. (8d) and (8e), respectively. A_j's defined through the $\alpha^{(j)}$ and c_1 parameters represent the polarization of the bright components. For the dark component, the parameter b_1 denotes the direction of the background and c_1 gives its amplitude. The one-soliton solution is characterized by nine real parameters $\alpha_{1R}^{(1)}$, $\alpha_{1I}^{(1)}$, $\alpha_{1R}^{(2)}$, $\alpha_{1I}^{(2)}$, k_{1R}, k_{1I}, c_{1R}, c_{1I}, and b_1 and is restricted by the condition $|c_1|^2 < |k_1-ib_1|^2$. Now the role of α parameters can be realized explicitly in the amplitude (intensity) of bright components and also through the nontrivial phase of all the components. This is shown in Fig. 2. In fact this has important consequences in the collision process as will be illustrated in the following sections. Thus the dark soliton part influences the bright part through the parameters c_1 and b_1 whereas the bright solitons influence the dark soliton phase (central position) through the α parameters.

(b) *Two-soliton solution.* Following Hirota's bilinearization method as in the case of $N=2$, here we obtain the two-soliton solution as

$$q_j = \frac{1}{D}(\alpha_1^{(j)}e^{\eta_1} + \alpha_2^{(j)}e^{\eta_2} + e^{\eta_1+\eta_1^*+\eta_2+\delta_{1j}} + e^{\eta_2+\eta_2^*+\eta_1+\delta_{2j}}),$$

$$j=1,2,$$
(15a)

$$q_3 = \frac{1}{D}[c_1 e^{i\zeta_1}(1 + e^{\eta_1+\eta_1^*+Q_{11}^{(1)}} + e^{\eta_1+\eta_2^*+Q_{12}^{(1)}} + e^{\eta_2+\eta_1^*+Q_{21}^{(1)}}$$

$$+ e^{\eta_2+\eta_2^*+Q_{22}^{(1)}} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+Q_3^{(1)}})],$$
(15b)

$$e^{\delta_{1j}} = (k_2-k_1)\mu_{11}\mu_{21}(\alpha_2^{(j)}\chi_{21} - \alpha_1^{(j)}\chi_{11}),$$
(15c)

$$e^{\delta_{2j}} = (k_2-k_1)\mu_{12}\mu_{22}(\alpha_2^{(j)}\chi_{22} - \alpha_1^{(j)}\chi_{12}),$$
(15d)

$$e^{R_3} = |k_1-k_2|^2\mu_{11}\mu_{12}\mu_{21}\mu_{22}(\chi_{12}\chi_{21} - \chi_{11}\chi_{22}), \quad j=1,2,$$
(15e)

where μ_{il}'s are now redefined as

$$\mu_{il} = \frac{1}{(k_i+k_l^*)\chi_{il}},$$
(15f)

$$\chi_{il} = \frac{(k_i+k_l^*)}{\displaystyle\sum_{j=1}^{2}(\alpha_i^{(j)}\alpha_l^{(j)*})}\left(1 - \frac{|c_1|^2}{(k_i-ib_1)(k_l^*+ib_1)}\right), \quad i,l=1,2.$$
(15g)

The form of D is given as in Eq. (11c) and the expressions for $e^{Q_{11}^{(1)}}$, $e^{Q_{12}^{(1)}}$, $e^{Q_{21}^{(1)}}$, $e^{Q_{22}^{(1)}}$, and $e^{Q_3^{(1)}}$ take the form as given in Eq. (12e) with the above redefinition of the μ_{il}'s. $\alpha_{iR}^{(j)}$, $\alpha_{iI}^{(j)}$,
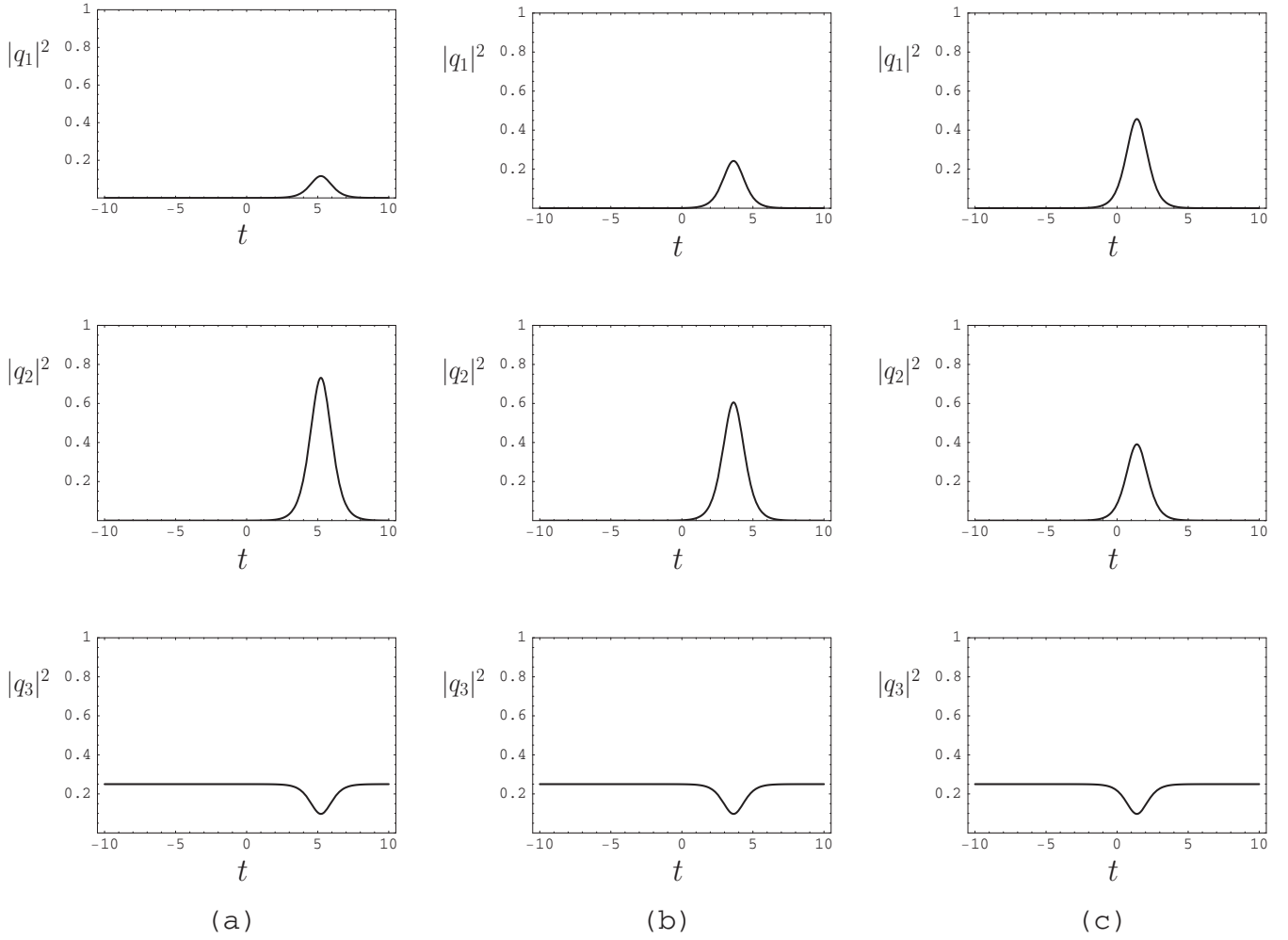
FIG. 2. Intensity plots of one-soliton of the mixed CNLS equations with $N=3$ for different values of the α parameters for a fixed value of z. (a) $\alpha_1^{(1)}=0.2+0.01i$, $\alpha_1^{(2)}=0.5+0.05i$, (b) $\alpha_1^{(1)}=1+i$, $\alpha_1^{(2)}=2+i$, and (c) $\alpha_1^{(1)}=13-13i$, $\alpha_1^{(2)}=17+0.3i$. The role of α parameters are shown in both the intensity and phase of the bright soliton while it affects the phase (central position) of the dark soliton. The parameters c_1, b_1, and k_1 are chosen as $|c_1|=0.56$, $b_1=0.2$, $k_1=1+i$.

k_{iR}, k_{iI}, $i,j=1,2$, c_{1R}, and b_1 are the 15 real parameters which characterize the above solution. The nature of this two-soliton solution will be discussed in Sec. IV B.

2. One-bright–two-dark soliton solution

Next we consider the case where the bright soliton appears in the q_1 component and the two dark solitons are found in the remaining two components (q_2,q_3). This gives us the possibility of introducing two background fields $c_1e^{i\zeta_1}$ and $c_2e^{i\zeta_2}$.

(a) *Mixed one-soliton solution.* The corresponding one-soliton solution obtained by using Hirota's method is

$$q_1 = \frac{\alpha_1^{(1)}e^{\eta_1}}{1+e^{\eta_1+\eta_1^*+R}}, \tag{16a}$$

that is

$$q_1 = A_1k_{1R}e^{i\eta_{1I}}\,\mathrm{sech}\!\left[\eta_{1R}+\frac{R}{2}\right], \tag{16b}$$

$$q_{l+1} = \frac{c_le^{i\zeta_l}\!\left[1-\left(\dfrac{k_1-ib_l}{k_1^*+ib_l}\right)e^{\eta_1+\eta_1^*+R}\right]}{1+e^{\eta_1+\eta_1^*+R}}, \quad l=1,2, \tag{16c}$$

where

$$\eta_1 = k_1t+i(k_1^2-\lambda)z, \quad \zeta_l=-(b_l^2+\lambda)z+b_lt,$$
$$\lambda = 2(|c_1|^2+|c_2|^2), \quad l=1,2.$$

$$e^R = \frac{|\alpha_1^{(1)}|^2}{(k_1+k_1^*)^2}\left(1-\frac{|c_1|^2}{|k_1-ib_1|^2}-\frac{|c_2|^2}{|k_1-ib_2|^2}\right)^{-1}, \tag{16d}$$

$$A_1 = \left(\frac{\alpha_1^{(1)}}{2k_{1R}}\right)e^{-R/2}. \tag{16e}$$

Now the one-soliton solution is characterized by 10 real parameters, $\alpha_{1R}^{(1)}$, $\alpha_{1I}^{(1)}$, k_{1R}, k_{1I}, c_{1R}, c_{1I}, c_{2R}, c_{2I}, b_1, and b_2.

(b) *Two-soliton solution.* As in the preceding section, here also we obtain the two-soliton solution as

$$q_1 = \frac{1}{D}(\alpha_1^{(1)}e^{\eta_1} + \alpha_2^{(1)}e^{\eta_2} + e^{\eta_1+\eta_1^*+\eta_2+\delta_{11}} + e^{\eta_2+\eta_2^*+\eta_1+\delta_{21}}),$$

(17a)

$$q_{l+1} = \frac{1}{D}[c_l e^{i\zeta_l}(1 + e^{\eta_1+\eta_1^*+Q_{11}^{(l)}} + e^{\eta_1+\eta_2^*+Q_{12}^{(l)}} + e^{\eta_2+\eta_1^*+Q_{21}^{(l)}}$$

$$+ e^{\eta_2+\eta_2^*+Q_{22}^{(l)}} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+Q_3^{(l)}})], \quad l = 1,2, \quad (17b)$$

where

$$e^{\delta_{11}} = (k_2 - k_1)^2 \frac{\mu_{11}\mu_{21}}{\alpha_1^{(1)*}}\rho, \quad e^{\delta_{21}} = (k_2 - k_1)^2 \frac{\mu_{12}\mu_{22}}{\alpha_2^{(1)*}}\rho.$$

(17c)

In Eqs. (17a) and (17b), the form of D, $e^{Q_{ij}^{(l)}}$, and $e^{Q_3^{(l)}}$ are the same as in Eqs. (11c) and (12e) with $i,j,l=1,2$. Also, the quantities e^{R_3}, μ_{il}, and ρ are redefined as

$$e^{R_3} = |k_1 - k_2|^2 \frac{\mu_{11}\mu_{12}\mu_{21}\mu_{22}}{|\alpha_1^{(1)}\alpha_2^{(1)}|^2}|\rho|^2, \quad (17d)$$

$$\mu_{il} = \frac{\alpha_i^{(1)}\alpha_l^{(1)*}}{(k_i + k_l^*)^2}\left(1 - \sum_{v=1}^{2}\frac{|c_v|^2}{(k_i - ib_v)(k_l^* + ib_v)}\right)^{-1}, \quad i,l = 1,2,$$

(17e)

and

$$\rho = \left(1 + \sum_{v=1}^{2}\frac{|c_v|^2}{(k_1 - ib_v)(k_2 - ib_v)}\right). \quad (17f)$$

The two-soliton solution is characterized by 14 real parameters. Again we will study the nature of this solution in Sec. IV B.

C. *N*-soliton solutions

The above procedure of obtaining soliton solutions can be extended to three- and *N*-soliton solutions with some effort, though the analysis is cumbersome. In this work, we restrict our analysis to the two-soliton solution only as the *N*-soliton collisions represented by *N*-soliton solution in general take place pairwise in soliton theory. Work is in progress in this direction and the results will be presented separately.

D. Bright-dark soliton solutions of the mixed *N*-CNLS case

After obtaining the two- and three-component mixed-soliton solutions, the next natural step is to generalize the results to the arbitrary *N*-component case, where $N=m+n$. For this purpose, we consider the case where the bright solitons appear in the first *m* components and the dark solitons appear in the remaining $n[\equiv(N-m)]$ components. So the resulting mixed-soliton solution can be denoted as *m*-bright–*n*-dark-type soliton solution, as pointed out in Sec. II.

m-bright–*n*-dark soliton solution

(a) *One-soliton solution.* The mixed one-soliton solution of the mixed *N*-CNLS case is found as

$$q_j = \frac{\alpha_1^{(j)}e^{\eta_1}}{1 + e^{\eta_1+\eta_1^*+R}}, \quad (18a)$$

that is

$$q_j = A_j k_{1R} e^{i\eta_{1I}} \operatorname{sech}\left(\eta_{1R} + \frac{R}{2}\right), \quad j = 1,2,\ldots,m, \quad (18b)$$

$$q_{l+m} = \frac{c_l e^{i\zeta_l}\left[1 - \left(\frac{k_1 - ib_l}{k_1^* + ib_l}\right)e^{\eta_1+\eta_1^*+R}\right]}{1 + e^{\eta_1+\eta_1^*+R}}, \quad l = 1,2,\ldots,n,$$

(18c)

where

$$e^R = \frac{\sum_{j=1}^{m}(\alpha_1^{(j)}\alpha_1^{(j)*})}{(k_1 + k_1^*)^2}\left(1 - \sum_{l=1}^{n}\frac{|c_l|^2}{|k_1 - ib_l|^2}\right)^{-1}, \quad (18d)$$

$$A_j = \left(\frac{\alpha_1^{(j)}}{2k_{1R}}\right)e^{-R/2}. \quad (18e)$$

Here

$$\eta_j = k_j t + i(k_j^2 - \lambda)z, \quad \zeta_l = -(b_l^2 + \lambda)z + b_l t,$$

$$\lambda = 2\sum_{l=1}^{n}|c_l|^2, \quad j = 1,2,\ldots,m \text{ and } l = 1,2,\ldots,n.$$

(18f)

The one-soliton solution is characterized by $(2m+3n+2)$ number of real parameters, $\alpha_{1R}^{(j)}, \alpha_{1I}^{(j)}, k_{1R}, k_{1I}, c_{lR}, c_{lI}, b_l$, $j=1,2,\ldots,m$, $l=1,2,\ldots,n$ with the condition $|c_l|^2 < |k_1 - ib_l|^2$, $l=1,2,\ldots,n$.

(b) *Two-soliton solution.* Generalization of the mixed two-soliton solution presented in the preceding sections for $N=2$ and $N=3$ cases yields the following *m*-bright–*n*-dark two-soliton solution of Eq. (1) with arbitrary *N*:

$$q_j = \frac{1}{D}(\alpha_1^{(j)}e^{\eta_1} + \alpha_2^{(j)}e^{\eta_2} + e^{\eta_1+\eta_1^*+\eta_2+\delta_{1j}} + e^{\eta_2+\eta_2^*+\eta_1+\delta_{2j}}),$$

$$j = 1,2,\ldots,m, \quad (19a)$$

$$q_{l+m} = \frac{1}{D}[c_l e^{i\zeta_l}(1 + e^{\eta_1+\eta_1^*+Q_{11}^{(l)}} + e^{\eta_1+\eta_2^*+Q_{12}^{(l)}} + e^{\eta_2+\eta_1^*+Q_{21}^{(l)}}$$

$$+ e^{\eta_2+\eta_2^*+Q_{22}^{(l)}} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+Q_3^{(l)}})], \quad l = 1,2,\ldots,n.$$

(19b)

Here the denominator D is given by Eq. (11c). The quantities e^{R_1}, e^{R_2}, e^{δ_0}, and $e^{\delta_0^*}$ are as defined in Eq. (12b) but with the following redefinitions:

$$e^{Q_{ij}^{(l)}} = -\frac{(k_i - ib_l)}{(k_j^* + ib_l)}\mu_{ij}, \quad i,j = 1,2,$$

$$e^{Q_3^{(l)}} = \left(\frac{(k_1 - ib_l)(k_2 - ib_l)}{(k_1^* + ib_l)(k_2^* + ib_l)}\right)e^{R_3}, \quad (19c)$$

$$e^{\delta_{1j}} = (k_2 - k_1)\mu_{11}\mu_{21}(\alpha_2^{(j)}\chi_{21} - \alpha_1^{(j)}\chi_{11}), \quad j = 1,2,\ldots,m,$$

$$e^{\delta_{2j}} = (k_2 - k_1)\mu_{12}\mu_{22}(\alpha_2^{(j)}\chi_{22} - \alpha_1^{(j)}\chi_{12}),$$

$$e^{R_3} = |k_1 - k_2|^2 \mu_{11}\mu_{12}\mu_{21}\mu_{22}[\chi_{12}\chi_{21} - \chi_{11}\chi_{22}], \quad (19d)$$

$$\mu_{ip} = \frac{1}{(k_i + k_p^*)\chi_{ip}},$$

and

$$\chi_{ip} = \frac{(k_i + k_p^*)}{\sum\limits_{j=1}^{m}(\alpha_i^{(j)}\alpha_p^{(j)*})}\left(1 - \sum\limits_{l=1}^{n}\frac{|c_l|^2}{(k_i - ib_l)(k_p^* + ib_l)}\right), \quad i,p = 1,2.$$

$$(19e)$$

The number of real parameters which characterize the two-soliton solution is $(4m+3n+4)$.

IV. SHAPE-CHANGING COLLISIONS OF MIXED SOLITONS

The fascinating property of the bright solitons of the integrable N-CNLS system with focusing-type nonlinearity is that they exhibit shape-changing collisions characterized by intensity redistribution, amplitude-dependent phase shift, and relative separation distances [12,13,16], which can then be used to construct collision-based logic gates for optical computation [28,29]. In this and in the following sections we analyze such collision dynamics of bright solitons in the presence of dark solitons in the mixed 2-CNLS and 3-CNLS equations and also its effect on the propagation and collision of dark solitons. In this regard, we perform an asymptotic analysis of the two-soliton solutions for mixed 2-CNLS and 3-CNLS equations.

A. Asymptotic analysis of mixed two-soliton solution of mixed 2-CNLS equations

To start with, we consider the collision properties associated with the mixed two-soliton solution (11) of the mixed 2-CNLS equations (6a) and (6b). Following this we carry out the analysis for mixed 3-CNLS equations. Without loss of generality, we take $k_{jR} > 0$ and $k_{1I} > k_{2I}$, $k_j = k_{jR} + ik_{jI}$, $j = 1,2$, and obtain the asymptotic forms of two colliding solitons (say S_1 and S_2). Similar analysis can be carried out for other choices of k_{jR} and k_{jI} also. Using the expression (11) for the bright-dark two-soliton solution of the mixed 2-CNLS system (6a) and (6b), we carry out a detailed asymptotic
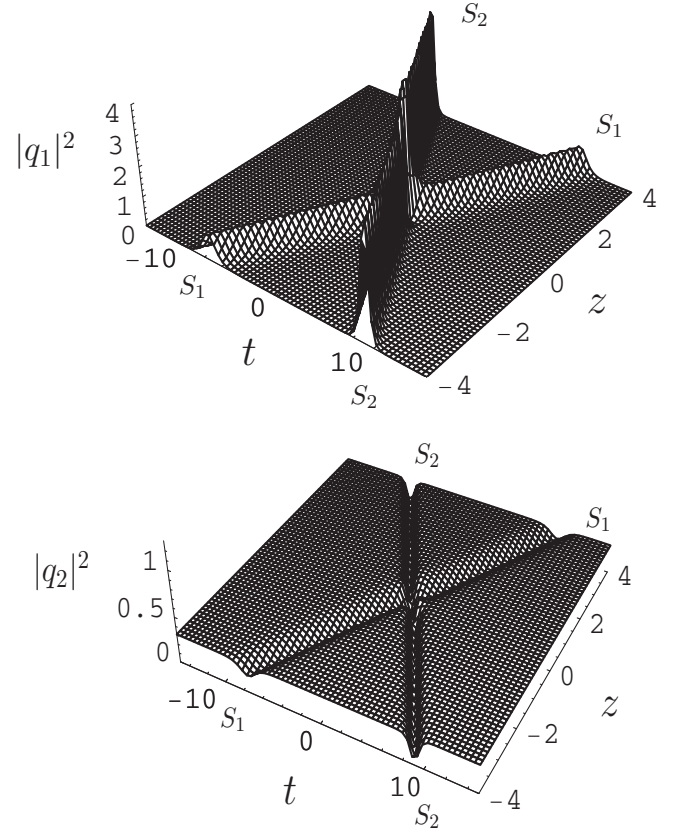

FIG. 3. Elastic collision of (bright-dark) two solitons in the mixed CNLS system for the $N=2$ case. The parameters are chosen as given in the text.

analysis in Appendix A for the two-soliton collision process. Based on this analysis we identify the following.

1. Role of dark soliton on bright soliton collision

The amplitudes of the two solitons S_1 and S_2 before (after) interaction are given by $A_1^{1-}k_{1R}(A_1^{1+}k_{1R})$ and $A_1^{2-}k_{2R}(A_1^{2+}k_{2R})$, respectively, in the q_1 component. Forms of $A_1^{j\pm}$, $j=1,2$, are given in Appendix A [see Eqs. (A1c), (A2c), (A3c), and (A4c)]. By rewriting these forms one can show that the intensities of solitons before and after interaction are the same (elastic), i.e., $|A_1^{j-}| = |A_1^{j+}|$, $j=1,2$, even though the complex amplitudes differ in phase. Also, the appearance of the background parameter c_1 in the expression for the bright soliton amplitudes before and after collision [see Eqs. (A1c), (A2c), (A3c), and (A4c)] shows that this parameter influences the bright soliton amplitudes throughout the collision.

Typical bright-dark soliton collision in the mixed 2-CNLS system is shown in Fig. 3 for the parametric choices $k_1 = 1 + i$, $k_2 = 2 - i$, $|c_1| = 0.56$, $b_1 = 0.2$, $\alpha_1^{(1)} = 1$, $\alpha_2^{(1)} = 1 + i$. For better understanding, first we plot the intensity profiles showing the bright soliton elastic collision scenario in the absence of dark soliton parameter ($c_1 = 0$), at $z = -4$ and $z = 4$. This is shown in Fig. 4(a). Then the same bright soliton collision is plotted for the above parametric choices as in Fig. 3 at $z = -4$ and $z = 4$, that is in the presence of the dark component, in Fig. 4(b). From these two figures, we observe that due to the presence of dark soliton the amplitudes of the colliding
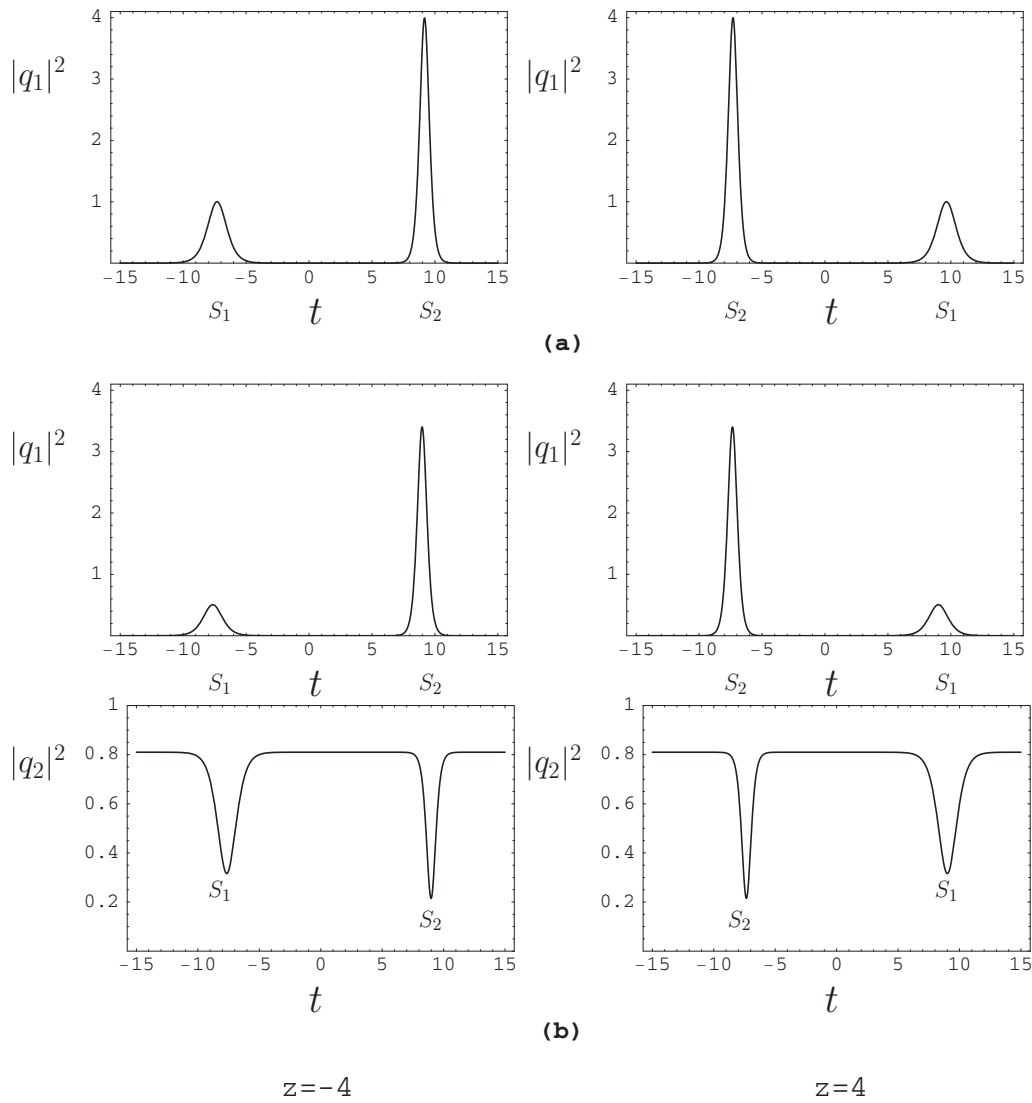
FIG. 4. Intensity profiles of two colliding bright solitons of the mixed CNLS equations with $N=2$, before ($z=-4$) and after ($z=4$) collision: (a) in the absence of dark component [$c_1=0$ in Eq. (11)]; (b) in the presence of dark component [$c_1 \neq 0$ in Eq. (11)]. The figure is plotted for a special choice of parameters (as given in the text) with $|c_1|=0.56$. Note the elastic nature of the collision.

bright solitons are reduced by the same amount throughout the collision.

The two colliding solitons S_1 and S_2 suffer phase shifts Φ_1 and Φ_2, respectively, in both the bright and dark components. These phase shifts for the bright and dark solitons are given by the expression

$$\Phi_1 = -\Phi_2 = \left(\frac{R_3 - R_2 - R_1}{2} \right)$$

$$= \frac{1}{2} \ln \left(\frac{|k_1 - k_2|^2}{(k_1 + k_2^*)^2 (k_2 + k_1^*)^2} \frac{\left(1 + \frac{|c_1|^2}{P_1 P_2}\right)\left(1 + \frac{|c_1|^2}{P_1^* P_2^*}\right)}{\left(1 - \frac{|c_1|^2}{P_2 P_1^*}\right)\left(1 - \frac{|c_1|^2}{P_1 P_2^*}\right)} \right),$$

(20a)

where $P_1 = k_1 - ib_1$ and $P_2 = k_2 - ib_1$. Note that the phase shifts appearing here [Eq. (20a)] and in the following [see Eqs.

(23a) and (26)] are real quantities as the terms appearing in the argument of the logarithmic function are products of complex conjugates. The role of dark component comes into picture through this phase shift due to the explicit appearance of the background parameters (c_1, b_1). Notice that in the absence of the dark component ($c_1=0$), the phase shift Φ_1 reduces to the standard phase shift experienced by colliding solitons in scalar nonlinear Schrödinger equations (NLS). Also, it is important to notice that the α parameters have no effect on the phase shift in the mixed 2-CNLS case. However, this is not true in the mixed N-CNLS case for more than two components ($N > 2$) as will be shown in the following sections. The relative separation distances between the two colliding solitons before interaction

$$t_{12}^- = \frac{R_3 k_{1R} - R_1 (k_{1R} + k_{2R})}{2 k_{1R} k_{2R}}$$

(20b)

and after interaction
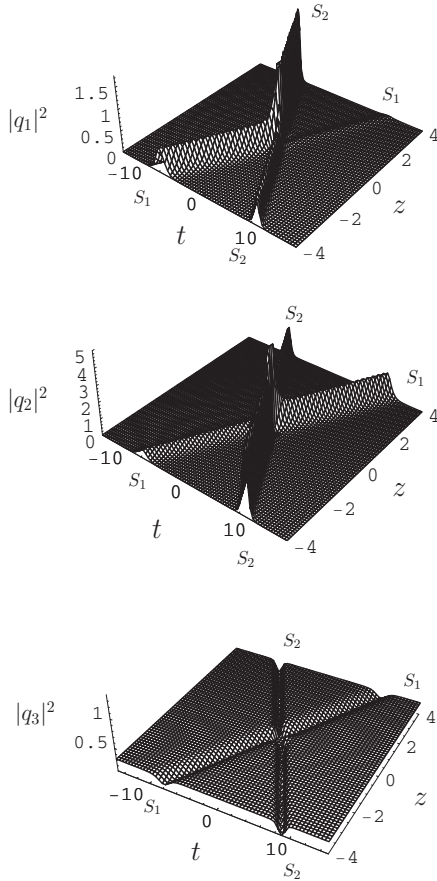
FIG. 5. Shape-changing collision of two solitons in the mixed CNLS system for the $N=3$ case. The parameters are as given in the text.
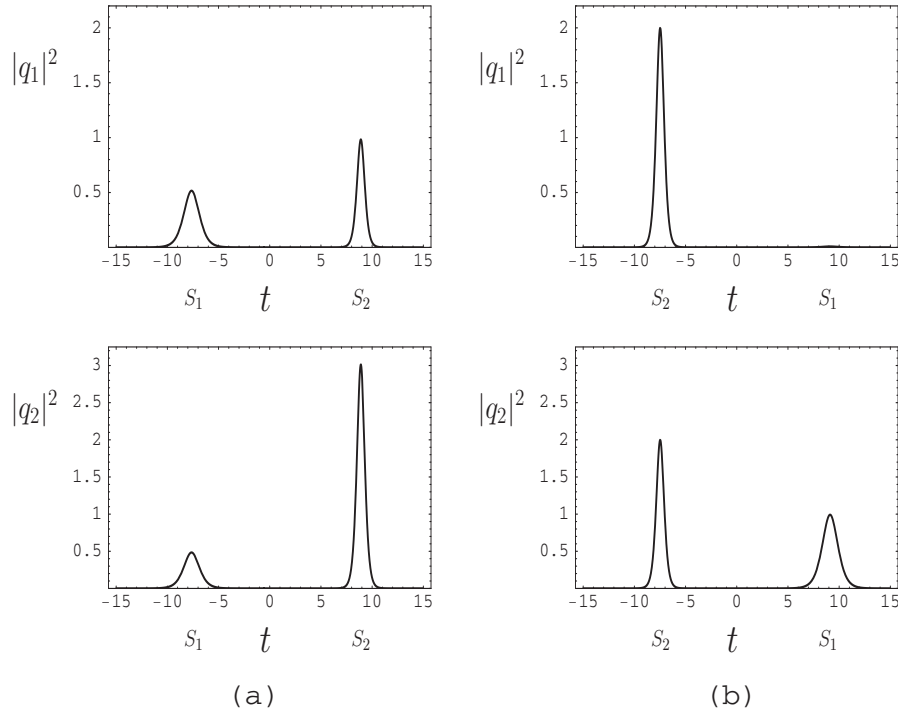
$$t_{12}^{+} = \frac{R_2(k_{1R} + k_{2R}) - R_3 k_{2R}}{2k_{1R}k_{2R}} \qquad (20c)$$

also remain unaffected by α parameters.

2. Role of bright soliton on dark soliton collision

The asymptotic expressions [Eqs. (A1b), (A2b), (A3b), and (A4b)] reveal the fact that the collision among dark solitons is elastic as the intensities of colliding solitons remain the same before and after interaction. From Fig. 3 it can be observed that the bright-component parameter α has no influence at all either in the amplitude or in the phase shift of the dark soliton during collision. Thus, the inclusion of α parameters in the bright soliton solution [see Eq. (11)] does not affect dark soliton collisions.

B. Asymptotic analysis of mixed two-soliton solution of mixed 3-CNLS equations

The next natural step is to study the collision process in the mixed 3-CNLS equations and one can generalize the results to the N-CNLS equations with arbitrary N. The asymptotic expressions of the solitons corresponding to the $N=3$ case are presented in Appendix B. This 3-CNLS system admits two distinct types of solution as mentioned in Sec. III B. First, let us consider the two-bright–one-dark soliton collision and then the one-bright–two-dark soliton collision.

1. Two-bright–one-dark soliton collision

We analyze two important physical quantities, namely (i) intensity and (ii) phase shift of both bright and dark solitons.
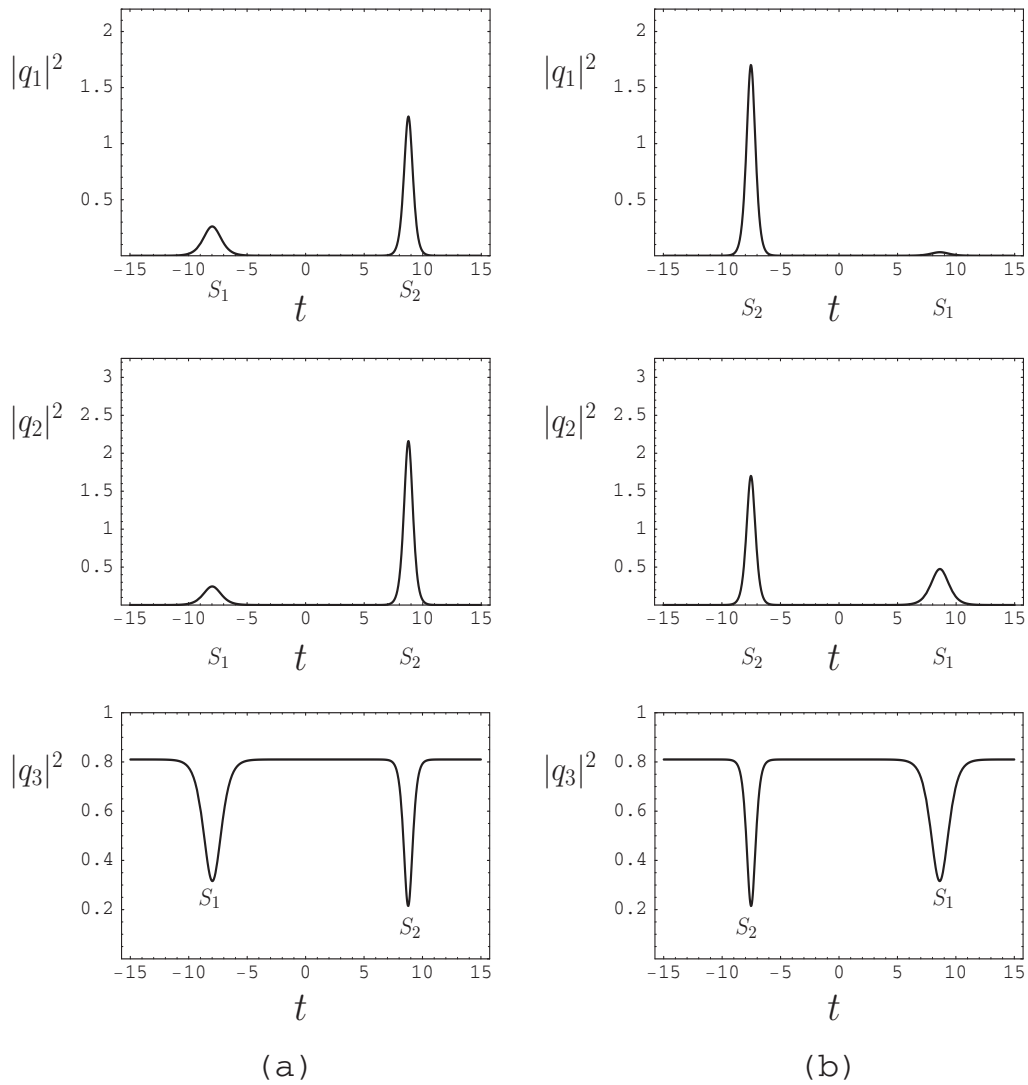

FIG. 6. Intensity profiles showing the collision scenario of two bright solitons of the mixed CNLS equations with $N=3$ in the absence of dark component, $(c_1=0)$ with $\alpha_1^{(1)}/\alpha_2^{(1)} \neq \alpha_1^{(2)}/\alpha_2^{(2)}$. (a) $z=-4$ and (b) $z=4$, given by the specific choice of parameters as given in the text.

FIG. 7. Intensity profiles showing the collision scenario of two bright solitons of the mixed CNLS equations with $N=3$ in the presence of dark component ($c_1 \neq 0$) with $\alpha_1^{(1)}/\alpha_2^{(1)} \neq \alpha_1^{(2)}/\alpha_2^{(2)}$. (a) $z=-4$ and (b) $z=4$, given by the specific choice of parameters as given in the text.

(i) Intensities of bright and dark solitons: Analyzing Eqs. (B1c), (B2c), (B3c), and (B4c), we find that the amplitudes (intensities) of bright solitons before and after interaction aredifferent. In fact, the intensities of the colliding bright solitons before and after interactions can be related through the expression

$$|A_j^{l+}|^2 = |T_j^l|^2 |A_j^{l-}|^2, \quad j,l = 1,2, \tag{21}$$

where the superscripts $l\pm$ represent the solitons designated as S_1 and S_2, at $z \to \pm\infty$. The transition intensities are identified from Appendix B as

$$|T_j^1|^2 = \frac{|1 - \kappa_2(\alpha_2^{(j)}/\alpha_1^{(j)})|^2}{|1 - \kappa_1\kappa_2|}, \tag{22a}$$

$$|T_j^2|^2 = \frac{|1 - \kappa_1\kappa_2|}{|1 - \kappa_1(\alpha_2^{(j)}/\alpha_1^{(j)})|^2}, \quad j = 1,2, \tag{22b}$$

$$\kappa_1 = \frac{\chi_{11}}{\chi_{21}}, \quad \kappa_2 = \frac{\chi_{22}}{\chi_{12}}, \tag{22c}$$

$$\chi_{il} = \frac{(k_i + k_l^*)}{2 \sum_{j=1}^{2}(\alpha_i^{(j)}\alpha_l^{(j)*})} \left(1 - \frac{|c_1|^2}{(k_i - ib_1)(k_l^* + ib_1)}\right), \quad i,l = 1,2. \tag{22d}$$

Note that the transition amplitudes T_j^l's now are also functions of dark soliton parameters c_1 and b_1. On the other hand, from Eqs. (B1b), (B2b), (B3b), and (B4b) we find that the intensities of dark solitons remain unchanged due to the collision process.

(ii) Phase shift of bright and dark solitons: The amplitude-dependent phase shift $\Phi_1 (= \frac{R_3 - R_2 - R_1}{2})$ for soliton S_1 can be expressed in the present case as

$$\Phi_1 = \frac{1}{2}\ln\left\{\frac{|k_1 - k_2|^2}{(k_1 + k_2^*)(k_2 + k_1^*)}\left[1\right.\right.$$

$$\left.\left.- U\left(\frac{(k_1 + k_1^*)(k_2 + k_2^*)}{(k_1 + k_2^*)(k_2 + k_1^*)}\frac{\left(1 - \dfrac{|c_1|^2}{P_1 P_1^*}\right)\left(1 - \dfrac{|c_1|^2}{P_2 P_2^*}\right)}{\left(1 - \dfrac{|c_1|^2}{P_1 P_2^*}\right)\left(1 - \dfrac{|c_1|^2}{P_2 P_1^*}\right)}\right)\right]\right\},$$

$$\text{(23a)}$$

where

$$U = \frac{(\alpha_1^{(1)}\alpha_2^{(1)*} + \alpha_1^{(2)}\alpha_2^{(2)*})(\alpha_2^{(1)}\alpha_1^{(1)*} + \alpha_2^{(2)}\alpha_1^{(2)*})}{(|\alpha_1^{(1)}|^2 + |\alpha_1^{(2)}|^2)(|\alpha_2^{(1)}|^2 + |\alpha_2^{(2)}|^2)}. \quad \text{(23b)}$$

Note that the soliton S_2 experiences an exactly opposite phase shift $\Phi_2(=-\Phi_1)$. The above phase shift in turn results in the following change in relative separation distance between the solitons before and after collision:

$$\Delta t_{12} = t_{12}^- - t_{12}^+ = \frac{(k_{1R} + k_{2R})}{k_{1R}k_{2R}}\Phi_1. \quad \text{(24)}$$

The expression for Φ_1 given in Eq. (23a) clearly indicates that now the phase shift depends on α parameters as well as c_1. Thus, the phase shift and ultimately the relative separation distance between the solitons can be altered during a two-soliton collision process for a given combination of k_j, c_1, and b_1, by just varying the α parameters and as a whole the combined soliton profile gets altered.

This kind of collision scenario is shown in Fig. 5. The corresponding intensity plots show that the bright solitons undergo shape-changing collisions characterized by intensity redistribution, amplitude-dependent phase shift, and relative separation distance for the parametric choice $k_1 = 1 + i, k_2 = 2 - i$, $b_1 = 0.2$, $|c_1| = 0.56$, $\alpha_1^{(1)} = 1$, $\alpha_1^{(2)} = (32 + i80)/89$, $\alpha_2^{(1)} = 1$, $\alpha_2^{(2)} = 1$. The solitons (say S_1 and S_2) are well separated before and after collision in both the components q_1 and q_2. In the q_1 component the intensity of soliton S_1 gets suppressed while that of soliton S_2 is enhanced after interaction, whereas in the q_2 component it gets reversed. Such shape-changing collisions occur for $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} \neq \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$, which is quite general. But when we choose $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} = \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$, the two solitons exhibit elastic collision. It is instructive to note that although the dark solitons appear in the q_3 component, they indirectly influence the shape-changing collisions through the carrier wave background parameters c_1 and b_1. From the asymptotic analysis it follows that these background parameters influence the intensities of the colliding bright solitons before and after collision by different amounts through their explicit appearance in the transition intensities [see Eq. (22)]. However, the nature of the collision is unaltered. We present below a detailed discussion to obtain a clear picture about the influence of dark solitons on bright soliton collision and vice versa.

(a) *Effect of dark soliton on the intensity of bright soliton.* For a better understanding, we present the shape-changing collision of bright solitons (i) in the absence of dark component, that is $c_1 = 0$, in Fig. 6 and (ii) in the presence of dark component, that is $c_1 \neq 0$, in Fig. 7 for the above-mentioned parametric choices. From these figures we observe that, in the presence of dark component, the intensities of solitons S_1 in the q_1 component is decreased (increased) before (after) collision (as compared with Fig. 6), but not by the same amount. Thus, the effect of the dark component on the intensity of the bright soliton before its collision is different from the effect on the intensity after the collision. But in the q_2 component there occurs decrement of intensity in both the solitons before and after collision due to the presence of dark soliton. This confirms the fact that the presence of dark solitons indeed influences the cross-phase coupling between the two components q_1 and q_2 which in turn affects the energy redistribution between those components as observed from the figures.

However, the nature of collision, that is enhancement (suppression) in S_1 and suppression (enhancement) in S_2 in the $q_1(q_2)$ component during the shape-changing collision process, is still preserved. We also notice that in the case of the standard elastic collision process resulting for the choice $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} = \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$, the role of the c_1 parameter on the amplitudes of colliding bright solitons before and after collision is the same. This is shown in Fig. 8 and Fig. 9, for $c_1 = 0$ and $c_1 \neq 0$, respectively. From the figures we observe that the intensities of bright solitons before and after collision are affected by the dark component by the same amount.

(b) *Collision behavior of dark solitons in the presence of bright solitons.* The intensities of dark solitons in the q_3 component are unaffected during the collision in the presence of the bright solitons in the q_1 and q_2 components. This is obvious from the analytic expressions (B1b), (B2b), (B3b), and (B4b). This type of collision scenario is shown in the third figure of Fig. 5 (see also Fig. 7). The analysis presented in Appendix B reveals the fact that the α parameters do affect the interaction of dark solitons through the shift in central position of the solitons which ultimately changes the separation distance between them after collision. This phase shift and the resulting change in the relative separation distance between the solitons can be obtained from Eq. (23a) and Eq. (24), respectively. In fact, the change in relative separation distance becomes more significant and displays interesting propagation and collision dynamics of solitons when the soliton velocities are moderately different and k_{jR}'s are equal. For illustrative purpose, we consider the propagation of such composite two dark solitons arising for the choice $k_1 = 0.6 - i$, $k_2 = 0.6 - 0.5i$, $b_1 = 0.2$, $|c_1| = 0.56$, $\alpha_1^{(1)} = 1$, $\alpha_1^{(2)} = i$, $\alpha_2^{(1)} = (22/55) - 45i$, $\alpha_2^{(2)} = 1$ in Fig. 10, at $z = -5$ and $z = 5$. The analytic expression corresponding to the above choice $\left(k_{1R}\right.$ $= k_{2R}, \frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} \neq \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}\right)$ is given by
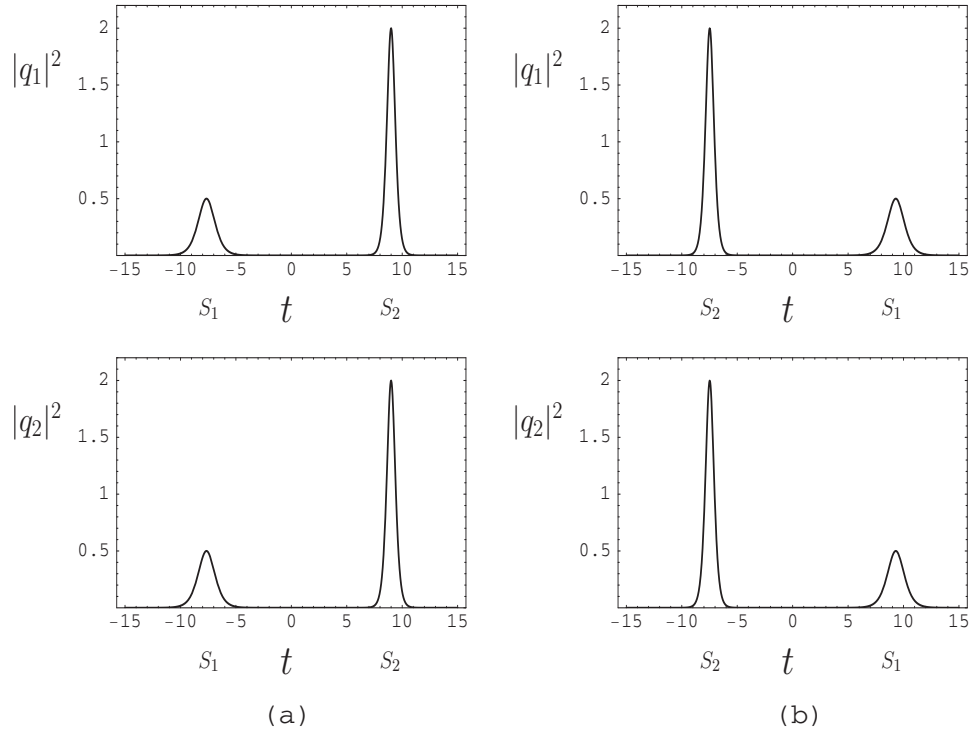
FIG. 8. Intensity profiles showing the collision scenario of two bright solitons of the mixed CNLS equations with $N=3$ in the absence of dark component ($c_1=0$), for $\alpha_1^{(1)}/\alpha_2^{(1)}=\alpha_1^{(2)}/\alpha_2^{(2)}$. (a) $z=-4$ and (b) $z=4$. The parameters are chosen as given in the text.

$$q_3 = c_1 e^{i\zeta_1} \left(\frac{e^{Q_3/2}\cosh\left(A+\dfrac{Q_3}{2}\right) + e^{(Q_{11}+Q_{22})/2}\cosh\left(B+\dfrac{Q_{22}-Q_{11}}{2}\right) + e^{(Q_{12}+Q_{21})/2}\cos\left(C+\dfrac{Q_{21}-Q_{12}}{2}\right)}{e^{R_3/2}\cosh\left(A+\dfrac{R_3}{2}\right) + e^{(R_1+R_2)/2}\cosh\left(B+\dfrac{R_2-R_1}{2}\right) + e^{(\delta_0^*+\delta_0)/2}\cos\left(C+\dfrac{\delta_0^*-\delta_0}{2}\right)} \right), \tag{25a}$$

where

$$A = 2k_{1R}[t - z(k_{1I}+k_{2I})], \tag{25b}$$

$$B = 2k_{1R}[z(k_{1I}-k_{2I})], \tag{25c}$$

$$C = (k_{2I}-k_{1I})t + (k_{1I}^2 - k_{2I}^2)z. \tag{25d}$$

Such a soliton solution can be viewed as soliton complex [30,31] and all the parameters in Eq. (25a) are defined in Eq. (15). Figure 10(a) shows that the dark soliton complex varies its profile during propagation within a finite distance. This is a consequence of change in the relative separation distance due to the presence of α parameters. However, for the same choice of k_j, b_1, and c_1 but with $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} = \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$, the dark soliton complex does not vary its profile after propagation through the same distance. This is shown in Fig. 10(b). This kind of behavior is a striking feature of multisoliton complexes [30,31]. The bright counterparts of this dark soliton also

form bright soliton complexes of variable shape and such bright soliton complexes in CNLS equations with focusing nonlinearity are discussed in Refs. [13,30–32]. Work is in progress on such bright-dark multisoliton complexes.

2. One-bright–two-dark soliton collision

The two-soliton solution given by Eq. (17) describes the collision of solitons in which bright soliton collision takes place in the q_1 component and the dark soliton collision occurs in the q_2 and q_3 components. The asymptotic expressions for the colliding solitons before and after collision are given in Appendix B. Equations (B5)–(B8) show that the intensities of bright solitons before and after collisions are the same, i.e., $|A_1^{l-}|=|A_1^{l+}|$, $l=1,2$. Similarly, the intensities of the dark solitons are also unaltered after collision. This indicates that the bright as well as the dark solitons undergo elastic collision. However, there occurs a dark soliton parameter-dependent phase shift due to collision. The phase shift of soliton S_1 in both the bright and dark components is given by
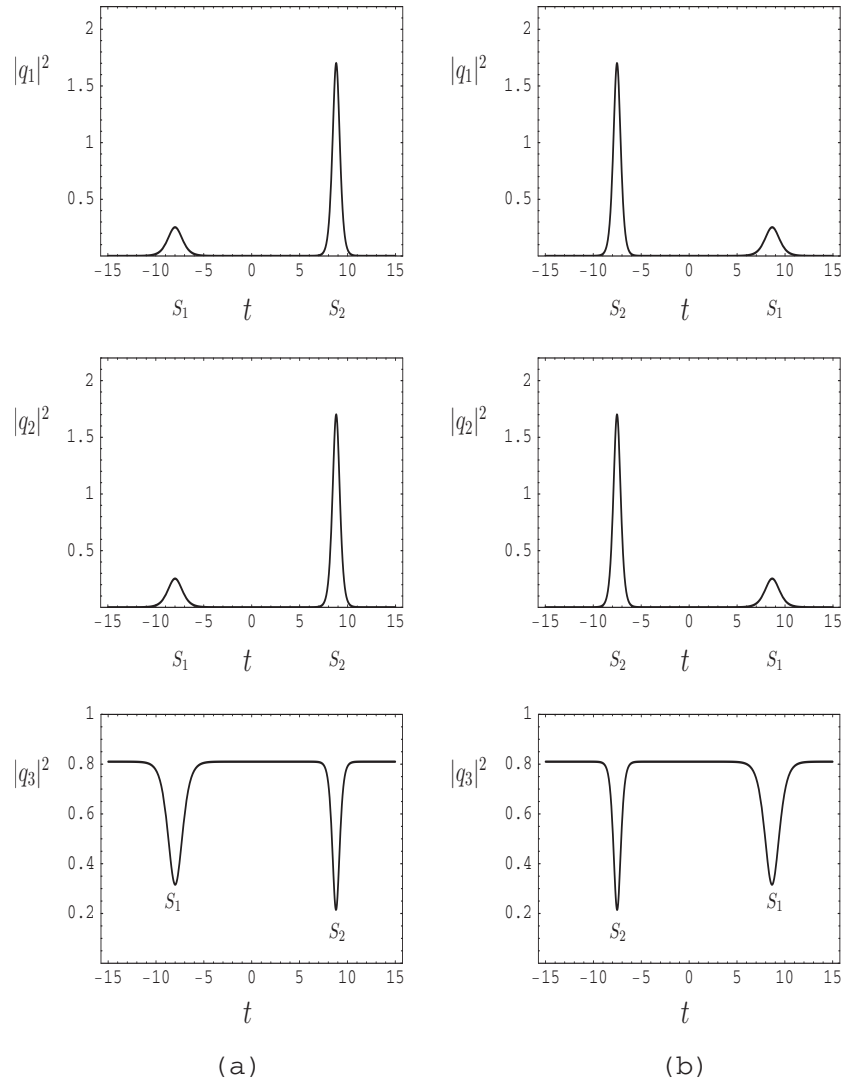
FIG. 9. Intensity profiles showing the collision scenario of two bright solitons of the mixed CNLS equations with $N=3$ in the presence of dark component ($c_1 \neq 0$), for $\alpha_1^{(1)}/\alpha_2^{(1)} = \alpha_1^{(2)}/\alpha_2^{(2)}$. (a) $z=-4$ and (b) $z=4$. The parameters are chosen as in the text.

$$\Phi_1 = \left(\frac{R_3 - R_2 - R_1}{2}\right) = \frac{1}{2}\ln\left(\frac{|k_1 - k_2|^2}{(k_1 + k_2^*)^2(k_2 + k_1^*)^2} \frac{\left|1 + \sum_{v=1}^{2}\frac{|c_v|^2}{(k_1 - ib_v)(k_2 - ib_v)}\right|^2}{\left|1 - \sum_{v=1}^{2}\frac{|c_v|^2}{(k_1 - ib_v)(k_2 - ib_v)}\right|^2}\right). \tag{26}$$

Similarly, soliton S_2 undergoes a phase shift $\Phi_2=-\Phi_1$. The dark-component parameters c_1, c_2, b_1, and b_2 influence the phase shift whereas the bright-component parameter α does not alter the phase shift.

The above asymptotic analysis of the two-soliton solution of 2-CNLS and 3-CNLS can be extended straightforwardly to N-CNLS equations, with arbitrary N. By generalizing the above study we also point out that for mixed N-CNLS equation (1), the bright and dark solitons can be split up in $(N-1)$ ways, starting from $(N-1)$-bright–one-dark soliton solu-

tion, $(N-2)$-bright–two-dark soliton solution, up to one-bright–$(N-1)$-dark soliton solution. We also arrive at an important conclusion that for m-bright–n-dark two-soliton solution, where $m+n=N$, the shape-changing collision of bright solitons takes place only when $m \geq 2$.

V. CONCLUSION

To conclude, we have obtained the explicit mixed-type (bright-dark) soliton solutions for the multicomponent mixed
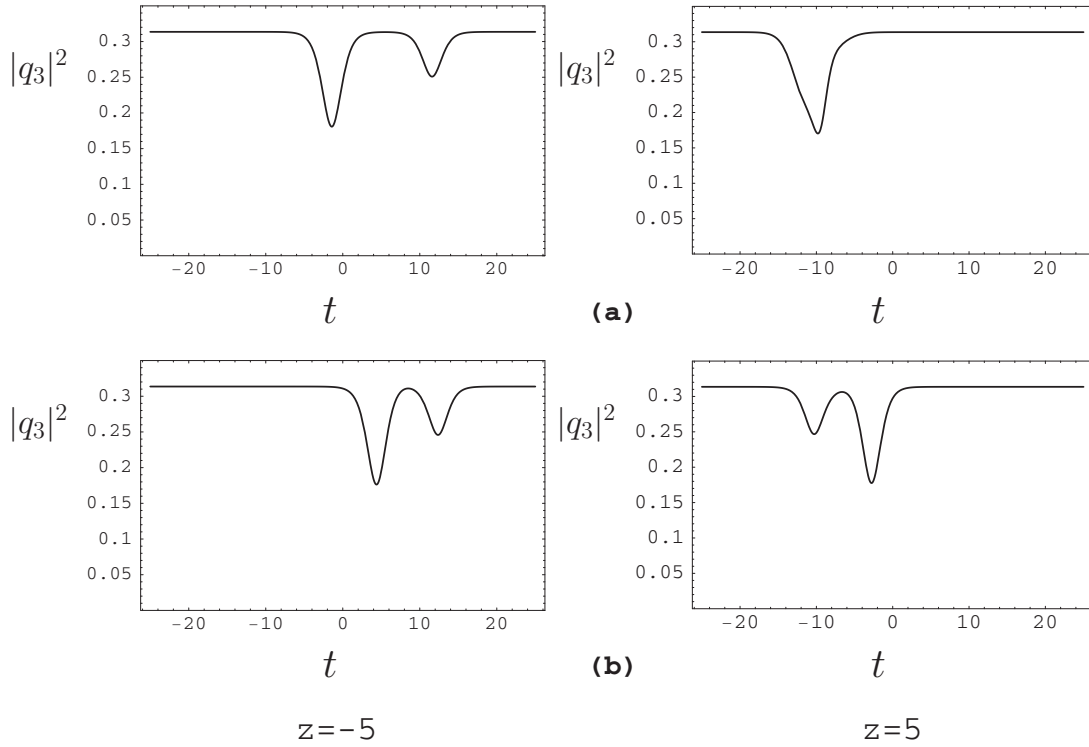
FIG. 10. Intensity profiles showing the propagation of two dark solitons in the mixed CNLS system, before ($z=-5$) and after ($z=5$) collision: (a) for the special choice of parameters (as given in the text) with $\alpha_1^{(1)}/\alpha_2^{(1)} \neq \alpha_1^{(2)}/\alpha_2^{(2)}$; (b) for the special choice of parameters (as given in the text) with $\alpha_1^{(1)}/\alpha_2^{(1)} = \alpha_1^{(2)}/\alpha_2^{(2)}$.

coupled nonlinear Schrödinger equations using Hirota's bi-linearization method. In particular, we have shown that these coupled bright and dark solitons possess rich structure and become more general than individual bright or dark solitons. Next, our analysis of their collision behavior reveals the fact that there exist elastic collisions of bright as well as dark solitons for the two component case. We observe that for this case the bright solitons are affected uniformly by the dark solitons. The bright soliton parameters α have no effect on the phase shift whereas it is influenced by the dark soliton parameters c_1 and b_1. The important observation of this study is that for more than two components, if the bright solitons appear in at least two components, then those bright solitons undergo shape-changing collisions characterized by intensity redistribution, amplitude-dependent phase shift, and relative separation distances, but their counterpart (dark solitons) exhibits only elastic collisions but with amplitude-dependent phase shift. This identification can find potential applications in optical as well as matter wave switching devices where the switching is performed through shape-changing collision of solitons. Further, in contrast to the N-component Manakov system here the phase shift of bright as well as dark solitons is characterized by dark soliton parameters c_1 and b_1 in addition to the α parameters. Also, we observe that the dark soliton parameter c_1 influences the intensity of bright solitons by a different amount when the bright soliton parameters, α, are such that $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} \neq \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$ while their amplitudes are affected by dark solitons by the same amount for $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} = \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$. One more noticeable observation is that the dark solitons vary their profiles depending on α parameters during propagation as in multisoliton complexes [31,32]. The various results obtained from the study will give further insight into the bright-dark paired solitons, soliton complex formation, collision in boson-fermion mixtures and in nonlinear left-handed materials and their applications in switching devices.

APPENDIX A: ASYMPTOCIC ANALYSIS OF BRIGHT-DARK TWO-SOLITON SOLUTION OF MIXED 2-CNLS EQUATIONS

In the limit $z \to \pm\infty$ the two-soliton solution (11) takes the following asymptotic forms for the choices k_{1R}, $k_{2R} > 0$, and $k_{1I} > k_{2I}$. For other choices of k_i's, $i=1,2$, similar analysis can be made.

(a) *Before collision (limit $z \to -\infty$).*

(i) Soliton 1 ($\eta_{1R} \simeq 0$, $\eta_{2R} \to -\infty$),

$$q_1 \simeq A_1^{1-} k_{1R} e^{i\eta_{1I}} \operatorname{sech}\left(\eta_{1R} + \frac{R_1}{2}\right), \tag{A1a}$$

$$q_2 \simeq -c_1 e^{i(\zeta_1+\phi_1^{(1)})}\left[\cos\,\phi_1^{(1)}\tanh\left(\eta_{1R}+\frac{R_1}{2}\right)+i\,\sin\,\phi_1^{(1)}\right],$$
(A1b)

where

$$A_1^{1-}=\left(\frac{\alpha_1^{(1)}}{\alpha_1^{(1)*}}\right)^{1/2}\frac{1}{k_{1R}}\sqrt{k_{1R}^2-|c_1|^2\cos^2\,\phi_1^{(1)}},\quad\text{(A1c)}$$

$$\phi_1^{(1)}=\tan^{-1}\left(\frac{k_{1I}-b_1}{k_{1R}}\right).\qquad\text{(A1d)}$$

(ii) Soliton 2 ($\eta_{2R}\simeq0,\ \ \eta_{1R}\to\infty$),

$$q_1 \simeq A_1^{2-}k_{2R}e^{i\eta_{2I}}\,\text{sech}\left(\eta_{2R}+\frac{(R_3-R_1)}{2}\right),\qquad\text{(A2a)}$$

$$q_2 \simeq c_1 e^{i(\zeta_1+\phi_2^{(1)})}\frac{(k_1-ib_1)}{(k_1^*+ib_1)}\left[\cos\,\phi_2^{(1)}\tanh\left(\eta_{2R}+\frac{(R_3-R_1)}{2}\right)\right.$$
$$\left.+\,i\,\sin\,\phi_2^{(1)}\right],\qquad\text{(A2b)}$$

where

$$A_1^{2-}=\left(\frac{\alpha_2^{(1)}}{\alpha_2^{(1)*}}\right)^{1/2}\frac{1}{k_{2R}}\sqrt{k_{2R}^2-|c_1|^2\cos^2\,\phi_2^{(1)}}\left(\frac{X}{X^*}\right)\quad\text{(A2c)}$$

and

$$\phi_2^{(1)}=\tan^{-1}\left(\frac{k_{2I}-b_1}{k_{2R}}\right).\qquad\text{(A2d)}$$

In the above expressions,

$$X=(k_1-k_2)(k_1+k_2^*)(k_1^*+ib_1)\{[(k_1-ib_1)(k_2^*+ib_1)-|c_1|^2]$$
$$\times[(k_1-ib_1)(k_2-ib_1)+|c_1|^2]\}^{1/2}.$$

(b) *After collision (limit $z\to\infty$).*
(i) Soliton 1 ($\eta_{1R}\simeq0,\ \ \eta_{2R}\to\infty$),

$$q_1 \simeq A_1^{1+}k_{1R}e^{i\eta_{1I}}\,\text{sech}\left(\eta_{1R}+\frac{(R_3-R_2)}{2}\right),\qquad\text{(A3a)}$$

$$q_2 \simeq c_1 e^{i(\zeta_1+\phi_1^{(1)})}\frac{(k_2-ib_1)}{(k_2^*+ib_1)}\left[\cos\,\phi_1^{(1)}\tanh\left(\eta_{1R}+\frac{(R_3-R_2)}{2}\right)\right.$$
$$\left.+\,i\,\sin\,\phi_1^{(1)}\right],\qquad\text{(A3b)}$$

where

$$A_1^{1+}=\left(\frac{\alpha_1^{(1)}}{\alpha_1^{(1)*}}\right)^{1/2}\frac{1}{k_{1R}}\sqrt{k_{1R}^2-|c_1|^2\cos^2\phi_1^{(1)}}\left(\frac{Y}{Y^*}\right).\quad\text{(A3c)}$$

Here

$$Y=(k_1-k_2)(k_2+k_1^*)(k_2^*+ib_1)\{[(k_1-ib_1)(k_2-ib_1)+|c_1|^2]$$
$$\times[(k_1^*+ib_1)(k_2-ib_1)-|c_1|^2]\}^{1/2}.$$

(ii) Soliton 2 ($\eta_{2R}\simeq0,\ \ \eta_{1R}\to-\infty$),

$$q_1 \simeq A_1^{2+}k_{2R}e^{i\eta_{2I}}\,\text{sech}\left(\eta_{2R}+\frac{R_2}{2}\right),\qquad\text{(A4a)}$$

$$q_2 \simeq -c_1 e^{i(\zeta_1+\phi_2^{(1)})}\left[\cos\,\phi_2^{(1)}\tanh\left(\eta_{2R}+\frac{R_2}{2}\right)+i\,\sin\,\phi_2^{(1)}\right],$$
(A4b)

where

$$A_1^{2+}=\left(\frac{\alpha_2^{(1)}}{\alpha_2^{(1)*}}\right)^{1/2}\frac{1}{k_{2R}}\sqrt{k_{2R}^2-|c_1|^2\cos^2\,\phi_2^{(1)}}.\quad\text{(A4c)}$$

Note that in the above expressions $A_j^{l+}\neq A_j^{l-}$, $j,l=1,2$, $|A_j^{l+}|=|A_j^{l-}|$, and thereby confirms that no intensity redistribution occurs in the case of bright soliton and also for dark soliton [see Eqs. (A1b), (A2b), (A3b), and (A4b)]. The phase shift $\Phi_1\ (\Phi_2)$ of soliton $S_1\ (S_2)$ in the bright and dark component obtained from the above asymptotic expressions is given by Eq. (20a).

APPENDIX B: ASYMPTOTIC ANALYSIS OF BRIGHT-DARK TWO-SOLITON SOLUTION OF MIXED 3-CNLS EQUATIONS

1. Two-bright–one-dark soliton solution

Here also we assume k_{1R}, $k_{2R}>0$, and $k_{1I}>k_{2I}$, then the two-soliton (two-bright–one-dark) solution (15) takes the following forms asymptotically ($z\to\pm\infty$).
(a) *Before collision (limit $z\to-\infty$).*
(i) Soliton 1 ($\eta_{1R}\simeq0,\ \ \eta_{2R}\to-\infty$),

$$\begin{pmatrix}q_1\\q_2\end{pmatrix}\simeq\begin{pmatrix}A_1^{1-}\\A_2^{1-}\end{pmatrix}k_{1R}\,\text{sech}\left(\eta_{1R}+\frac{R_1}{2}\right)e^{i\eta_{1I}},\qquad\text{(B1a)}$$

$$q_3 \simeq -c_1 e^{i(\zeta_1+\phi_1^{(1)})}\left[\cos\,\phi_1^{(1)}\tanh\left(\eta_{1R}+\frac{R_1}{2}\right)+i\,\sin\,\phi_1^{(1)}\right],$$
(B1b)

where

$$\begin{pmatrix}A_1^{1-}\\A_2^{1-}\end{pmatrix}\simeq\begin{pmatrix}\alpha_1^{(1)}\\\alpha_1^{(2)}\end{pmatrix}\frac{\sqrt{k_{1R}^2-|c_1|^2\cos^2\,\phi_1^{(1)}}}{k_{1R}(|\alpha_1^{(1)}|^2+|\alpha_1^{(2)}|^2)^{1/2}}.\quad\text{(B1c)}$$

(ii) Soliton 2 ($\eta_{2R}\simeq0,\ \ \eta_{1R}\to\infty$),

$$\begin{pmatrix}q_1\\q_2\end{pmatrix}\simeq\begin{pmatrix}A_1^{2-}\\A_2^{2-}\end{pmatrix}k_{2R}\,\text{sech}\left(\eta_{2R}+\frac{(R_3-R_1)}{2}\right)e^{i\eta_{2I}},\quad\text{(B2a)}$$

$$q_3 \simeq c_1 e^{i(\zeta_1+\phi_2^{(1)})}\frac{(k_1-ib_1)}{(k_1^*+ib_1)}\left[\cos\,\phi_2^{(1)}\tanh\left(\eta_{2R}+\frac{(R_3-R_1)}{2}\right)\right.$$
$$\left.+\,i\,\sin\,\phi_2^{(1)}\right],\qquad\text{(B2b)}$$

where

$$\begin{pmatrix} A_1^{2-} \\ A_2^{2-} \end{pmatrix} \simeq \begin{pmatrix} \alpha_2^{(1)}\left(\dfrac{1-(\alpha_1^{(1)}/\alpha_2^{(1)})\kappa_1}{\sqrt{1-\kappa_1\kappa_2}} \right) \\[4mm] \alpha_2^{(2)}\left(\dfrac{1-(\alpha_1^{(2)}/\alpha_2^{(2)})\kappa_1}{\sqrt{1-\kappa_1\kappa_2}} \right) \end{pmatrix}$$
$$\times \left(\frac{(k_2-k_1)|k_1+k_2^*|}{(k_2+k_1^*)|k_1-k_2|} \frac{\sqrt{k_{2R}^2-|c_1|^2\cos^2\phi_2^{(1)}}}{k_{2R}(|\alpha_2^{(1)}|^2+|\alpha_2^{(2)}|^2)^{1/2}} \right).$$
$$\text{(B2c)}$$

(b) *After collision (limit $z\to\infty$).*
(i) Soliton 1 ($\eta_{1R}\simeq 0$, $\eta_{2R}\to\infty$),

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \simeq \begin{pmatrix} A_1^{1+} \\ A_2^{1+} \end{pmatrix} k_{1R}\,\mathrm{sech}\left(\eta_{1R}+\frac{(R_3-R_2)}{2} \right) e^{i\eta_{1I}}, \quad \text{(B3a)}$$

$$q_3 \simeq c_1 e^{i(\zeta_1+\phi_1^{(1)})} \frac{(k_2-ib_1)}{(k_2^*+ib_1)}\left[\cos\phi_1^{(1)}\tanh\left(\eta_{1R}+\frac{(R_3-R_2)}{2} \right) \right.$$
$$\left. +\, i\sin\phi_1^{(1)} \right], \quad \text{(B3b)}$$

where

$$\begin{pmatrix} A_1^{1+} \\ A_2^{1+} \end{pmatrix} \simeq \begin{pmatrix} \alpha_1^{(1)}\left(\dfrac{1-(\alpha_2^{(1)}/\alpha_1^{(1)})\kappa_2}{\sqrt{1-\kappa_1\kappa_2}} \right) \\[4mm] \alpha_1^{(2)}\left(\dfrac{1-(\alpha_2^{(2)}/\alpha_1^{(2)})\kappa_2}{\sqrt{1-\kappa_1\kappa_2}} \right) \end{pmatrix}$$
$$\times \left(\frac{(k_1-k_2)|k_1+k_2^*|}{(k_1+k_2^*)|k_1-k_2|} \frac{\sqrt{k_{1R}^2-|c_1|^2\cos^2\phi_1^{(1)}}}{k_{1R}(|\alpha_1^{(1)}|^2+|\alpha_1^{(2)}|^2)^{1/2}} \right).$$
$$\text{(B3c)}$$

(ii) Soliton 2 ($\eta_{2R}\simeq 0$, $\eta_{1R}\to-\infty$),

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \simeq \begin{pmatrix} A_1^{2+} \\ A_2^{2+} \end{pmatrix} k_{2R}\,\mathrm{sech}\left(\eta_{2R}+\frac{R_2}{2} \right) e^{i\eta_{2I}}, \quad \text{(B4a)}$$

$$q_3 \simeq -c_1 e^{i(\zeta_1+\phi_2^{(1)})}\left[\cos\phi_2^{(1)}\tanh\left(\eta_{2R}+\frac{R_2}{2} \right) + i\sin\phi_2^{(1)} \right], \quad \text{(B4b)}$$

where

$$\begin{pmatrix} A_1^{2+} \\ A_2^{2+} \end{pmatrix} \simeq \begin{pmatrix} \alpha_2^{(1)} \\ \alpha_2^{(2)} \end{pmatrix} \frac{\sqrt{k_{2R}^2-|c_1|^2\cos^2\phi_2^{(1)}}}{k_{2R}(|\alpha_2^{(1)}|^2+|\alpha_2^{(2)}|^2)^{1/2}}. \quad \text{(B4c)}$$

In the above equations $\phi_1^{(1)}$ and $\phi_2^{(1)}$ are defined in Eqs. (A1d) and (A2d), respectively. Note that $A_j^{l+}\neq A_j^{l-}$, and also in general the intensities $|A_j^{l+}|^2\neq|A_j^{l-}|^2$, $j,l=1,2$, except when $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}}=\frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$. The explicit relation between the intensities before and after collision is given by Eq. (22) in the text. The phase shifts Φ_1 and Φ_2 suffered by solitons S_1 and S_2, respectively, are given in Eq. (23a).

2. One-bright–two-dark soliton solution

Considering the one-bright–two-dark two-soliton solution (17), the analysis in the asymptotic limits can be performed as follows (with k_{1R}, $k_{2R}>0$, and $k_{1I}>k_{2I}$).
(a) *Before collision (limit $z\to-\infty$).*
(i) Soliton 1 ($\eta_{1R}\simeq 0$, $\eta_{2R}\to-\infty$),

$$q_1 \simeq A_1^{1-}k_{1R}e^{i\eta_{1I}}\,\mathrm{sech}\left(\eta_{1R}+\frac{R_1}{2} \right), \quad \text{(B5a)}$$

$$q_{j+1} \simeq -c_j e^{i(\zeta_j+\phi_1^{(j)})}\left[\cos\phi_1^{(j)}\tanh\left(\eta_{1R}+\frac{R_1}{2} \right) \right.$$
$$\left. +\, i\sin\phi_1^{(j)} \right], \quad j=1,2, \quad \text{(B5b)}$$

where

$$A_1^{1-} = \left(\frac{\alpha_1^{(1)}}{\alpha_1^{(1)*}} \right)^{1/2} \frac{1}{k_{1R}}\sqrt{k_{1R}^2-(|c_1|^2\cos^2\phi_1^{(1)}+|c_2|^2\cos^2\phi_1^{(2)})}, \quad \text{(B5c)}$$

$$\phi_1^{(j)} = \tan^{-1}\left(\frac{k_{1I}-b_j}{k_{1R}} \right). \quad \text{(B5d)}$$

(ii) Soliton 2 ($\eta_{2R}\simeq 0$, $\eta_{1R}\to\infty$),

$$q_1 \simeq A_1^{2-}k_{2R}e^{i\eta_{2I}}\,\mathrm{sech}\left(\eta_{2R}+\frac{(R_3-R_1)}{2} \right), \quad \text{(B6a)}$$

$$q_{j+1} \simeq c_j e^{i(\zeta_j+\phi_2^{(j)})}\frac{(k_1-ib_j)}{(k_1^*+ib_j)}\left[\cos\phi_2^{(j)}\tanh\left(\eta_{2R}+\frac{(R_3-R_1)}{2} \right) \right.$$
$$\left. +\, i\sin\phi_2^{(j)} \right], \quad j=1,2, \quad \text{(B6b)}$$

where

$$A_1^{2-} = A_1^{2+}\left(\frac{Q_1 Q_2}{Q_1^* Q_2^*} \right), \quad \phi_2^{(j)} = \tan^{-1}\left(\frac{k_{2I}-b_j}{k_{2R}} \right), \quad \text{(B6c)}$$

$$Q_1 = (k_1-k_2)\left(1 + \frac{|c_1^2|}{(k_1-ib_1)(k_2-ib_1)} \right.$$
$$\left. + \frac{|c_2^2|}{(k_1-ib_2)(k_2-ib_2)} \right)^{1/2}, \quad \text{(B6d)}$$

$$Q_2 = (k_1+k_2^*)\left(1 - \frac{|c_1^2|}{(k_1-ib_1)(k_2^*+ib_1)} \right.$$
$$\left. + \frac{|c_2^2|}{(k_1-ib_2)(k_2^*+ib_2)} \right)^{1/2}. \quad \text{(B6e)}$$

Here A_1^{2+} is defined in Eq. (B8c) given below.
(b) *After collision (limit $z\to\infty$).*
(i) Soliton 1 ($\eta_{1R}\simeq 0$, $\eta_{2R}\to\infty$),

$$q_1 \simeq A_1^{1+}k_{1R}e^{i\eta_{1I}}\,\mathrm{sech}\left(\eta_{1R}+\frac{(R_3-R_2)}{2} \right), \quad \text{(B7a)}$$

$$q_{j+1} \simeq c_j e^{i(\zeta_j + \phi_1^{(j)})} \frac{(k_2 - ib_j)}{(k_2^* + ib_j)} \left[\cos \phi_1^{(j)} \tanh\left(\eta_{1R} + \frac{(R_3 - R_2)}{2} \right) \right.$$
$$\left. + i \sin \phi_1^{(j)} \right], \quad j = 1,2, \tag{B7b}$$

where

$$A_1^{1+} = A_1^{1-} \left(\frac{Q_1 Q_2^*}{Q_1^* Q_2} \right). \tag{B7c}$$

(ii) Soliton 2 ($\eta_{2R} \simeq 0, \quad \eta_{1R} \to -\infty$),

$$q_1 \simeq A_1^{2+} k_{2R} e^{i\eta_{2I}} \operatorname{sech}\left(\eta_{2R} + \frac{R_2}{2} \right), \tag{B8a}$$

$$q_{j+1} \simeq - c_j e^{i(\zeta_j + \phi_2^{(j)})} \left[\cos \phi_2^{(j)} \tanh\left(\eta_{2R} + \frac{R_2}{2} \right) \right.$$
$$\left. + i \sin \phi_2^{(j)} \right], \quad j = 1,2, \tag{B8b}$$

where

$$A_1^{2+} = \left(\frac{\alpha_2^{(1)}}{\alpha_2^{(1)*}} \right)^{1/2} \frac{1}{k_{2R}} \sqrt{k_{2R}^2 - (|c_1|^2 \cos^2 \phi_2^{(1)} + |c_2|^2 \cos^2 \phi_2^{(2)})}. \tag{B8c}$$

Note that in the above expressions $|A_1^{l+}| = |A_1^{l-}|$, $l = 1,2$, and thereby confirms that no intensity redistribution occurs in the case of bright solitons, and also for dark solitons [see Eqs. (B5b), (B6b), (B7b), and (B8b)].

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