Molecular superfluid phase in systems of one-dimensional multicomponent fermionic cold atoms

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(Received 11 June 2007; published 30 January 2008)

We study a simple model of N-component fermions with contact interactions which describes fermionic atoms with $N=2F+1$ hyperfine states loaded into a one-dimensional optical lattice. We show by means of analytical and numerical approaches that, for attractive interaction, a quasi-long-range molecular superfluid phase emerges at low density. In such a phase, the pairing instability is strongly suppressed and the leading instability is formed from bound states made of N fermions. At small density, the molecular superfluid phase is generic and exists for a wide range of attractive contact interactions without an SU(N) symmetry between the hyperfine states.

DOI: 10.1103/PhysRevA.77.013624 PACS number(s): 03.75.Mn, 03.75.Ss, 71.10.Fd, 71.10.Pm

Because of rapid progress in recent years, cold atom systems have become a major field of research for investigating the physics of strong correlations in a widely tunable range and in unprecedentedly clean systems [1]. Ultracold atomic systems also offer direct access to the study of spin degeneracy since the hyperfine spin F can be larger than $1/2$, resulting in $2F+1$ hyperfine states. In nonmagnetic traps, such as optical traps, this high degeneracy might give rise to novel exotic quantum phases. The superfluid state of optically trapped alkali fermions with hyperfine spin $F>1/2$ has been studied with an emphasis on the general structure of the large-spin Cooper pairs [2]. The spin degeneracy in fermionic atoms is also expected to give rise to more complex superfluid phases. In particular, a molecular superfluid (MS) phase might be stabilized where more than two fermions form a bound state. Such a nontrivial superfluid behavior has already been found in different contexts. In nuclear physics, a four-particle condensate—the α particle—is favored over deuteron condensation at low densities [3] and it may have implications for light nuclei and asymmetric matter in nuclear stars [4]. This quartet condensation can also occur in semiconductors with the formation of biexcitons [5]. A quartetting phase, which stems from the pairing of Cooper pairs, has also been found in a model of one-dimensional (1D) Josephson junctions [6]. A similar phase has also been reported in exact-diagonalization calculations of the two-dimensional t-J model at low doping [7]. More recently, the emergence of quartets and triplets (three-fermion bound states) has been proposed to occur in the context of ultracold fermionic atoms [8–11].

In view of this increasing interest in the formation of complex superfluid condensates, it would be highly desirable to have at one's disposal a simple paradigmatic N-component fermionic model which displays this exotic physics. It will be shown in this letter that such a model is provided by the 1D N-component fermionic Hubbard model with attractive contact interaction

$$\mathcal{H} = -t\sum_{i,\alpha}[c^{\dagger}_{\alpha,i}c_{\alpha,i+1} + \text{H.c.}] + \frac{U}{2}\sum_{i} n_i^2, \tag{1}$$

where $c^{\dagger}_{\alpha,i}$ is the fermion creation operator corresponding to the $N=2F+1$ hyperfine states $\alpha=1,\ldots,N$ and $n_i=\Sigma_\alpha c^{\dagger}_{\alpha,i}c_{\alpha,i}$ is the density at site i. Model (1) displays an extended $\mathrm{U}(N)=\mathrm{U}(1)\times\mathrm{SU}(N)$ symmetry and it has been recently introduced in the context of ultracold fermionic atoms [12]. A possible experimental realization of this model (1) for $N=3$ would be a system of ^{6}Li atoms loaded into a 1D optical lattice with a carefully tuned combination of external magnetic and optical fields to make three internal states exhibit SU(3) symmetry [13]. The $N=4$ case might also be relevant to the optical trap of four hyperfine states of ^{40}K ($F=9/2$ atoms) [14]. The SU(N) symmetry of Eq. (1) has an important consequence since, when $N>2$, even for $U<0$ there can be no pairing between fermions: there is no way to form a SU(N) singlet with only two fermions. The only superfluid instability that can be stabilized is a molecular one where N fermions form a SU(N) singlet: $M_i^{\dagger}=c^{\dagger}_{1,i}c^{\dagger}_{2,i}\cdots c^{\dagger}_{N,i}$. In this paper, we shall show, by means of a combination of analytical and numerical results obtained by the density-matrix renormalization group (DMRG) technique [15], that this MS phase emerges in the phase diagram of model (1) for $U<0$ and at at small enough density n. The latter phase is not an artifact of the extended SU(N) symmetry of model (1)—it is robust to symmetry breaking terms toward more realistic situations. In this respect, we believe that the Hubbard model (1) captures the main generic features responsible for the formation of the MS phase.

Low-energy approach. The low-energy effective field theory corresponding to the SU(N) Hubbard chain (1) can be derived, as usual, from the linearization at the two Fermi

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points ($\pm k_F$) of the dispersion relation of free N-component fermions [16,17]. The derivation of the low-energy Hamiltonian is straightforward (see, for instance, Ref. [18] for details) and, away from half-filling, it separates into two commuting charge and spin pieces $\mathcal{H}=\mathcal{H}_c+\mathcal{H}_s$. This is the famous spin-charge separation which is the hallmark of 1D electronic systems [16,17]. Within this low-energy description, the U(1) charge excitations are described by a free massless bosonic field Φ with Hamiltonian density

$$\mathcal{H}_c=\frac{v}{2}\left[\frac{1}{K}(\partial_x\Phi)^2+K(\partial_x\Theta)^2\right], \quad (2)$$

where v and K are, respectively, the charge velocity and the Luttinger parameter. A perturbative estimate gives $v=v_F[1+U(N-1)/(\pi v_F)]^{1/2}$ and $K=[1+U(N-1)/(\pi v_F)]^{-1/2}$ with the Fermi velocity $v_F=2ta\sin(k_F a)$ and lattice spacing a. For generic fillings, no umklapp terms appear and the charge degrees of freedom display metallic properties in the Luttinger liquid universality class [16,17].

The hyperfine spin sector is described by the SU(N) Thirring model which is an integrable field theory [19]. For attractive interaction ($U<0$) a spectral gap m opens. The low-energy spectrum in the hyperfine spin sector consists of $N-1$ branches with masses $m_r=m\sin(\pi r/N)$ $(r=1,\ldots,N-1)$ [19]. The dominant instability which governs the physics of this phase is the one with the slowest decaying correlations at zero temperature. Both the one-particle Green function $G(x)=\langle c^\dagger_{\alpha,i}c_{\alpha,i+x}\rangle$ and (onsite) pairing correlations $P(x)=\langle c^\dagger_{\alpha,i}c^\dagger_{\beta,i}c_{\beta,i+x}c_{\alpha,i+x}\rangle$ are short range. On the contrary, the equal-time density correlation $N(x)=\langle n_i n_{i+x}\rangle$ associated with a charge-density wave (CDW) and the equal-time MS correlations $M(x)=\langle M_i M^\dagger_{i+x}\rangle$ have the following power-law decay at long distance [20]:

$$N(x)\sim\cos(2k_F x)x^{-2K/N}, \quad (3)$$

$$M(x)\sim x^{-N/(2K)} \quad \text{for } N \text{ even}, \quad (4)$$

$$M(x)\sim\sin(k_F x)x^{-(K+N^2/K)/(2N)} \quad \text{for } N \text{ odd}. \quad (5)$$

We thus see that CDW and MS instabilities compete and the key point of the analysis is the one which dominates. At issue is the value of the Luttinger parameter K. In particular, a dominant MS instability requires $K>N/2$ ($K>N/\sqrt{3}$) for N even (odd, respectively) and thus a fairly large value of K which, with only short range interaction, is not guaranteed. However, a simple argument suggests that this may be realized at sufficiently small density at large negative U. Indeed, when $n\ll 1$ and $|U|/t\gg 1$, a dilute gas of strongly bound N-fermion objects forms and Eq. (1) behaves as essentially free hardcore bosons (N even) or free fermions (N odd) with an effective hopping $t^N/|U|^{N-1}$. One can therefore estimate $M(x)$ in this limit as the free bosonic Green function $M(x)\sim x^{-1/2}$ [17,21], when N is even and, as the free fermion Green function, $M(x)\sim\sin(k_F x)/x$, when N is odd. By comparing with Eqs. (4) and (5), we deduce an upper bound for K which is $K_{\max}=N$ [22]. From the perturbative estimate we see that $K>1$ and K increases with $|U|$, so that there is room

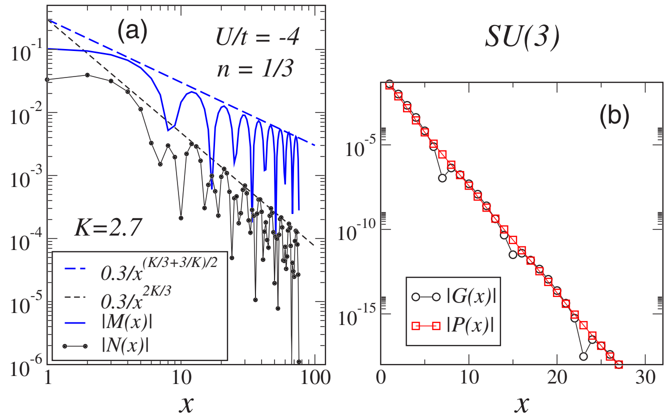

FIG. 1. (Color online) SU(3) model: triplet and density correlations vs distance obtained by DMRG with $L=153$, $n=1/3$, and $U/t=-4$. (a) Dominant triplet over CDW correlations can both be fitted with $K=2.7$. We also see the k_F ($2k_F$) oscillations of $M(x)$ [$N(x)$]. (b) One-particle Green function $G(x)$ and pairing correlations $P(x)$ vs distance. Both are short range and with the same correlation length $\xi=0.68$.

to stabilize an MS phase for sufficiently strong attractive interaction and small density. In addition, at zero density, the N-component Fermi gas with an SU(N) symmetry is known to be exactly solvable by means of the Bethe-ansatz approach and bound states of N fermions are formed for attractive interaction [23]. Outside these cases of infinite attractive interaction or vanishing density, the existence and stability of this MS phase stem from the full nonperturbative behavior of the Luttinger parameter K as a function of the density n and the interaction U. We shall now evaluate numerically this parameter in the simplest odd and even cases $N=3,4$ by computing dominant correlations with the DMRG technique to conclude on the extension of the MS phase.

Numerical results. We have performed extensive DMRG calculations for both the $N=3$ and $N=4$ cases and for a wide range of densities n and interactions U [24,25]. We show in Figs. 1(a) and 1(b) and Figs. 2(a) and 2(b) our data for $N=3$ and 4, respectively, at typical values of n and $U=-4t$. In both cases, and in agreement with the low-energy approach, we find that a gap opens in the hyperfine spin sector and that the one-particle and pairing correlations are always short ranged. From Figs. 1(b) and 2(b), one can compute the one- and two-particle correlation lengths ξ that is expected to vary as the inverse gap: $\xi\sim 1/m$. We find that the ratio $R=m_1/m_2$ is close to 1 and $1/\sqrt{2}$, respectively, for $N=3$ and $N=4$ as expected from the low-energy approach. In contrast, we see in Figs. 1(a) and 2(a) that the density and MS correlations $N(x)$ and $M(x)$ display power-law behavior. Clearly, triplet and quartet correlations dominate over CDW at these densities ($n=1/3$ for $N=3$ and $n=0.5$ for $N=4$). The phase diagrams for both SU(3) and SU(4) models are presented in Figs. 3 and 4 which give a map of K vs interaction and density. The values of K were obtained from the power-law behavior of the molecular correlation $M(x)$ using Eqs. (4) and (5). We find that triplet and quartet superfluid phases emerge in a wide portion of the phase diagrams (gray area) separated from a CDW phase by a crossover line $n_c(U)$. As

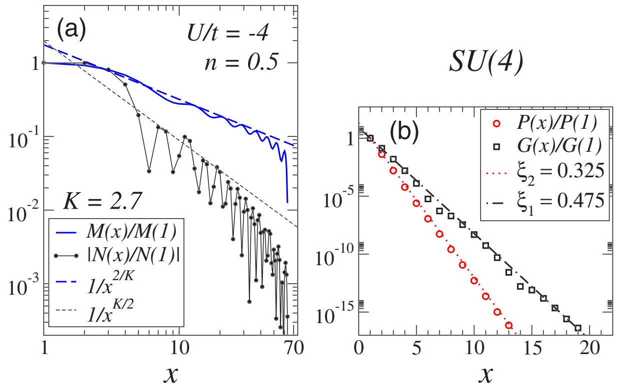

FIG. 2. (Color online) SU(4) model: (a) quartet and density correlations vs distance obtained by DMRG with $L=128$ and $U/t=-4$ at filling $n=0.25$. The same $K\simeq 2.7$ is used to give a rough estimate of the exponent. (b) One- and two-particle correlations vs distance. Both are short-range and the ratio of the two correlation lengths is $\xi_2/\xi_1=0.68\simeq 1/\sqrt{2}$.

the density decreases from half-filling ($n=N/2$) to 0, K increases from $N/4$ to N for moderate or strong $|U|/t$, as expected from the strong-coupling argument. Interestingly enough, the MS phase extends to small values of U at sufficiently small densities. We also observe that the curve $n_c(U)$ is likely to saturate in the strong coupling limit [25]. For example in the SU(4) case, the $K(n)$ function is almost U-independent for $|U|/t>2$ so that lines of equal K are parallel to the U axis.

Effect of symmetry breaking perturbations. At this point, the natural question is whether the molecular superfluid phases survives to the breaking of the SU(N) symmetry. This is an important question since in most of the realistic situations, the actual symmetry is expected to be much smaller. Part of the answer is given in 1D systems by the accepted

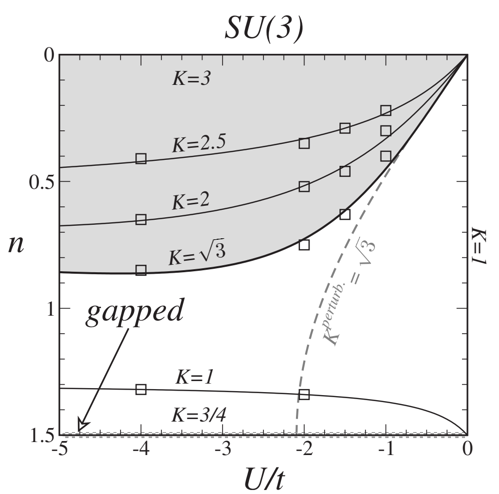

FIG. 3. SU(3) model. Phase diagram showing the Luttinger parameter K vs filling n and interaction U. The gray area is the superfluid triplet phase. Lines are guide for the eyes which behavior satisfies the perturbative limit close to the point ($U/t=0$, $n=0$). We also plot the perturbative estimate separating both regions (see text), that agrees with our numerical finding at small $|U|$ but deviates for larger values.

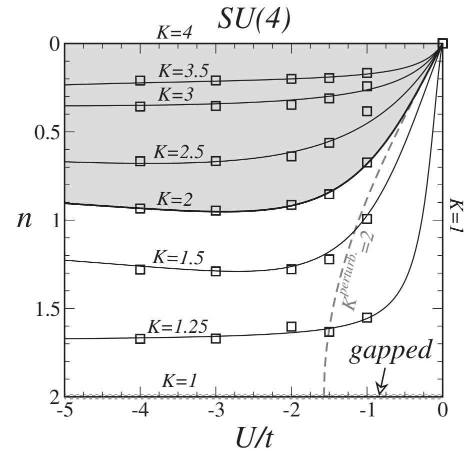

FIG. 4. Same as Fig. 3 for the SU(4) case.

view that, at sufficiently low energies and for generic interactions, the dynamical symmetry is most likely to be enlarged [26]: though the SU(N) symmetry is not an exact symmetry, it is physically meaningful as an effective low-energy theory. As an example, we consider the SU(4) case relevant for spin-3/2 cold atoms and add to the Hubbard Hamiltonian (1) a singlet-pairing coupling $V\Sigma_i P^\dagger_{00,i}P_{00,i}$, where $P^\dagger_{00,i}=c^\dagger_{3/2,i}c^\dagger_{-3/2,i}-c^\dagger_{1/2,i}c^\dagger_{-1/2,i}$. As shown in Ref. [27], the pairing term reduces the SU(4) symmetry down to SO(5). We show typical data for $U/t=-4$ and $V/t=-2$ at the density $n=1/2$ in Fig. 5.

We clearly see that the equal-time pairing correlation function $P(x)=\langle P^\dagger_{00,i}P_{00,i+x}\rangle$ admits an exponential decay, i.e., there is no BCS instability. In contrast, quartet correlations are (quasi-) long ranged and dominate over CDW ones. Remarkably, we observe from Fig. 5 that the gap ratio R is very similar to the one for the full SU(4) symmetric model when $V=0$. This means that the SU(4) model (1) is a very good starting point to explore the main features of the quartet phase. Of course, for large negative V, a BCS phase does appear [28] but the main point here is to show that the quartet molecular phase is not an artifact of the SU(4) symmetry and does exist in more realistic models [29]. A more detailed study will be presented elsewhere.

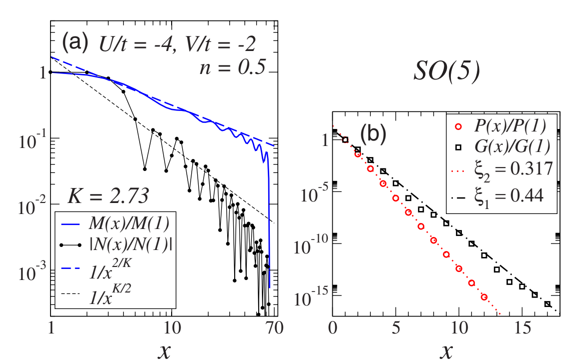

FIG. 5. (Color online) SO(5) model. (a) Quartet and density correlations for $U/t=-4$ and $V/t=-2$ at the density $n=0.5$. The value of the Luttinger parameter is $K=2.7$. (b) One- and two-particle correlations vs distance. Both are short range and the ratio of the two correlation lengths is $\xi_2/\xi_1=0.72\simeq 1/\sqrt{2}$.

Concluding remarks. We have shown that a quasi-long-range general MS phase can emerge in 1D for attractive interactions at low density. This 1D phase, characterized by a bound-state made of N fermions, can be viewed as a nematic Luttinger liquid and a simple paradigmatic model to describe its main physical properties is the attractive SU(N) Hubbard chain (1). The triplet and quartet phases in the simplest $N=3,4$ cases might be explored experimentally in the context of spinor ultracold fermionic atoms. As a first step, we have assumed here a homogeneous optical lattice and neglect in the first approximation the parabolic confining potential of the atomic trap. We expect that this potential will not affect the properties of this molecular phase at low density. Such an effect could be investigated by DMRG calculations for quantitative comparisons [30]. In the context of cold atoms experiments, the triplet and quartet phases can be probed by radio-frequency spectroscopy to measure the excitation gaps of the successive triplet-quartet dissociation process. We hope that future experiments in ultracold fermionic atoms will reveal the existence of these triplet and quartet phases.

We would like to thank F. H. L. Essler, P. Schuck, G. V. Shlyapnikov, and A. M. Tsvelik for useful discussions and their interest. S.C. and G.R. thank IDRIS (Orsay, France) and CALMIP (Toulouse, France) for use of supercomputers. SRW acknowledges the support of the NSF under Grant No. DMR-0605444.

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