Index of refraction of gases for matter waves: Effect of the motion of the gas particles on the calculation of the index

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Two different formulas relating the index of refraction n of gases for atom waves to the scattering amplitude have been published. We show here that these two formulas are not consistent with the definition of the total scattering cross section while the formula developed by one of us [C. Champenois, Ph.D. thesis, Université P. Sabatier, 1999 (unpublished)] is in agreement with this standard knowledge. We discuss this result, in particular in the neutron case for which such an index was first introduced. We finally evaluate the index of refraction as a function of well-known quantities and we discuss the order of magnitude of the ratio of $(n-1)/n_t$, where n_t is the gas density.

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I. INTRODUCTION

To describe the propagation of matter waves through a medium, it is possible to use an index of refraction, as commonly done for light. This idea was first introduced around 1940 for the propagation of neutrons through matter (see the review papers by Foldy [1] and Lax [2]).

Atom interferometry has permitted the study of the propagation of an atom wave through a dilute gas and the first measurements of the index of refraction of gases for sodium waves were made in 1993 at MIT by Schmiedmayer *et al.* [3,4], who measured the attenuation and the phase shift of the transmitted wave. Further works in the same laboratory led to the observation of glory oscillations of the index as a function of the sodium atom velocity [5–7]. More recently, our group has measured the index of refraction of several gases for lithium waves [8].

Whatever the nature of the wave and the medium, the index of refraction describes the modification of the propagation of an incident wave due to the waves scattered in the forward direction by the particles of the medium: The scattered waves interfere with the incident wave and modify its phase and amplitude. The modification of the phase induces a modification of the wave velocity, described by the real part of the index of refraction, while the modification of the amplitude is described by its imaginary part.

In practice, the index of refraction is proportional to the complex forward scattering amplitude [9,10]. The imaginary part of this amplitude is related to the total cross section, which is traditionally measured by beam scattering experiments, whereas its real part can be measured only by atom interferometry. This amplitude exhibits resonances, for a collision energy comparable to the potential well depth, and glory oscillations, for larger energies. Glory oscillations are the quantum consequence of the existence of an undeflected

classical trajectory due to the compensation of attractive and repulsive forces [11].

For atom waves, the forward scattering amplitude can be calculated if the interaction potential between a particle of the wave and a particle of the target gas is known. Several papers [12–20] have discussed the theory of the index of refraction for atom waves and their equations have been used to compare calculated values of the index of refraction [13–17,21–23] with experimental results. A detailed comparison is possible only if the thermal motion of the target gas is not neglected and we show here that the formulas [15,16] introduced to take into account this motion disagree with collision physics. We propose a formula developed by one of us [17]: this formula agrees with standard results of collision physics as well as with a recent calculation made by Hornberger and Vacchini [24], using the formalism of the quantum linear Boltzmann equation.

In the present paper, we first recall the previous results concerning the index of refraction. Then, we explain why these formulas are not in agreement with the Beer-Lambert law and we extract from this discussion a formula giving the imaginary part of the index of refraction, which is generalized to the real part of the index of refraction. We then discuss the differences between the old and new formulas. We verify that our formula is in agreement with the results concerning the index of refraction for slow neutrons. Finally, we compare the order of magnitude of the index of refraction of gases for light and atomic waves.

In the cases where the index of refraction for atom waves has been measured, the values of the ratio $(n-1)/n_t$, n_t being the target gas density, are very close to the value of this ratio for the index of refraction for ordinary gases for visible light. This similarity is in fact a coincidence, without any physical meaning.

II. ORIGIN OF THE PROBLEM

When one compares light waves and atom waves interacting with an atomic (or molecular) target gas, there is a large

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difference, which is precisely related to the motion of the target particles. In the case of light, the photon velocity, almost equal to c in dilute matter, is considerably larger than the target particle velocity. At the same time, the photon momentum is usually considerably smaller than the target particle momentum. The atom motion induces Doppler effect and the photon momentum induces atom recoil. Because the velocity of light is usually so much larger than the one of the target particles, these two effects have small consequences on the index of refraction, if we except the frequency range close to a sharp resonance line. This simple idea remains true, even if the target particles move rapidly in the laboratory, because one can always study the problem in their rest frame.

In the case of atomic waves, the velocity of an atom of the wave is usually comparable to the velocity of a target particle and, at the same time, the scattering properties depend rapidly on the relative velocity. Moreover, the momentum of an atom of the wave and the one of a target particle are usually of comparable magnitudes. In practice, it is absolutely necessary to take into account the target particle motion to make a realistic calculation of the index of refraction.

III. NOTATIONS

In the laboratory frame, a plane wave of wave vector $\mathbf{k}_p$ describes the propagation in vacuum of a projectile p of mass m_p and velocity $\mathbf{v}_p$:

$$\hbar \mathbf{k}_p = m_p \mathbf{v}_p. \tag{1}$$

The projectile can be any massive particle described by quantum mechanics, a neutron, an electron, an atom, or a molecule. This wave propagates through a gas of density n_t made of target particles t of mass m_t and velocity $\mathbf{v}_t$, the target particles being also described by quantum mechanics. The wave vector k_r describes the relative motion of the projectile p in the center-of-mass frame for a collision with a target particle t: $k_r = \mu v_r / \hbar$, where $v_r = |\mathbf{v}_p - \mathbf{v}_t|$ is the relative velocity and $\mu = m_p m_t / (m_p + m_t)$ is the reduced mass. The present notations will be used to write all previous formulas in order to facilitate comparison.

IV. INDEX FOR FIXED SCATTERING CENTERS

When a plane wave of wave vector $\mathbf{k}_p$ enters in a medium made of target particles t, its propagation is modified, with a different wave vector $\mathbf{k}_{p,m}$ given by

$$\mathbf{k}_{p,m} = n \mathbf{k}_p, \tag{2}$$

where n is the index of refraction. If the medium is described by a random distribution of fixed scattering centers, the index of refraction n is given by

$$n = 1 + 2\pi n_t \frac{f(k_p)}{k_p^2}, \tag{3}$$

where $f(k_p)$ is the forward scattering amplitude of the wave scattered by one scattering center. $f(k_p)$ has the dimension of a length and, as usual, the index n is a dimensionless quan-

tity. A general derivation of this formula is found in the paper of Foldy [1] (see also the review of Lax [2], and references therein). Equation (3) is the first order term of an expansion in powers of the gas density n_t and it is a good approximation if the following conditions are fulfilled:

(1) The mean distance $n_t^{-1/3}$ between nearest scattering centers must be considerably larger than the projectile de Broglie wavelength $\lambda_p = 2\pi / k_p$, i.e., $n_t^{-1/3} \lambda_p \ll 1$.

(2) The mean distance $n_t^{-1/3}$ between scattering centers must be considerably larger than the range of the interaction potential. Unfortunately, as realistic atom-atom interaction potentials $V(r)$ vanish only for an infinite distance r, the range does not have a clear definition for such potentials.

(3) The density n_t of scattering centers is low enough that the mean-field correction is negligible, i.e., $(n-1) \ll 1$. This last condition, which involves a condition on k_p and $f(k_p)$, depends on the interaction potential and collision energy.

Practically, for thermal atoms waves and a target gas near room temperature, with relative velocities of the order of 10^3 m/s and target densities up to $n_t \approx 10^{19}$ m^{-3} used in the experiments [3–8], the mean interatomic distance $n_t^{-1/3}$ is larger than 5×10^{-7} m, the index of refraction is of the order of $|n-1| \lesssim 10^{-10}$ and these three conditions are well fulfilled.

In the early days of atom interferometry, the target particles were treated as fixed scattering centers and the index was given by Eq. (3): This was the case of the first paper [3] dealing with the index of refraction for atom waves, written by Schmiedmayer *et al.* and of the paper written by one of us (J.V.) [12]. As soon as experimental values [4] of the index of refraction of gases for atomic waves became available in 1995, it appeared necessary to take into account the target gas thermal motion. The next section presents the equations used by different groups.

V. FORMULAS GIVING THE INDEX OF A GAS AT THERMAL EQUILIBRIUM

Three research groups have been involved in the calculation of the index of refraction of gas for atomic waves and have worked on the proper way to take into account the motion of the targets. For convenience and clarity, the formulas used by each group are presented separately.

A. Publications of Pritchard and co-workers

After a first paper where targets were considered at rest [3], this research group described a set of measurements of the index of refraction of gases for sodium waves [4]. To interpret their experiment, these authors use the following equation:

$$n = 1 + 2\pi n_t \frac{f(k_r)}{k_p k_r}, \tag{4}$$

where the thermal average is not explicitly discussed. The same equation is also used in the review paper [5] written by this group. A thermal average is taken into account in the paper by Hammond *et al.* [6], with an index of refraction given by

$$n = 1 + 2\pi \frac{n_t}{k_p} \left\langle \frac{f(k_r)}{k_r} \right\rangle, \tag{5}$$

where the brackets $\langle \cdots \rangle$ mean the average over the velocity distribution of the target particles.

B. Publications of Dalgarno and co-workers

Equation (5) appears in the paper of Forrey *et al.* [15]. In their paper, Forrey *et al.* [15] calculate the distribution of the relative wave vector k_r and, for sake of completeness, we reproduce this calculation in the Appendix. The group of Dalgarno published a series of papers on the index of refraction for atom waves [20,23] or for electron waves [19] and this work was continued by Forrey and co-workers [22]. In Ref. [18], Forrey *et al.* discusssed the statistical averaging procedure for the refractive index of matter waves. They did not give a complete derivation of their formula but they state that a key step is the Lorentz invariance of the ratio $f(k_r)/k_r = f(k_p)/k_p$.

C. Publications of our research group in Toulouse

Audouard *et al.* published a calculation of the index of refraction of argon gas for sodium waves [13]. This work by our group took into account the effect of thermal averaging. We had made an error in the algebra which was corrected in Ref. [14]. The derivation of the thermal average formula was given in a following paper by Champenois *et al.* [16]. Our calculation was based on the Fizeau effect: This effect is a frame dragging effect well known for light, which has also been studied in the case of neutron matter waves [25–27]. We wrote

$$\mathbf{k}_{p,m} = \mathbf{k}_p + \langle (n_{\text{c.m.}} - 1) \mathbf{k}_r \rangle \tag{6}$$

with the center of mass index of refraction $n_{\text{c.m.}}$ given by

$$n_{\text{c.m.}} = 1 + 2\pi n_t \frac{f(k_r)}{k_r^2}. \tag{7}$$

As above, the brackets $\langle \cdots \rangle$ stand for the average over the velocity distribution of the target particles. We will not recall the final result [16] corresponding to a Boltzmann distribution of the target gas, because this result is not correct. After discussion with the group of Dalgarno, whose results are presented above, we were convinced that our formula was wrong, in particular because our description of the Fizeau effect is not correct. In [17], Champenois derived the formula (see Eqs. V.78 and V.81)

$$n = 1 + 2\pi n_t \frac{m_p + m_t}{m_t} \left\langle \frac{f(k_r)}{k_p^2} \right\rangle. \tag{8}$$

As the derivation of this formula was very involved, we will not reproduce it here but, in the next section, we use a simpler argument to convince the reader that this formula is the right one.

VI. DISAGREEMENT WITH CLASSIC RESULTS AND A NEW FORMULA

In this part, we show that neither Eq. (5) proposed by Forrey *et al.* nor our Eqs. (6) and (7) are in agreement with

well-accepted atomic collision results. We first introduce the total scattering cross section and relate it to the imaginary part of the index of refraction.

A. The Beer-Lambert law

In nonrelativistic mechanics, the total scattering cross section $\sigma(v_r)$ is related to the number of collisions dN_{coll} occurring during a time $d\tau$ in a volume dV between a projectile p and a target t:

$$\frac{dN_{\text{coll}}}{d\tau dV} = n_p n_t \sigma(v_r) v_r. \tag{9}$$

From this equation, we deduce the mean number of collisions $dN_{\text{coll},p}/d\tau$ encountered by a projectile p per unit time

$$\frac{dN_{\text{coll},p}}{d\tau} = n_t \sigma(v_r) v_r. \tag{10}$$

We now consider a beam of projectiles p crossing a slab of target gas, with the velocity $\mathbf{v}_p$ perpendicular to the slab. A slab of thickness dL is crossed by a projectile p in a time $d\tau = dL/v_p$ and the mean number of collisions for a projectile is given by

$$dN_{\text{coll},p} = n_t \sigma(v_r) \frac{v_r}{v_p} dL. \tag{11}$$

From this equation, one can deduce the transmission T of the slab, i.e., the fraction of the incoming flux which has crossed the slab without any collision. For a finite thickness L, the transmission T is obtained by a straightforward integration

$$T = \exp\left[-n_t \sigma(v_r) \frac{v_r}{v_p} L \right]. \tag{12}$$

This equation is the Beer-Lambert law, usually written with an effective cross section $\sigma_{\text{eff}}(v_p) = \sigma(v_r) v_r / v_p$:

$$T = \exp[-n_t \sigma_{\text{eff}}(v_p) L]. \tag{13}$$

If the target velocity is spread, with a normalized distribution $P(\mathbf{v}_t)$ [i.e., verifying $\int P(\mathbf{v}_t) d^3 \mathbf{v}_t = 1$], we must replace the effective cross section $\sigma_{\text{eff}}(v_p)$ in Eq. (13) by its average $\langle \sigma_{\text{eff}}(v_p) \rangle$ given by

$$\langle \sigma_{\text{eff}}(v_p) \rangle = \int P(\mathbf{v}_t) \sigma(v_r) \frac{v_r}{v_p} d^3 \mathbf{v}_t. \tag{14}$$

B. Wavelike description of the attenuation of a beam by a slab

We are going to calculate the transmission T of the beam through the same slab of length L, using the wave point of view. Let ψ_{inc} be the incident wave and ψ_{trans} the transmitted wave given by

$$\psi_{\text{trans}} = e^{i(n-1)k_p L} \psi_{\text{inc}} = t e^{i\varphi} \psi_{\text{inc}} \tag{15}$$

with $t = \exp[-\text{Im}(n-1)kL]$ and $\varphi = \text{Re}(n-1)kL$. The transmission in intensity is $T = t^2$, which depends solely on the imaginary part of $(n-1)$

$$T = \exp[- 2\mathrm{Im}(n - 1)k_p L]. \qquad (16)$$

C. Consequences for the index of refraction

As the attenuation of the beam calculated in the two formalisms must be the same, the imaginary part of the index of refraction is related to the effective cross section by the following equation:

$$\mathrm{Im}(n - 1) = \frac{n_t \langle \sigma_{\mathrm{eff}}(v_p) \rangle}{2k_p}. \qquad (17)$$

The total cross section is related by the optical theorem to forward scattering amplitude [10]

$$\sigma(v_r) = 4\pi \frac{\mathrm{Im}[f(k_r)]}{k_r}. \qquad (18)$$

Using this relation and Eqs. (14) and (17), we obtain a formula giving the imaginary part of the index of refraction in agreement with the Beer-Lambert law:

$$\mathrm{Im}(n - 1) = 2\pi n_t \frac{m_p + m_t}{m_t} \left\langle \frac{\mathrm{Im}[f(k_r)]}{k_p^2} \right\rangle, \qquad (19)$$

where the brackets $\langle \cdots \rangle$ mean the average over the target velocity distribution $P(\mathbf{v}_t)$. Once we have an expression for the imaginary part of $(n-1)$, we get the real part by a simple generalization

$$n = 1 + 2\pi n_t \frac{m_p + m_t}{m_t} \left\langle \frac{f(k_r)}{k_p^2} \right\rangle. \qquad (20)$$

This equation agrees with the result derived by Champenois [17]. It also agrees with a result recently obtained by Hornberger and Vacchini [24]. We want to point out that Eq. (20) can be applied with any type of velocity distributions, as could be produced by a gas flow or a molecular beam, and not only with a Maxwell-Boltzmann distribution corresponding to thermal equilibrium.

D. Comparison of the new and previous formulas

Equation (20) differs from Eq. (5) established by Forrey *et al.* [15,18]: The denominator is $k_r k_p$ in Eq. (5) and k_p^2 in Eq. (20) and the mass ratio $(m_p+m_t)/m_t$ present in Eq. (20) is absent from Eq. (5). From Eq. (5), we deduce an effective cross section given by

$$\langle \sigma_{\mathrm{eff}}(v_p) \rangle = \int P(\mathbf{v}_t)\sigma(v_r)d\mathbf{v}_t. \qquad (21)$$

This expression of $\langle \sigma_{\mathrm{eff}}(v_p) \rangle$ differs from Eq. (14) and in the $v_p \ll \alpha$ limit, where α is the thermal velocity defined in the Appendix, the effective cross section given by Eq. (21) is independent of the projectile velocity v_p, while the correct behavior given by Eq. (14) is $\langle \sigma_{\mathrm{eff}}(v_p) \rangle \propto 1/v_p$. This is a well-known result recognized in everyday life: We run when it rains to get less wet.

However, in the opposite limit when the target gas temperature vanishes, $\mathbf{v}_t = \mathbf{0}$ so that $v_r = v_p$ and $k_r = \mu v_p$, it is easy

to verify that Eqs. (20) and (5) are then equivalent. This equivalence suggests that, even when the temperature does not vanish, the index calculated by these two formulas will not differ strongly as long as $\alpha \ll v_p$.

VII. DISCUSSION OF THE NEUTRON CASE

It is also interesting to apply Eq. (20) to the well known case of neutron waves. Neutrons are scattered only by nuclei, if we except the case of ferromagnetic materials in which the magnetic interactions of the neutron spin cannot be neglected. In the low-energy domain where the formalism of the index of refraction is useful, the neutron-nucleus scattering process is almost always dominated by *s*-wave scattering [28,29].

The index of refraction is frequently calculated as the consequence of an effective potential which is related to the scattering length of the neutron-nucleus interaction potential. However, there are few papers devoted to the theoretical relation between scattering theory and the index of refraction for neutrons, in which the motion of the nuclei is taken into account. This is the case of the papers by Lippmann and Schwinger [30] and also of a brief paper by Kleinman and Snow [31], who state: "This derivation of the index clearly shows that there is no Doppler effect due to the motion of the nuclei, because the λ in the formula is the neutron wavelength relative to the boundary of the slab."

Obviously, the neutron case differs from the case of atom waves only by the fact that *s*-wave ($l=0$) scattering dominates the forward scattering amplitude, which, in this case, is given by

$$f(k_r) = \frac{\exp(i\delta_0)\sin \delta_0}{k_r} \approx - a(1 - ik_r a), \qquad (22)$$

where δ_0 is the *s*-wave phase shift and a is the scattering length defined by $a = -\lim (\tan \delta_0/k_r)$ when $k_r \to 0$. If we keep only the leading term in Eq. (22), $f(k_r) \approx -a$, our formula (20) is equivalent to the result of Kleinman and Snow [31]:

$$n = 1 - 2\pi n_t \frac{m_p + m_t}{m_t} \frac{a}{k_p^2}. \qquad (23)$$

As k_r is absent from the result, there is no Doppler effect on the index of refraction. However, if we take into account the first order term in k_r in Eq. (22), the forward scattering amplitude has a nonvanishing imaginary part $\mathrm{Im}[f(k_r)] \approx k_r a^2$. As this imaginary part is linear in k_r, the imaginary part of the index of refraction is sensitive to the motion of the target particles but this imaginary part, which is very small, is usually ignored. The k_r dependence of the imaginary part of the index of refraction has no practical consequences but this remark proves that the absence of Doppler effect on the index of refraction of matter for neutrons is a very special property valid only for the real part of the index of refraction in the *s*-wave limit.

VIII. COMPARISON OF THE ORDER OF MAGNITUDE OF THE INDEX OF REFRACTION OF GASES FOR LIGHT AND MATTER WAVES

Up to now, we have not discussed the numerical value of the index of refraction of gases for matter waves. A somewhat surprising feature is that the index of refraction of gases for matter waves [4,8] and the index of refraction of transparent gases for light have similar values, when the gas density n_t is the same. In this section, we calculate the value of the $(n-1)/n_t$ ratio from first principles in both cases and we compare the results, in order to understand the origin of this similarity.

Schmiedmayer *et al.* [4] give the values of the real and imaginary parts of $(n-1)$ of several gases at $T=300$ K for sodium matter waves with a velocity $v_p=1000$ m/s. All the values of $\mathrm{Re}(n-1)$ or of $\mathrm{Im}(n-1)$ are in the range $(0.14-2.49)\times 10^{-10}$ for a gas pressure equal to 1 mTorr at $T=300$ K. This pressure corresponds to a gas density $n_t \approx 3.2\times 10^{19}$ m^{-3} from which we find that the ratios $\mathrm{Re}(n-1)/n_t$ and $\mathrm{Im}(n-1)/n_t$ are in the range $(0.4-8)\times 10^{-30}$ m^3. Our results for lithium waves [8] are somewhat larger, in the range $(1-2)\times 10^{-29}$ m^3.

It is well known that the index of refraction of air for visible light is almost purely real with its value given by $(n-1)\approx 2.8\times 10^{-4}$ at ordinary pressure and a temperature of 288 K, with a density $n_t \approx 2.55\times 10^{25}$ m^{-3}. From these values, we calculate a ratio $(n-1)/n_t \approx 1.1\times 10^{-29}$ m^3 for air. Similar values will be obtained for other transparent gases.

The ratio $(n-1)/n_t$ has comparable values for light or matter waves, although the only common feature is that they involve an index of refraction. We now evaluate these two ratio from first principles.

In optics, the index of refraction of an atomic or molecular gas is dominated by the electric dipole transitions, with its value given by

$$n(\omega) = \left[1 + 4\pi n_t \alpha(\omega)\right]^{1/2} \approx 1 + 2\pi n_t \alpha(\omega), \qquad (24)$$

where n_t is the gas density and $\alpha(\omega)$ the atomic or molecular electric polarizability for an angular frequency ω. In the visible region of the spectrum, where ω is usually smaller than the angular frequency of the main resonance transitions, $\alpha(\omega)$ is close to its static value $\alpha(0)$. We now use atomic units, with $\alpha(0)=a_0^3 \alpha_{\mathrm{a.u.}}(0)$, where a_0 is the Bohr radius ($a_0 \approx 0.529\times 10^{-10}$ m) and we get

$$\frac{n(\omega)-1}{n_t} \approx 2\pi a_0^3 \alpha_{\mathrm{a.u.}}(0). \qquad (25)$$

Tabulated values of $\alpha_{\mathrm{a.u.}}$ for atoms [32] vary from 1.4 a.u. for helium up to 450 a.u. for francium and the polarizability of small molecules has similar values [for instance, nitrogen dimer N_2 has a polarizability $\alpha_{\mathrm{a.u.}}(0)\approx 11.9$ [33]].

For matter waves, the index of refraction is given by Eq. (20). The imaginary part $\mathrm{Im}(n-1)$ is related to the total cross section $\sigma(v_r)$ and, for a purely attractive $-C_6/r^{-6}$ potential, there is a closed form expression of $\sigma(v_r)$ [9]:

$$\sigma(v_r) = 8.08 \left[\frac{C_6}{\hbar v_r}\right]^{2/5}. \qquad (26)$$

The 8.08 factor is the numerical value of a complicated expression involving the Gamma function. This result is valid in an intermediate range of energy, with many partial waves contributing to the scattering amplitude [11]. This result does not explain the glory oscillations, which exists only when the potential is attractive at long range and repulsive at short range, but this result gives the correct value of the cross section averaged over the glory oscillations. As we want only an order of magnitude, we will make some simplifications, by neglecting the thermal motion of the target particle and by assuming that $m_p \ll m_t$, so that we can replace v_r by v_p. We thus get an expression of the imaginary part of $(n-1)/n_t$:

$$\frac{\mathrm{Im}(n-1)}{n_t} = \frac{4.04}{k_p}\left[\frac{C_6}{\hbar v_p}\right]^{2/5}. \qquad (27)$$

For a purely attractive $-C_6/r^{-6}$ potential, the real and imaginary parts have similar values: Schmiedmayer *et al.* [4] calculated the ratio $\rho = \mathrm{Re}(n-1)/\mathrm{Im}(n-1)$ and found that it is constant and equal to $\rho=0.7265$ (see also Ref. [16]). We can now express $\mathrm{Im}(n-1)/n_t$, using atomic units for $C_6 = C_{6\ \mathrm{a.u.}} m_e c^2 \alpha_{\mathrm{FS}}^2 a_0^6$ (m_e is the electron mass and $\alpha_{\mathrm{FS}} \approx 1/137.037$ is the fine-structure constant). We obtain

$$\frac{\mathrm{Im}(n-1)}{n_t} = 4.12\times 10^{-3} a_0^3 \left(\frac{m_e}{m_p}\right)\left(\frac{c}{v_p}\right)^{7/5}(C_{6\ \mathrm{a.u.}})^{2/5}. \qquad (28)$$

Tabulated values of the atom-atom $C_{6\ \mathrm{a.u.}}$ coefficient span the range from 1.47 for helium up to around 7260 a.u. for cesium [33,34] (we quote these references which provide a large set of C_6 values). The comparison of Eqs. (25) and (28) proves that the index of refraction of gases for light and matter waves have very different expressions and, if they have comparable values as in the cases discussed above, this must be considered as a pure coincidence. The index for matter waves has a rapid velocity dependence in $v_p^{-7/5}$ in a large velocity range and it would be considerably larger for lower projectile velocities, provided that the target gas velocity can still be neglected.

IX. CONCLUSION

In this paper, we have shown that the equations proposed by Forrey *et al.* [15,18] and by our group [16] to take into account the motion of the gas particles in the calculation of the index of refraction for matter waves are not consistent with the traditional definition of a cross section. Following the result obtained by Champenois in her thesis [17], we propose a formula for the index of refraction:

$$n = 1 + 2\pi n_t \frac{m_p + m_t}{m_t}\left\langle \frac{f(k_r)}{k_p^2}\right\rangle. \qquad (29)$$

This formula is consistent with classic results of collision theory and it is also in agreement with a result recently obtained by Hornberger and Vacchini [24]. We have used this

formula when comparing our measurement of the index of refraction of gases for lithium waves with theoretical values [8]. Equation (29) also agrees with the formula giving the index of refraction of matter for neutrons. Finally, in the limiting case where the thermal velocity α of the gas is considerably smaller than the projectile velocity v_p, our Eq. (29) and Eq. (5) proposed by Forrey *et al.* [15,18] are equivalent but they differ considerably in the opposite limit $v_p \ll \alpha$.

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APPENDIX

We calculate the distribution $P(v_r)$ of the relative velocity v_r, starting from a normalized Maxwell-Boltzmann distribution of the target velocity $\mathbf{v}_t$:

$$P_{MB}(\mathbf{v}_t)d^3\mathbf{v}_t = \frac{1}{\pi^{3/2}\alpha^3}\exp(-\mathbf{v}_t^2/\alpha^2)d^3\mathbf{v}_t \quad (A1)$$

with $\alpha=\sqrt{2k_BT/m_t}$, k_B being the Boltzmann constant and T the temperature. Using $\mathbf{v}_t=\mathbf{v}_p-\mathbf{v}_r$, we can write

$$P(\mathbf{v}_r)d^3\mathbf{v}_r = \frac{1}{\pi^{3/2}\alpha^3}\exp[-(\mathbf{v}_p-\mathbf{v}_r)^2/\alpha^2]d^3\mathbf{v}_r. \quad (A2)$$

We take the z axis along to the projectile velocity $\mathbf{v}_p$, we use spherical coordinates for $\mathbf{v}_r$, v_r being its modulus, θ its angle with the z axis, and φ its azimuth. Integration over φ and θ is easy and we obtain

$$P(v_r)dv_r = \frac{2v_r}{\pi^{1/2}\alpha v_p}\exp\left[-\frac{v_p^2+v_r^2}{\alpha^2}\right]\sinh\left[\frac{2v_pv_r}{\alpha^2}\right]dv_r \quad (A3)$$

which is normalized, $\int_0^\infty P(v_r)dv_r=1$.

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