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Super-Tonks-Girardeau regime in trapped one-dimensional dipolar gases

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Possible signatures are discussed of a super-Tonks-Girardeau gas in bosonic systems of trapped quasi-one-dimensional dipoles at zero temperature. We provide an estimation of the frequency of the lowest compressional mode and compare it to analytical results derived using the harmonic approach in the high-density regime. We construct an exact mapping of the ground-state wave function of a one-dimensional dipolar system of bosons, fermions, and a Bose-Fermi mixture and conclude that the local properties and energy are the same at zero temperature. The question as to what extent the dipolar potential can be treated as long or short range is discussed.

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I. INTRODUCTION

The recent realization of chromium condensates [1] has attracted much interest to the unusual properties of dipolar condensates. The use of chromium has proved to be very advantageous as it has a very large permanent dipolar moment of $\mu=6\mu_B$. Other possible realizations of dipolar gases include polar molecules [2–4] and laser-induced dipoles [5,6]. The study of collective oscillations is a very sensitive tool for the investigation of cold gas properties [7]. Dipolar forces compete with the short-range s-wave scattering interaction. Although in the first experiments the s-wave interaction gave the dominant contribution to the energy, the effects of dipolar interactions have been detected in the shape of an expanding cloud [8]. During the expansion of an initially trapped gas, dipolar forces might lead to an anisotropic shape of the expanding cloud [9]. Use of the Feshbach resonance technique has brought the experiments to a new level. Indeed, by means of the Feshbach resonance technique, it is possible to reduce s-wave isotropic contact interaction, such that the anisotropic magnetic dipole-dipole interaction between ^{52}Cr atoms becomes comparable in strength [10]. This induces large changes of the aspect ratio of the cloud, and, for strong dipolar interaction, the inversion of ellipticity during expansion can even be suppressed [10]. Chromium atoms also have a fermionic ^{53}Cr isotope and Bose-Fermi cold mixtures have recently been realized [11].

Furthermore, dipolar interactions are very interesting as they contain both repulsive and attractive parts. Having attractive interactions between bosonic particles at temperatures so low that a condensate is formed might lead to instability and a violent collapse process [12–14]. Collapse of large dipolar condensates in traps has been studied using nonlocal Gross-Pitaevskii [15–19] theory and the more precise diffusion Monte Carlo (DMC) [17] approach. Study of repulsive dipolar gases in one-dimensional optical lattices within the Bose-Hubbard model using the density matrix renormalization group approach has revealed [20] that an unusual insulator phase described by highly nonlocal string correlations might be formed in dipolar systems in different phases.

One-dimensional (1D) cold systems have received great attention in recent years [21–23]. The role of quantum fluctuations is increased in reduced dimensionality, sometimes leading to very different behavior. For example, a peculiarity of the one-dimensional world is the absence of a true Bose condensate in 1D homogeneous systems even at zero temperature [24]. There is a certain trend to study low-dimensional systems in recent years.

At the same time, the past years have brought new developments in experimental tools, thus permitting very precise measurements of cold gas properties to be made. One of the most precise techniques, which can be used for testing the equation of state, is the measurement of the frequency of the "breathing" mode produced by a sudden change of the frequency of the harmonic trapping. We note that differences of several percent can be resolved in present high-precision experiments with cold gases (see, for example, Ref. [25]).

In this paper, we address properties of a quasi-one-dimensional dipolar system at zero temperature. We provide an explicit mapping of the fermionic ground-state wave function and a wave function of the Bose-Fermi mixture to a bosonic wave function in one dimension. Due to this mapping, the energy of dipolar systems containing fermions can be predicted using results previously obtained for bosons. We calculate the frequency of the lowest breathing mode in the trap and compare it to analytic predictions obtained in the high-density limit. Measurement of the frequencies of collective oscillations can provide the signature of a super-Tonks-Girardeau (STG) regime [26,27] described by very strong correlations (stronger than in the Tonks-Girardeau gas [28] in which the coupling constant is infinitely large). It is very difficult to reach the STG regime in systems with short-range interactions, while its realization is feasible in dipolar systems. We address the problem of superfluidity in one-dimensional systems and discuss the question of to what extent the dipolar interaction is short or long range.

II. MODEL

A system of N dipoles in one-dimensional geometry is described by the following model Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2M}\sum_{i=1}^{N}\frac{\partial^2}{\partial z_i^2} + \frac{1}{2}\sum_{i=1}^{N} M\omega_z^2 z_i^2 + \frac{C_{dd}}{4\pi}\sum_{i<j}\frac{1}{|z_i - z_j|^3}. \quad (1)$$

Here we assume that all dipoles are polarized and are oriented perpendicularly to the one-dimensional line. This stabilizes the system (as only the repulsive part of dipolar interactions is relevant) and avoids collapses due to attraction. The expression for the coupling constant C_{dd} depends on the nature of the dipolar interaction. For example, the following realizations are possible.

(1) Cold bosonic atoms, with induced or static dipole momenta, in a transverse trap so tight that excitations of the levels of the transverse confinement are not possible and the system is dynamically one dimensional. The longitudinal confinement is described by the frequency ω_z of the harmonic trapping potential. The dipoles themselves can be either induced or permanent. In the case of dipoles induced by an electric field E, the coupling has the form $C_{dd}=E^2\alpha^2$, where α is the static polarizability. For permanent magnetic dipoles aligned by an external magnetic field one has $C_{dd}=m^2$, where m is the magnetic dipole moment. We suppose that the s-wave scattering length is tuned to zero by applying the Feshbach resonance and only dipolar forces are relevant.

(2) Spatially indirect excitons in two coupled quantum wires. A quantum wire is a semiconductor nanostructure where an electron or a hole is allowed to move only in one direction and excitations of the transverse quantization levels are negligible. In two parallel quantum wires, one containing only holes and the other only electrons, holes and electrons couple, forming indirect excitons. If such a system is dilute enough, it constitutes a 1D set of dipoles. In this case, $C_{dd}=e^2d^2/\varepsilon$, where e is an electron's charge, ε is the dielectric constant of the semiconductor, and d is the distance between the centers of the quantum wires. This system is the 1D counterpart of a 2D indirect exciton system in coupled quantum wells.

III. SUPER-TONKS-GIRARDEAU REGIME

The properties of a homogeneous system ($\omega_z=0$) have been studied numerically in [29] by means of the diffusion Monte Carlo method. The equation of state and correlation functions have been calculated as a function of the guiding parameter nr_0, where

$$r_0 = \frac{MC_{dd}}{4\pi\hbar^2} \quad (2)$$

is a characteristic length, n being the linear density. It has been found that the system is extremely correlated and shows crystal-like properties as the strength of dipolar interactions nr_0 is increased. This behavior is very different from that of systems with short-range potentials, like the ones recently realized experimentally [30,31]. One-dimensional gases with short-range interaction potential are conveniently described by the δ-interaction Lieb-Liniger Hamiltonian [32]

$$\hat{H}_{LL} = -\frac{\hbar^2}{2M}\sum_{i=1}^{N}\frac{\partial^2}{\partial z_i^2} + g_{1D}\sum_{i<j}\delta(z_i - z_j), \quad (3)$$

where the one-dimensional coupling constant g_{1D} is inversely proportional to the one-dimensional scattering length a_{1D}: $g_{1D}=-2\hbar^2/ma_{1D}$. The correlation functions of dipolar systems, Hamiltonian (1), have completely different behavior from those described by the short-range Hamiltonian (3) which in this case [33,34] are much more similar to those of a weakly interacting Bose gas. For example, a typical shape of the static structure factor $S(k)$ is a smooth function which goes from zero for $k=0$ to an asymptotic constant value $S(k)=1$ for large $|k|$. In the most strongly interacting limit of the Lieb-Liniger system $g_{1D}\to+\infty$ (Tonks-Girardeau regime [28]), the static structure factor is equal to that of an ideal Fermi gas and has a discontinuity in the derivative at momentum $|k|=2k_F=2\pi n$. Even stronger correlations might be achieved in a system with short-ranged potential by quickly crossing the confinement-induced resonance [35] as proposed in [26]. In this resulting super Tonks-Girardeau regime, bosonic atoms will interact with attractive potential $g_{1D}\to-\infty$. The true ground state in this regime is a solitonlike state with large and negative energy [36] ("attractive collapse" of the system). Instead, if the sweep across resonance is fast and does not significantly perturb the position of the particles, the system will still remain in a gaslike state (which is metastable). As shown in Ref. [26], this state is dynamically stable if the gas parameter na_{1D} is relatively small. In the following we will say that a system is in the super-Tonks-Girardeau regime if correlations are stronger than in the Tonks-Girardeau regime. The static structure factor in the STG regime has a peak at $|k|=2k_F$, and the height of the peak increases for larger values of the gas parameter na_{1D}. The super-Tonks-Girardeau regime also exists in a number of one-dimensional systems (bosonic and fermionic) with infinite strength potentials, namely, hard rods [37] and the Calogero-Sutherland model [38]. So far this regime has never been observed experimentally, with the best candidate for its observation being a system of dipoles. The advantage of dipolar systems is that they are long living in the super-Tonks-Girardeau regime, as it is a true ground state and not a metastable state as in systems with short-range interactions. In addition, any restriction on the gas density nr_0 is removed, contrary to the requirement $na_{1D}\ll 1$ in the case of short-range gases. An experimental signature of the STG regime in a trapped system is the frequency of the lowest compressional mode Ω_z larger than $2\omega_z$ [26]. In Sec. V we calculate explicitly the dependence of Ω_z on the parameters of a trap.

The super-Tonks-Girardeau regime is expected to have quite unusual properties. It has been shown [39–41] for the Lieb-Liniger model (3) that the dynamic form factor has a power law at the point where the excitation spectrum touches zero, $S(\omega,2k_F)\propto\omega^{\eta-2}$, $\omega\to 0$, where $\eta=2\hbar k_F/mc$. In the regime of repulsive δ interaction $\eta>2$ and $S(0,2k_F)=0$. For the marginal case of the Tonks-Girardeau regime $S(\omega,2k_F)=\text{const}$. Instead, for attractive δ interaction (i.e., in the super-Tonks-Girardeau regime) this expression predicts a weak (power law) divergence in the dynamic form factor

$S(\omega,2k_F)\rightarrow\infty$, $\omega\rightarrow 0$. For additional information on singularities in the dynamic structure factor for different values of momenta, see Refs. [42,43].

IV. SUPERFLUID VS NORMAL BEHAVIOR IN ONE-DIMENSIONAL SYSTEMS

The problem of superfluidity has peculiarities in a one-dimensional system. While the different ways of defining the superfluid part in three- and two-dimensional systems are consistent, this is not the case in a one-dimensional world. Indeed, its calculation as the response of a liquid to sample boundary motion (*winding-number* method [44]) for the highly correlated states described by a translationarily invariant gaslike wave function (for example, the exact wave function for the Tonks-Girardeau [28], Calogero-Sutherland [45,46], and hard-rod [28] systems; and DMC evaluation for Lieb-Liniger [33] and dipolar [29] systems) would provide the result that such systems are completely superfluid. This argument would apply even to the TG regime, where the bosonic system has many similar properties to an ideal Fermi gas. At the same time, one should keep in mind that exposure of a one-dimensional ideal Fermi gas to a tiny perturbation will change the ground-state wave function in a dramatic way: the overlap of the new wave function and the old is essentially zero. This effect, known as the orthogonality catastrophe (see, for example, the textbook [47]), shows that in a one-dimensional system stability to external perturbations has to be carefully checked. Contrary to the winding-number approach, the Landau argument would lead to the exactly opposite result, classifying systems as completely normal. Indeed, in one-dimensional Luttinger liquids the excitation spectrum always touches zero at a finite value of momentum, $|k|=2k_F$ (see, for example, Ref. [41]; for the Lieb-Liniger model (3) the "type II" excitation [48] that touches zero can be identified as a dark soliton [49]). Another way to calculate the superfluid density by the response to a transverse probe (transverse current-current response) is not applicable within the one-dimensional description as no transverse direction is included in the model. Contradictions in the results arise from the different definitions of the superfluid part and reflect the nonstandard nature of the system. Probably the most natural and appropriate way to test the superfluidity in a 1D world is by dragging a small impurity (perturbation) through the system and seeing if this leads to energy dissipation. The force F_V, experienced by the system, depends on the interaction parameter η as $F_V \propto V^{\eta-1}$ [40], where $V \ll 1$ a.u. is the velocity with which a small δ perturbation moves through the system. Thus in the mean-field limit $\eta\rightarrow\infty$ the force is vanishing, and from the practical point of view the system behaves analogously to a superfluid. On the contrary, in the Tonks-Girardeau limit $F_V \propto V$ and the system behaves, from the point of view of friction, as a normal system, where the drag force is proportional to the velocity. In between there is a smooth crossover. The dipolar one-dimensional systems are expected to behave like normal ones.

Although Bose-Einstein condensation is absent in a one-dimensional system even at zero temperature, its reminiscence can still be observed in the divergence of the momentum distribution for $k=0$. This divergence is present in a homogeneous system of bosons in the Tonks-Girardeau regime. From the Luttinger liquid theory, it is possible to show that deep in the super-Tonks-Girardeau regime this divergence will be removed. This happens for $\eta=1$ (similarly to the Calogero-Sutherland [38] and hard-rod [37] systems) or, in terms of the Luttinger parameter, $K=1/2$. The Luttinger parameter can be extracted from the equation of state or from the phononic part of the static structure factor and is explicitly given in Ref. [50].

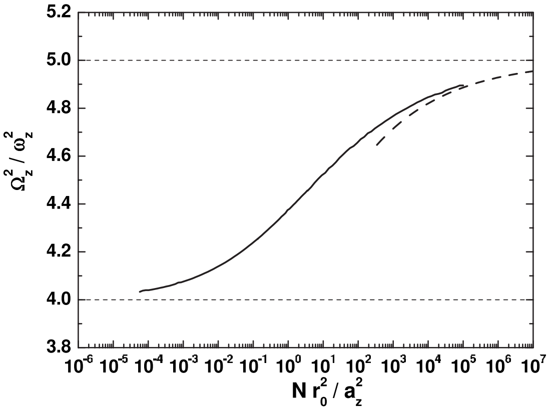

FIG. 1. Square of the lowest breathing mode frequency, Ω_z^2, as a function of the coupling strength Nr_0^2/a_z^2. Solid line, using equation of state of one-dimensional dipoles (data are taken from [51], Fig. 6.5); dashed line, high-density expansion, Eq. (5).

V. FREQUENCIES OF COLLECTIVE OSCILLATIONS

The equation of state obtained in [29] can be used to predict properties of a trapped one-dimensional gas within the local density approximation (LDA) (see, for example, [52]) by assuming that the chemical potential in a trap is the sum of the local chemical potential, taken to be the same as in a homogeneous system $\mu_{hom}(n)$, and the external harmonic potential: $\mu=\mu_{hom}(n(z))+(1/2)M\omega_z^2 z^2$. Within the LDA number of particles N and oscillator length $a_z=\sqrt{\hbar/M\omega_z}$ come in a single combination, Nr_0^2/a_z^2. The LDA is expected to describe correctly the properties of a gas in the trap if the size of the cloud is large compared to a_z. From the knowledge of the density profile $n(z)$, one can obtain the mean square radius of the cloud $\langle z^2\rangle=\int_{-R}^{R} n(z)z^2 dz/N$ and thus, making use of the result [53] $\Omega_z^2=-2\langle z^2\rangle/(d\langle z^2\rangle/d\omega_z^2)$, to calculate the frequency Ω_z^2 of the lowest breathing mode.

Figure 1 shows the square of the lowest breathing mode frequency, Ω_z^2, as a function of the coupling strength Nr_0^2/a_z^2. In the regime of small densities, $Nr_0^2/a_z^2 \ll 1$, the distance between particles is large and the potential energy due to particle-particle interactions is small. At the same time, the wave function vanishes when two particles meet. This behavior is similar to that of the Tonks-Girardeau or ideal Fermi gas. Indeed, in the regime, $Nr_0^2/a_z^2 \ll 1$, the frequency reaches the typical result of an ideal Fermi gas $\Omega_z=2\omega_z$. The observation of a breathing mode with a frequency larger than $2\omega_z$ would be a clear signature of the super-Tonks-Girardeau regime.

The attempt to calculate analytically the equation of state in the regime $nr_0 \ll 1$ in terms of the s-wave scattering length and to obtain the expansion for the frequencies of collective oscillations encounters problems, as will be explained in Sec. VII. Instead, some analytical results can be obtained in the regime of high densities $nr_0 \gg 1$. In this regime, the dipolar system has crystal-like properties [29]. At the same time the difference in the energy between gaslike and crystal-like descriptions is extremely small. This justifies an attempt to derive in a perturbative way the equation of state of a crystal and use it for an approximate description of the gaslike phase at the same density. We use the classical crystal harmonic approach to calculate the equation of state in a perturbative manner. Details of the calculation are provided in the Appendix. The result for the equation of state is

$$\frac{E}{N} = \zeta(3)\frac{\hbar^2}{Mr_0^2}(nr_0)^3 + C\frac{\hbar^2}{Mr_0^2}(nr_0)^{5/2} + \cdots, \quad nr_0 \gg 1, \tag{4}$$

with $C=2.26\ldots$. Application of the LDA to "perturbative" equations of state has been studied in detail in [52]. The density profile and total and release energy are easily obtained by using expansion (4) in the high-density $Nr_0^2/a_z^2 \gg 1$ regime. Here we report the frequency of the lowest compressional mode:

$$\frac{\Omega_z^2}{\omega_z^2} = 5 - \frac{1.1358\ldots}{(Nr_0^2/a_z^2)^{1/5}} + \cdots. \tag{5}$$

The obtained dependence is shown in Fig. 1 as a dashed line and provides a good description for large densities.

VI. BOSE-FERMI MAPPING FOR THE GROUND-STATE WAVE FUNCTION

In a previous paper [29] we calculated the zero-temperature equation of state for the homogeneous Hamiltonian (1) with $\omega_z=0$ for bosonic particles and sampled correlation functions over the ground-state wave function $\Psi_B(z_1,\ldots,z_N)$. As in experiments dipoles can be not only bosons but also fermions, or even a mixture of bosons and fermions, we note that the equation of state we have obtained is applicable also to systems containing fermions. In order to prove that, we construct an exact mapping of the wave function of bosonic dipoles to a fermionic wave function by analogy to what Girardeau did in his classical work [28]. For simplicity we start with a system of same-spin fermions. Such a system is described by the Hamiltonian (1). The main difference from the bosonic case is that the fermionic wave function must be antisymmetric with the respect to exchange of any two particles. This can be obtained as

$$\Psi_F(z_1,\ldots,z_N) = \prod_{i<j} \mathrm{sgn}(z_i - z_j)\Psi_B(z_1,\ldots,z_N). \tag{6}$$

It is easy to check that the symmetry of wave function (6) is correct. Furthermore, due to the Pauli exclusion principle, two fermions are not permitted to stay in the same place $\Psi_F(z_1,\ldots,z_N)=0$ if $z_i=z_j$. This is already satisfied in the construction of the bosonic wave function $\Psi_B(z_1,\ldots,z_N)$ due to divergence of the $1/|z|^3$ interaction for small z. This means that the diffusion Monte Carlo calculation performed for bosons is equivalent to the fixed-node diffusion Monte Carlo calculation for fermions with exactly known nodal structure as far as the energy and local quantities (pair correlation function, static structure factor, etc.) are concerned. The trick (6) has been successfully used for study the fermionic Calogero-Sutherland model in Ref. [38].

Another system that will have a similar zero-temperature equation of state is a system of two-component fermions, where the mass M_σ of different-spin atoms $\sigma=\uparrow,\downarrow$ is the same. The Hamiltonian of such a system is given by

$$\hat{H} = -\frac{\hbar^2}{2M}\sum_{\sigma;i=1}^{N^\sigma}\frac{\partial^2}{\partial z_{i,\sigma}^2} + \frac{1}{2}M\omega_z^2\sum_{\sigma;i=1}^{N^\sigma} z_{i,\sigma}^2 + \frac{C_{dd}}{8\pi}\left(\sum_{i\neq j}^{N^\uparrow}\frac{1}{|z_i^\uparrow - z_j^\uparrow|^3} + \sum_{i\neq j}^{N^\downarrow}\frac{1}{|z_i^\downarrow - z_j^\downarrow|^3} + \sum_{i,j}\frac{1}{|z_i^\uparrow - z_j^\downarrow|^3}\right). \tag{7}$$

The ground-state wave function of a system with totally $N=N^\uparrow+N^\downarrow$ atoms is then mapped onto a system of N bosons with a Hamiltonian (1) as

$$\Psi_F(z_1^\uparrow,\ldots,z_{N^\uparrow}^\uparrow,z_1^\downarrow,\ldots,z_{N^{w}\downarrow}^\downarrow) = \prod_{i<j}^{N^\uparrow}\mathrm{sgn}(z_i^\uparrow - z_j^\uparrow)\prod_{i<j}^{N^\downarrow}\mathrm{sgn}(z_i^\downarrow - z_j^\downarrow)\Psi_B(z_1^\uparrow,\ldots,z_N^\uparrow,z_1^\downarrow,\ldots,z_N^\downarrow). \tag{8}$$

Thus we conclude that a system of one-dimensional fermionic dipoles has the same ground-state equation of state as a system of bosonic dipoles. Consequently, the LDA density profile in a trap is the same. This means that the frequencies of the lowest breathing mode for fermions follow the dependence shown in Fig. 1 with N being the total number of dipoles. The limit $Nr_0^2/a_z^2 \to 0$ corresponds to ideal fermions and ideal fermions have the spherical breathing mode $\Omega = 2\omega_{ho}$ in any dimension (see, for example, [52]). For a finite value of Nr_0^2/a_z^2 the frequency is increased due to repulsive interactions. Similar effects have been predicted in Ref. [54] using Bethe-ansatz theory for systems of two-component fermions with δ-pseudopotential attractive [55,56] and repulsive [57] interactions between atoms of different spin. While a homogeneous system of bosons collapses if the interaction is attractive, this is not the case for fermions, where the Pauli principle stabilizes the system. Adding weak attraction between atoms leads to softening of the breathing mode $\Omega_z < 2\omega_z$ [54].

The Bethe-ansatz method permits us to find the ground-state energy and, thus, study collective oscillations within the LDA in a mixture of one-dimensional bosons and fermions with δ-pseudopotential repulsive interactions and arbitrary boson-fermion density ratio [58]. Such a mixture was found to be always stable against demixing [58]. The ground-state wave function of a mixture of one-dimensional bosons and fermions can be obtained with the same reasoning as for (6) and (8), thus leading to the same frequency of oscillations as in Fig. 1. This kind of mapping turns out to be quite general. Indeed, it works for one-dimensional systems where the one-dimensional interaction potential diverges when two particles meet and shows that the energy and local quantities (pair-correlation function, three-particle correlation function, etc.) are the same for bosons and fermions at zero temperature.

VII. ARE DIPOLAR INTERACTIONS LONG RANGE?

It is common to oppose long-range dipolar interactions to short-range interactions described by the *s*-wave scattering length. But are dipolar interactions really long range? Or to what extent are they long range? There are at least two ways to approach this question.

The first way is to classify the potential $V_{\text{int}}(r)$ as long or short range depending if the chemical potential is an extensive or intensive quantity. If the homogeneous properties of a large system can be defined by density n only (in appropriate units), $\mu=\mu(n)$, the potential is short range. If instead the number of particles N has to be explicitly specified, $\mu=\mu(N,n)$, due to strong (diverging) dependence on N, the potential is of long range. This can be immediately checked by testing the convergence of the potential energy at large distances:

$$I=\int_{L_{\min}}^{\infty} V_{\text{int}}(r) r^{D-1} dr, \qquad (9)$$

where $L_{\min}$ is some cutoff length and D is the dimensionality. Following this definition, the potential is short range if it decays at large distances faster than r^{-D} in D dimensions (see, for example, [59]). From this point of view, the $1/|r|^3$ potential is long range in 3D, while it is short range in 1D and 2D.

Alternatively, short-range potentials can be defined as potentials that can be described by an asymptotic phase shift. This means that a short-range potential of range R can be approximated at large distances $r \gg R$ by a free wave with an appropriate phase shift, or, which is the same, the *s*-wave scattering length a. At sufficiently small densities $na^D \to 0$, the only relevant length is a and properties (for example, energy) can be expressed in terms of the gas parameter na^D. We use the definition of the *s*-wave scattering length as the position of the node of analytic continuation of the scattering solution from distances larger than the range of the potential in the zero-energy scattering limit. This definition works well in three-dimensional systems, but also it is applicable to low-dimensional systems.

It is possible to solve the two-body scattering problem for a $1/|r|^3$ potential at zero energy and find the scattering solution $f(r)$. One needs to look for a regular solution $[f(0)=0]$ of the following differential equation:

$$-\frac{\partial^2 f(r)}{\partial r^2}-\frac{D-1}{r}\frac{\partial f(r)}{\partial r}+\frac{f(r)}{r^3}=0, \qquad (10)$$

where length is expressed in units of r_0, Eq. (2), and energy in units of $\hbar^2/Mr_0^2$. Solutions of (10) can be written explicitly as $f^{3D}(r)\propto r^{1/2}K_1(2r^{-1/2})$, $f^{2D}(r)\propto K_0(2r^{-1/2})$, and $f^{1D}(r) \propto r^{-1/2}K_1(2r^{-1/2})$, where $K_n(r)$ denotes the modified Bessel function of the second kind. In order to find the *s*-wave scattering length one has to expand $f(r)$ far from the range of the potential $(r\to\infty)$ and compare it to the similar expansion of a plane wave in an appropriate number of dimensions: $f_{free}^{3D}(r)\propto 1-a_{3D}/r$, $f_{free}^{2D}(r)\propto \ln(r/a_{2D})$, $f_{free}^{1D}(r)\propto r-a_{1D}$. Expanding the solutions of Eq. (10), we find $f^{3D}(r)\propto 1+(2\gamma-1)/r-(\ln r)/r+O(r^{-2})$, $f^{2D}(r)\propto 2\gamma-\ln r+O(r^{-1})$, and $f^{1D}(r)\propto r+(2\gamma-1)-\ln r+O(r^{-1})$, where $\gamma=0.577\ldots$ is Euler's constant. There are logarithmic terms appearing in all dimensions. In a two-dimensional system such a term is compatible with the asymptotic behavior of a free-wave solution and it is possible to define a finite scattering length $a_{2D}=e^{2\gamma}r_0=3.172\ldots r_0$. Indeed, numerical evaluation of the equation of state in a two-dimensional dipolar system [60] is in agreement with the equation of state for hard disks with the same values of a_{2D} [61]. Also, Bogoliubov theory at small na_{2D}^2 provides correct predictions for the correlation functions and condensate fraction. Thus, in two dimensions the dipolar $1/|r|^3$ potential can be treated as a short-range one. Due to the large spatial extent of the dipolar interaction potential, the universal description (in terms of the gas parameter na_{2D}^2) starts being precise at larger values of the gas parameter for a dipolar potential than for the "usual" short-range potentials (for example, soft disks [61])

Instead, in a one-dimensional system the free-wave solution does not contain a logarithmic term and this makes the dipolar solution incompatible with it. For example, for short-range potentials the *s*-wave scattering length can be calculated through the limit $\lim_{r\to\infty}[r-f(r)/f'(r)]=a_{1D}$. As it is easy to see, the corresponding expression diverges in the case of a dipolar interaction potential. This means that it is not possible to describe the properties of a one-dimensional dipolar system by a short-range potential. The same conclusion can be reached by comparing the ground-state energy of a repulsive dipolar system [29,50] to the ground-state energy of the repulsive short-range δ pseudpotential [32] in the dilute regime $na_{1D}\ll 1$. The leading term in the energy E is that of the Tonks-Girardeau gas $E_{TG}/N=\pi^2\hbar^2 n_{1D}^2/6M$ [28]. The first correction should include (if the short-range description is possible) terms na_{1D}. Instead, it is clear that this is not the case, as for a Lieb-Liniger gas $a_{1D}<0$ and the energy is lowered $E<E_{TG}$. By simply neglecting the logarithmic term for dipoles one would find a negative scattering length $a_{1D}=(1-2\gamma)r_0=-0.154\ldots$, and the energy would be lower. But we definitely know that this is not the case; instead the energy is higher, $E>E_{TG}$. Thus we find problems in trying to describe the dipolar interaction potential by a short-range model. The same will happen in a three-dimensional system.

VIII. CONCLUSIONS

In conclusion, we have studied possible signatures of a super-Tonks-Girardeau gas in a system of trapped quasi-one-dimensional dipoles at zero temperature. This regime can be entered by exploiting a confinement-induced resonance of the effective 1D scattering amplitude. Using the previously calculated equation of state we provide predictions for the frequency of the lowest compressional mode. Properties in the high-density regime are calculated within the harmonic approximation. We provide an explicit mapping of the ground-state wave function of a one-dimensional dipolar system of bosons; fermions, and a Bose-Fermi mixture and conclude that the local properties and energy are the same at zero temperature. The question as to what extent the dipolar potential can be treated as long or short range is addressed. We also discussed different possible definitions of superfluidity in one-dimensional systems.

Recently, a related article [62] studied frequencies of collective oscillations. The findings there for the lowest breathing model are in agreement with ours. In addition frequencies of higher modes are calculated.

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APPENDIX: EQUATION OF STATE IN THE HIGH-DENSITY REGIME

In the high-density regime, $nr_0 \gg 1$, potential energy dominates and the properties of the system can be compared to those of a classical crystal with the lattice spacing a defined by the density $a = n^{-1}$. Several terms of the expansion of the equation of state can be found using the harmonic approximation (for a 2D dipolar crystal this has been done in Ref. [63]). The leading term is given by the potential energy of a classical lattice,

$$\frac{E^{(0)}}{N} = \frac{\hbar^2 r_0}{M}\sum_{j=1}^{\infty}\frac{1}{(ja)^3} = \zeta(3)\frac{\hbar^2}{Mr_0^2}(nr_0)^3. \qquad \text{(A1)}$$

Particles move close to lattice sites $u_j = z_j - ja \ll a$. The potential energy can be expanded up to quadratic terms in u_j and the classical equation of motion can be solved by looking for the wave solution $u_j = r_0 e^{i(kja-\omega t)}$. For the dipolar potential this leads to the following expression for the frequency ω:

$$\omega^2(k) = \frac{12\hbar^2 r_0}{M^2 a^5}\left(\sum_{j=1}^{\infty}\frac{2}{j^5} - \sum_{j=1}^{\infty}\frac{e^{j(ika)}}{j^5} - \sum_{j=1}^{\infty}\frac{e^{j(-ika)}}{j^5}\right)$$
$$= \frac{12\hbar^2 r_0}{M^2 a^5}[2\zeta(5) - \mathrm{Li}_5(e^{ika}) - \mathrm{Li}_5(e^{-ika})], \qquad \text{(A2)}$$

where $\zeta(z)$ is the Riemann zeta function and $\mathrm{Li}_n(z)$ is a polylogarithm function. The dispersion relation (A2) is shown in Fig. 2.

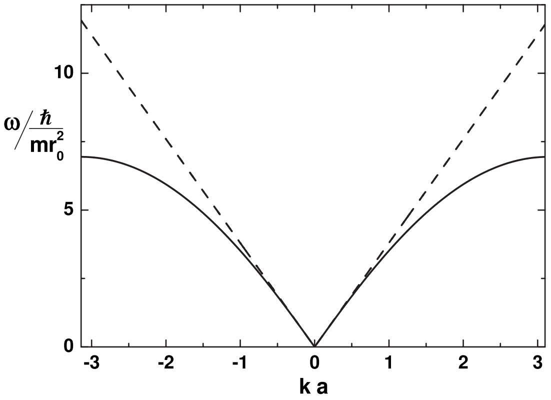

FIG. 2. Dispersion curve $\omega(k)$ in the first Brillouin zone $-\pi/a < k < \pi/a$ in the high-density limit $nr_0 \gg 1$. Solid line, harmonic approximation, Eq. (A2); dashed line, phonons, Eq. (A3).

The low-momentum behavior $|k| \to 0$ can obtained by expanding exponents in the sums in (A2) and noticing that zero- and first-order terms get canceled,

$$\omega(k) \approx \sqrt{\frac{12\hbar^2 r_0}{M^2 a^5}\sum_{j=1}^{\infty}\frac{k^2 a^2}{j^3}} = 2\sqrt{3\zeta(3)}\frac{\hbar}{Mr_0}(nr_0)^{3/2}|k|. \qquad \text{(A3)}$$

The small-$|k|$ behavior corresponds to phonons. The same result can be recovered by calculating the compressibility in the system, $mc^2 = n\,\partial\mu/\partial n$, using the leading term in the energy (A1) to calculate the chemical potential $\mu = \partial E^{(0)}/\partial N$. Then the phononic spectrum calculated as $\omega(k) = c|k|$ exactly coincides with expression (A3). In Fig. 2 the phonon excitation spectrum is plotted against the solution (A2). The latter solution has a characteristic rounding close to the boundary of the Brillouin zone. This feature, of course, is missed in the phononic description which becomes inapplicable for such large values of $|k|$.

The contribution $E^{(1)}$ to the lattice energy (A1) is obtained by summation of the energy of the zero-point motion of atoms. Thus, one has to integrate the dispersion (A2) over the first Brillouin zone (BZ):

$$\frac{E^{(1)}}{N} = \int_{BZ}\frac{\hbar\omega(k)}{2}\frac{dk}{V_{BZ}} = C\frac{\hbar^2}{Mr_0^2}(nr_0)^{5/2}. \qquad \text{(A4)}$$

The contribution obtained is positive, as it adds positive kinetic energy and describes displacement of atoms from the minimum of potential energy. The correction (A4) scales with density as $(nr_0)^{5/2}$, compared to the $(nr_0)^3$ dependence of the dominant term. The coefficient of proportionality $C = 2.26\ldots$ was obtained by numerical integration of (A4) with the dispersion relation (A2).

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