Quantum Cayley networks of the hypercube

Christopher Facer, Jason Twamley, and James Cresser
Centre for Quantum Computer Technology, Physics Department, Macquarie University, Sydney NSW 2109, Australia
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We develop the work of Christandl *et al.* [Phys. Rev. A **71**, 032312 (2005)], to show how a d-hypercube homogenous network can be dressed by additional links to perfectly route quantum information between any given input and output nodes in a duration that is independent of the routing chosen and, surprisingly, the size of the network.

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I. INTRODUCTION

The study of quantum networks has potential applications in many areas of quantum information science such as quantum communications within multiparty quantum protocols (quantum cryptography, quantum secret sharing, etc.), within quantum computer architectures, and quantum algorithms. In this paper we focus on a specific class of quantum network, called the "dressed hypercube," as a means for perfectly transporting quantum information between nodes within the network. These networks are similar to other networks capable of transporting qubits perfectly but they have the added quality that the destination of the qubits may be controlled by an external user.

Any implementation of quantum information processing which is not based on optical qubits will require a mechanism for transporting qubits between gates and processors. There have been several theoretical proposals for qubit transport which are based on a chain of spin-half particles that are coupled by Heisenberg or *XY* interactions. The first proposal [1] was a homogenous chain of particles coupled by homogeneous, nearest-neighbor interactions. A qubit is encoded at one end of the chain and the system evolves. The probability of retrieving an encoded qubit from the destination end of the chain was found to diminish as the length of the chain increased. Later, Christandl *et al.* [2] found that chains of any length were able to transport qubits perfectly but only if the coupling between neighboring particles was inhomogeneous and carefully engineered in such a way as to be strong at the middle of the chain and weaker towards the ends of the chain. More recently it was shown that relaxing the degree of control to encompass global addressing of all the particles in the chain also allows for perfect transport [3,4].

Some work has also been done in examining the properties of quantum networks, which are more complicated than linear chains. It was found in [5] that hypercubic networks of any dimension are capable of transporting qubits between pairs of antipodal nodes. In fact, if only a single pair of input and output nodes is considered, the hypercube reduces to the inhomogeneous chain. Below we show that by introducing additional links into a hypercubic network in a specific way, the destination node of a qubit can be changed. Thus if a user were able to choose which extra links in the network were "switched on," they would be able to route a qubit to any desired destination within the network and in a duration that is independent of the network size.

There have also been several studies that have looked at using quantum graph or network structures in quantum algorithms. One important example is the work in [6], where the authors consider the evolution of a quantum state on a graph as a method for searching a spatially arranged database. They show that in certain cases, the quantum algorithm achieves a speedup over the classical counterpart. Additionally, in [7], quantum graph structures are used in a quantum algorithm for evaluating Boolean formulas composed of NAND expressions.

We take a quantum network to be a collection of N spin-half particles, each of which is situated on one of the nodes of an undirected graph $G \coloneqq \{V(G), E(G)\}$, made up of nodes $V(G)$, and connecting edges $E(G)$. Theses ideas can be found in books about graph theory, for example [8]. The edges of the graph represent the allowed couplings between these particles, i.e., if two nodes i and j are connected on the graph, then $(i,j) \in E(G)$, and the two particles are coupled by an *XY* interaction $H_{ij}=J_{ij}[\hat{\sigma}_i^x\hat{\sigma}_j^x+\hat{\sigma}_i^y\hat{\sigma}_j^y]$. In what follows we will choose the coupling strengths $J_{ij}=1$. The total Hilbert space of the system is $\mathcal{H}_G=\otimes_{k\in V(G)}\mathcal{H}_k=(\mathbb{C}^2)^N$, where N is the cardinality of $V(G)$, the number of nodes in G.

The adjacency matrix $A(G)$ of a graph G captures all of the connections of the graph and is defined by

$$A_{ij}(G) = \begin{cases} 1 & \text{if } i \text{ is connected to } j \\ 0 & \text{if } i \text{ and } j \text{ are not connected.} \end{cases} \tag{1}$$

Using the adjacency matrix we can write down the Hamiltonian for the network of interacting particles as:

$$\hat{H}_{XY} = \frac{1}{2}\sum_{i,j} A_{ij}(\hat{\sigma}_x^i\hat{\sigma}_x^j + \hat{\sigma}_y^i\hat{\sigma}_y^j), \tag{2}$$

where the factor of $\frac{1}{2}$ accounts for the fact that the summation includes all pairs of interacting particles twice. Crucially, Hamiltonians of the form (2) (and more generally with any additional terms of the form $\hat{\sigma}_z^i\hat{\sigma}_z^j$, which then encompass Heisenberg coupled Hamiltonians) conserve the total z spin of the particles in the network. That is, $[\hat{H},\hat{\sigma}_z^{tot}]=0$, where $\hat{\sigma}_z^{tot}=\sum_{n=1}^N\hat{\sigma}_z^n$ is the total z spin. Thus the evolution occurs in separate invariant eigenspaces of the total Hilbert space, each labeled by the eigenvalue of $\hat{\sigma}_z^{tot}$. In the case where we allow a single excitation the evolution can be easily studied in a basis of N node states. We will represent these single excitation basis states for the single excitation subspace as $|k\rangle$, where all spins are down except the kth spin, which is up. Further, in this restricted single excitation case the *XY* Hamiltonian (2) is proportional to the adjacency matrix of the network. Similarly, for a Heisenberg coupled Hamiltonian, the total Hamiltonian becomes proportional to a related matrix known as the Laplacian of the network.

This model now allows us to explore the quantum dynamics of a particular network via the adjacency matrix of the network. In the case of networks based on the hypercube and the "dressed hypercube" (which we define below), we will show that the time evolution of the system performs a permutation of the states of the nodes in the network, and it does so periodically. This means that any qubit encoded on an "input" node becomes swapped with the spin state of the particle at an "output" node, effectively transporting the qubit through the network. Moreover, the permutation that is performed can be changed by dressing the underlying hypercube network in different ways.

The paper is organized as follows. In Sec. II we review the work of [5] to show how single-link and double-linked hypercubes can admit perfect quantum transport. We also review their construction of a more general network which admits perfect transport between very particular antipodal nodes. In Sec. III we expand their analysis to a class of Cayley graphs, known as dressed hypercubes, which are basically hypercube networks with specifically chosen additional links. We prove that the structure of the adjacency matrices of these dressed networks can always be written as a Kronecker product between two simple matrices. In Sec. IV we develop methods to characterize the spectrum of these dressed networks and thus determine the quantum dynamics on these networks. We show that the evolution, at specific times, permutes the quantum states of the nodes in the network in the single excitation subspace. We further find that the times at which the evolution corresponds to a permutation are independent of the specific permutation and N, the number of nodes in the network. We finally show that this new class of perfect transport networks does not fit into the very general category discovered in [5].

II. STATE TRANSFER IN HYPERCUBES

As mentioned above, our model consists of dressing a basic hypercube network with extra links. Before examining these dressed hypercubes it is instructive to review the perfect quantum transport of single excitations between antipodes on a d-dimensional hypercube. We follow [5], but later on we develop another proof that we can also apply to dressed hypercubes. One considers the network initialized with one overall excitation localized at one node $|A\rangle$, which evolves over the network and recoheres at another node $e^{\imath\phi}|B\rangle=\exp(-iH_G\tau)|a\rangle$, up to a global phase ϕ. One now assumes that the network G appears identical from both viewpoints of nodes A and B, i.e., we say that the network is *mirror symmetric*, when viewed by A and B. Under this special condition the subsequent evolution for a time τ will cause the wave function to recohere back again at A (up to a global phase), and thus $|\langle A|\exp(-2iH_G\tau)|A\rangle|=1$. One can show that for this to be possible, H_G must possess an energy eigenvalue spectrum E_k, such that the difference ratios are all rational fractions, i.e., $(E_i-E_j)/(E_{i'}-E_{j'})\in\mathbb{Q}$, $\forall$ $(i,j,i',j',i'\neq j')$. In [5] they consider the energy spectra of homogeneously coupled nearest-neighbor spin chains of length N and prove that the above eigenvalue condition for perfect transfer is only possible for single and double linked chains, i.e., when $N=2,3$. To get perfect transport over larger graphs they consider the Cartesian product of small (single link), perfect transfer chains. Considering the Cartesian product $L=G\times H$, and denoting the eigenvalues of the component graphs, $\{\lambda_i(G),1\leqslant i\leqslant|V(G)|\}$, and $\{\lambda_j(H),1\leqslant j\leqslant|V(H)|\}$, then $\lambda_k(L)=\lambda_i(G)+\lambda_j(H)$. To show this, one considers how the adjacency representation of the Cartesian product is formed. The adjacency matrix of the product graph is given by $A(G\times H)=A(G)\otimes\mathbb{I}_{|V(H)|}+\mathbb{I}_{|V(G)|}\otimes A(H)$. Using these properties, one can show that the eigenspectrum condition for perfect transport is satisfied for a graph $G^d=G\times G\times\cdots\times G$, if G itself does. The authors in [5] also presented another method of constructing a perfect transport network via its reduction to a 2D column representation. More specifically they consider graphs that can be arranged into columns of nodes and where edges connect only adjacent columns. Each node in a column i possesses an identical number of backward links to nodes in the previous column $i-1$, and an identical number of forward links to nodes in the next column $i+1$. Further, there are no links connecting nodes within a column i. With this construction they find examples of perfect transport (via a correspondence with hypercubic graphs), and then quantum transport on a one-dimensional spin chain with engineered coupling strengths. Below we will show a new construction of a perfect transport spin network which is not of this (already quite general) columnar form.

III. CAYLEY NETWORKS

We now introduce another method of constructing hypercube networks which also encompasses more general "dressed" hypercube networks called Cayley networks. By going to a binary labeling of the nodes we can find a group representation decomposition of the adjacency matrix of Cayley networks that allows us to prove the perfect transfer properties of hypercubes and dressed hypercubes. We first define the Cayley network $\mathrm{Cay}(G,S)$ of a finite group G (with identity element e), with respect to the generating set $S\subset G$, to be the network $\mathrm{Cay}(G,S)=\{V,E\}$, where the vertex set V corresponds to the elements of G, while the edge set E is given by

$$E=\{(x,y)|y=xg,\ \text{for some } g\in S\}. \tag{3}$$

We now consider $G=\mathbb{Z}_2^d$, the group whose elements are binary strings of length d. G is an elementary commutative group under bitwise addition modulo 2, and has order 2^d. The identity element of this group is $e\equiv\{(0,0,\ldots,0)\}$. In this analysis, we consider the family of generating subgroups $S_d^l=H_d^1\cup H_d^2$, where

$$H_d^1=\{(x_1,\ldots,x_d)\in\mathbb{Z}_2^d|\text{only one of } x_1,\ldots,x_d \text{ is } 1\} \tag{4}$$

is the set of elements of G containing a single 1, and

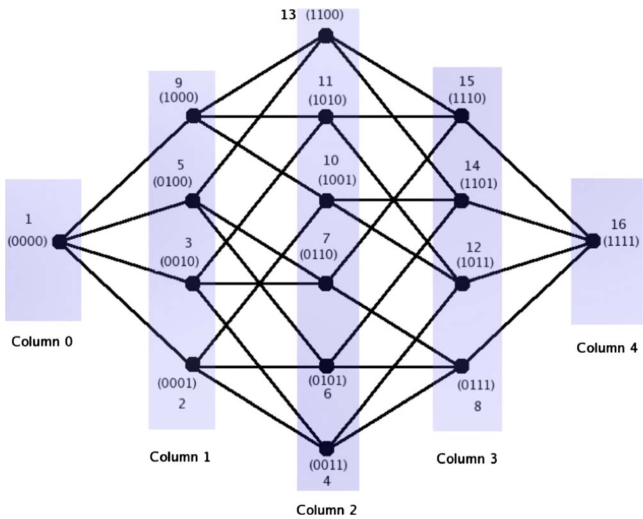

FIG. 1. (Color online) Illustration of the binary labeling of the nodes of a d-dimensional hypercube (here $d=4$). Nodes with equal Hamming weight can be arranged into columns. Edges only have unit Hamming length.

$$H_d^2 = \{(x_1, \ldots, x_l, 0^{d-l}) | (x_1, x_2, \ldots, x_l) \in \mathbb{Z}_2^l \setminus e\} \tag{5}$$

is the set of elements for which the trailing $d-l$ entries are zero.

One can compute that $|H_d^1| = d$, while $|H_d^2| = 2^l - 1$, and $|S_d^l| = 2^l + d - l - 1$. Using these tools one can form the Cayley binary network $\mathbb{Z}_2^d(l) \equiv \mathrm{Cay}(\mathbb{Z}_2^d, S_d^l)$. Initially we will set $l=1$, to consider d-dimensional hypercubes, and as an illustration we set $d=4$. In this case, the components of the generating set are $H_4^2 = \{(1,0,0,0)\}$, $H_3^1 = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$, and $|S_4^1| = 4$. The 2^4 nodes of $\mathbb{Z}_2^4(1)$ are connected via the Cayley edge relation $g_j = s_k \oplus g_i$, where $g_l \in V(G)$, and $s_k \in S_3^1$, and the group multiplication operation is addition modulo 2. Thus the node (0, 0, 0, 0) is connected to the nodes (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), while the node (1, 1, 0, 0) is connected to the nodes (1, 0, 0, 0), (0, 1, 0, 0), (1, 1, 1, 0), (1, 1, 0, 1) (see Fig. 1). From this construction it is clear that $\mathbb{Z}_2^d(l)$ are regular networks, i.e., the number of edges meeting at a node is identical throughout the network. The regular hypercubic networks, $\mathbb{Z}_2^d(l=1)$, exhibit d edges per node, and it is clear from simple examples that one can arrange the nodes into columns, labeled by the Hamming weights of the corresponding elements in G, and connected by edges of unit length Hamming distance. Such a columnar arrangement satisfies the general construction of [5], and thus, by their proof, exhibits perfect quantum transport between nodes (0, 0,…, 0) and (1, 1,…, 1). We now exhibit an alternative proof that will be used later when $l \neq 1$.

We make use of the following decomposition of the overall adjacency matrix of the graph G into components within the generating set

$$A(G) = \sum_{a \in S_d^l} \rho(a), \tag{6}$$

where ρ is the fundamental adjacency representation of an element a, in the generating set S_d^l, given by $\rho[a = (x_1, x_2, \ldots, x_d)] = X_1 \otimes X_2 \otimes \cdots \otimes X_d$, where

$$X_i = \begin{cases} \mathbb{I}_2 & \text{if } x_i = 0, \\ C & \text{if } x_i = 1, \end{cases} \tag{7}$$

and where

$$\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the fundamental matrix representation of a swap permutation. Thus for a $d=3$ (hyper)cube, $A(G) = C \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes C \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes C$. We further note that all matrices of the form $\{X_1 \otimes X_2 \otimes \cdots \otimes X_d$: each X_j is either $\mathbb{I}_2$ or $C\}$ form a much larger group under matrix multiplication that we will call the Kronecker product group of dimension d. This group is isomorphic to $\mathbb{Z}_2^d$.

The eigenstructure of $A(G)$, can now be broken down via the decomposition (6), as the eigenvectors of $D = A \otimes B$, are of the form $|a\rangle \otimes |b\rangle$, where $|a\rangle$ is an eigenvector of A, and similarly for $|b\rangle$. Thus the eigenvectors of the adjacency matrix of the d-hypercube must then take the form

$$|\lambda\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_d\rangle, \tag{8}$$

where $|x_i\rangle = |1\rangle$ or $|-1\rangle$. These two vectors correspond to the eigenvectors of the matrix C: (1, 1) with eigenvalue 1, and $(-1,1)$ with eigenvalue -1. Given that $A(G) = C \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} + \mathbb{I} \otimes C \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes C$, with d terms for the d hypercube, we can see that the maximal eigenvector is $|\lambda_d\rangle = |1\rangle \otimes |1\rangle \cdots |1\rangle$, with eigenvalue d. Since switching an x_i in $|\lambda\rangle$, from $x_i = 1$ to $x_i = -1$, reduces the overall eigenvalue by 2, we can see that there is a ladder of eigenvalues of $A(G)$. We can classify this ladder into three categories: (H1) The maximal eigenvector, with an eigenvalue $\lambda_{H1}^d = d$; (H2) the nth even group of eigenvectors with $2n|-1\rangle$ components in each eigenvector, with an overall eigenvalue of $\lambda_{H2}^d(n) = d - 4n$; (H3) the nth odd group of eigenvectors with $(2n-1)|-1\rangle$ components in each eigenvector with an overall eigenvalue of $\lambda_{H3}^d(n) = d - 4n + 2$. The eigenvalues form a ladder from $+d, +d-2, \ldots, -d+2, -d$, with minimum eigenstate $|\lambda_{-d}\rangle = |-1\rangle \otimes \cdots \otimes |-1\rangle$. Crucially, we now note that given d, we can always choose an integer k, such that $d-k$, $\lambda_{H2}^d(n) - k$, and $\lambda_{H3}^d(n) - k + 2$, are all multiples of 4. This fact now allows us to express the quantum evolution of a state initially localized on node $|m\rangle = \sum_{j=1}^N c_j |\lambda_j\rangle$, when decomposed over the eigenstates of $A(G)$, over a period of time $\tau = \pi/2$, to be

$$\hat{P}|m\rangle = e^{-i\hat{H}\pi/2}|m\rangle = \sum_{j=1}^{N} c_j e^{-i\lambda_j \pi/2}|\lambda_j\rangle \tag{9}$$

$$= e^{-ik\pi/2} \sum_{j=1}^{N} c_j e^{-i(\lambda_j - k)\pi/2}|\lambda_j\rangle. \tag{10}$$

From the above we can see that the phase factor within the sum will take the value $+1\{-1\}$ for all eigenvectors in the categories (H2){(H3)}, and thus $\hat{P}^2 = \mathbb{I}$ (up to a possible global phase factor). All of the eigenstates are either symmetric or antisymmetric under $\hat{P}$ (up to a global phase factor). From our category analysis above there are equal numbers of symmetric and antisymmetric eigenstates.

To show that $\hat{P}$ generates a permutation of the nodes, and in particular, that $\hat{P}=C^{\otimes d}$, which swaps the quantum state between antipodes, we recall that for d hypercubes the Hamiltonian and adjacency matrix is a sum of matrices from the Kronecker product group of dimension d. Since $\hat{P}=\exp(ik\pi/2)\exp(-i\hat{H}\pi/2)=\Sigma_{j=0}^{\infty}d_j(\hat{H})^j$, this power series expansion of the operator exponential must also be expressible as a sum of elements of the d-Kronecker product group. However, since $\hat{P}^2=\mathbb{I}$, the sum must only contain one term. The only such term that possesses the appropriate symmetry conditions (an equal number of symmetric and antisymmetric $|\lambda_j\rangle$) is $\hat{P}=C^{\otimes d}$. Thus by direct computation we have shown that the Hamiltonian, evolved for a duration $\tau=\pi/2$, yields a permutation of the single excitation subspace exchanging antipodes of the hypercube.

IV. DRESSED HYPERCUBES

We now consider the Cayley networks where $l>1$. From the definition of $\mathbb{Z}_2^d(l)$, the set of generators now expands, introducing new edges into the network. For example, in the case for $d=3$, when l is increased from 1 to 2, we obtain the single extra generator (1, 1, 0) (see Fig. 2). This method introduces edges such that the network remains regular, and its degree is increased by 1. The Kronecker decomposition of the adjacency matrix $A(G)$ into products of $\mathbb{I}_2$ and C via Eq. (6) still holds, and thus the eigenstates of $A(G)$ are composed of Kronecker products of $|\pm 1\rangle$. We again use this to map out the eigenspace of the overall adjacency matrix. As before we find a ladder of eigenvalues ranging from the maximum $\lambda_{max}=2^l+d-l-1$ (also the degree of the network) to a minimum, but now crucially $\lambda_{min}\neq-\lambda_{max}$. One can follow the same arguments as for the hypercube to find that eigenvectors fall into two categories, even and odd, which determine their symmetries under the quantum evolution operator $\hat{P}=\exp(-i\hat{H}\pi/2)$. We also find $\hat{P}^2=\mathbb{I}$ (again up to some global phase factor), and thus $\hat{P}$ is again expressible as one element of the Kronecker product group. This time, however, there are unequal numbers of symmetric and antisymmetric eigenstates of $\hat{H}\sim A(G)$, and we are led to identify $\hat{P}$ with a different permutation of the nodes of the network, depending on the value of l.

The choice of permutation is dictated by which eigenvectors are symmetric and which are antisymmetric. More specifically, we find that the eigenvalues of $A(G)$ ensure that the permutation operation is such that eigenvectors with even numbers of $|-1\rangle$'s in the last $d-l$ terms of their Kronecker product expansion are symmetric while those with an odd number are antisymmetric, e.g., for $d=3$, $l=2$, the symmetric eigenstates are those with a $|1\rangle$ in the rightmost slot. When $l=1$, categorizing these eigenvectors was relatively straightforward but this becomes more complex when $l>1$. It is

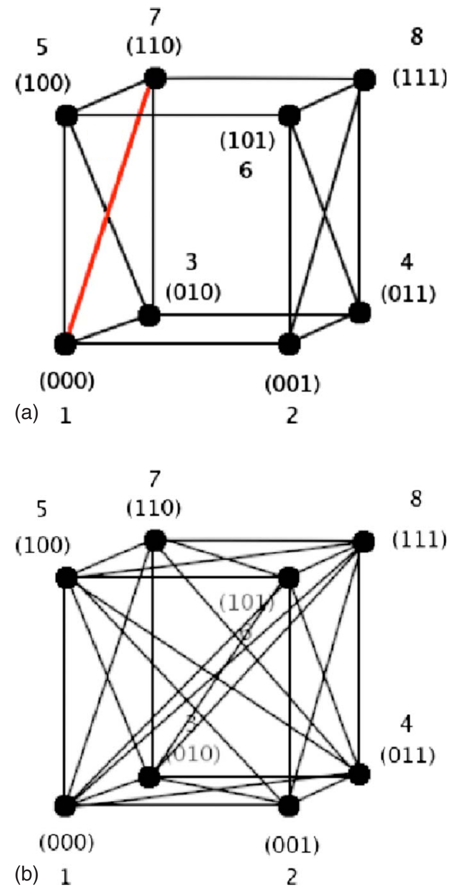

FIG. 2. (Color online) (i) The graph of the dressed hypercube $\mathbb{Z}_2^3(2)$. The link labeled (a) is the link generated from (000) by the generating set element (110). (ii) The graph of the dressed hypercube $\mathbb{Z}_2^3(3)$, a complete graph with eight nodes.

useful to break up the terms in the Kronecker product sum expansion of $A(G)$ into two classes. The first, called S_1, is the sum of elements generated by generators with $(d-l)$ 0's at the end, which corresponds to Kronecker terms that end with $\mathbb{I}^{\otimes d-l}$. The remaining class S_2 is the sum of all other terms that are not in S_1. For example, $d=3$, $l=2$, $A(G)=S_1+S_2$ where

$$S_1 = C\otimes\mathbb{I}_2\otimes\mathbb{I}_2+\mathbb{I}_2\otimes C\otimes\mathbb{I}_2+C\otimes C\otimes\mathbb{I}_2, \tag{11}$$

$$S_2=\mathbb{I}_2\otimes\mathbb{I}_2\otimes C. \tag{12}$$

One can compute the eigenvalue for any given eigenvector by considering the terms in the adjacency expansion in the two categories $S_{1,2}$. Each term in the adjacency sum expansion will contribute ± 1 to the overall eigenvalue. For each term in this expansion, if the number of C's that coincide with a $|-1\rangle$ in the product expansion of that term is even (odd) then this term contributes +1 (−1) to the overall eigenvalue.

Using this, and after some work, one can show that the contribution of all terms in the sum expansion of the eigenvector in the category S_1 contribute a factor of 2^l to the overall eigenvalue if that eigenvector contains no $|-1\rangle$ in the first l slots, and contributes only −1 otherwise. Similarly the

TABLE I. Eigenvectors of the network $\mathbb{Z}_2^3(2)$ and their eigenvalues. Those eigenvectors marked with an S are symmetric under time evolution for $\tau=\pi/2$, whereas those marked with an A are antisymmetric under the same evolution.

Eigenvector	Eigenvalue in $\mathbb{Z}_2^3(2)$	Symmetry
$\|1\rangle\otimes\|1\rangle\otimes\|1\rangle$	4	A
$\|-1\rangle\otimes\|1\rangle\otimes\|1\rangle$	0	A
$\|1\rangle\otimes\|-1\rangle\otimes\|1\rangle$	0	A
$\|1\rangle\otimes\|1\rangle\otimes\|-1\rangle$	2	S
$\|-1\rangle\otimes\|-1\rangle\otimes\|1\rangle$	0	A
$\|-1\rangle\otimes\|1\rangle\otimes\|-1\rangle$	-2	S
$\|1\rangle\otimes\|-1\rangle\otimes\|-1\rangle$	-2	S
$\|-1\rangle\otimes\|-1\rangle\otimes\|-1\rangle$	-2	S

terms in the expansion in S_2 give an overall contribution to the eigenvalue of $-P_1+P_2$, where P_1 is the number of $|-1\rangle$'s in the rightmost $d-l$ slots of the eigenvector, and likewise P_2 is the number of $|+1\rangle$'s occurring in those positions. Using these rules one finds that the eigenvalues for $\mathbb{Z}_2^d(l)$, form a ladder and can be classified into four groups:

$$\text{even}\begin{cases}\lambda^a_{\text{even}}(n)=2^l+d-l-1-4n\\ \lambda^b_{\text{even}}(n)=d-l-1-4n\end{cases}, \tag{13}$$

$$\text{odd}\begin{cases}\lambda^a_{\text{odd}}(n)=2^l+d-l+1-4n\\ \lambda^b_{\text{odd}}(n)=d-l+1-4n\end{cases}. \tag{14}$$

Here, the parameter n begins at zero and increases in integer steps until the minimal eigenvalue is reached. In the networks for which $l>1$, the minimal eigenvalue is $\lambda_{min}=l-d-1$. The groups labeled "a" in the above are those eigenvalues that correspond to eigenvectors that have no $|-1\rangle$'s in the first l positions of their Kronecker product expansion, and the groups that are labeled "b" have at least one $|-1\rangle$ in those positions.

We can again find an integer k such that $\lambda^d_{\text{even}}(n)-k$ and $\lambda^d_{\text{odd}}(n)-k+2$ are all multiples of 4 and thus all these eigenstates have eigenvalues ±1, under $\hat{P}$. However, we find now that if the eigenvector $|\lambda_j\rangle$ has an even number of $|-1\rangle$'s in the last $d-l$ slots, then it is symmetric under the permutation $\hat{P}|\lambda_j\rangle=|\lambda_j\rangle$, otherwise it is antisymmetric, i.e., $\hat{P}|\lambda_j\rangle=-|\lambda_j\rangle$. As $\hat{P}^2=\mathbb{I}$, the only single term in the Kronecker product group possible which respects these symmetries is, apart from a global phase factor

$$\hat{P}=\mathbb{I}^{\otimes l}\otimes C^{\otimes d-l},\quad 1<l\leq d. \tag{15}$$

For example, in the case where $d=3$, $l=2$, the eigenvalue structure and eigenvector symmetries can be found in Table I. The only permutation that preserves these symmetries is $\hat{P}=\mathbb{I}\otimes\mathbb{I}\otimes C$.

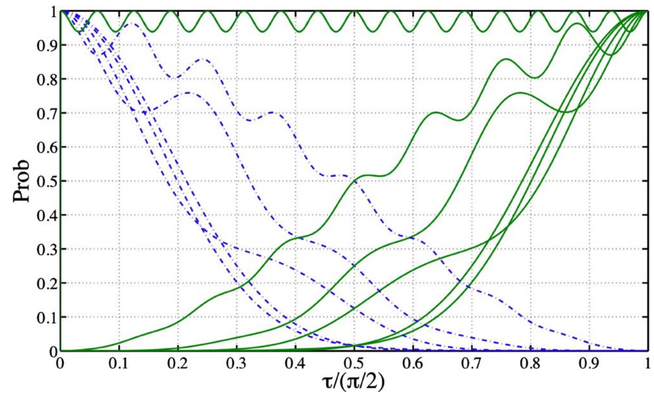

FIG. 3. (Color online) Graphs of $|\langle 0|U(\tau)|0\rangle|^2$ (dotted) and $|\langle T|U(\tau)|T\rangle|^2$ (solid) for $\mathbb{Z}_2^6(l)$, where $l=1,\ldots,6$, and T is the node targeted for perfect transport.

It is interesting to note that this predicts that for $l=d$ (a fully connected graph) $\hat{P}=\mathbb{I}^{\otimes d}$, the identity. This means that the excitation, after evolving for a period $\tau=\pi/2$, returns perfectly to its starting node. When $d=3$, and $l=1$, we get the permutations (1, 8)(2, 7)(3, 6)(4, 5), and when $l=2$ we obtain (1, 2)(3, 4)(5, 6)(7, 8). Figure 3 shows the evolution of the probability distribution of finding the excitation in the network $\mathbb{Z}_2^6(l)$.

V. CONCLUSION

We have investigated a complex class of quantum networks and found that it allows for the perfect transport of a quantum state. We have seen that the natural evolution of the network performs a permutation of the states of the spin-half particles located on each of the nodes. This arises from the structure of the eigenvalues of the adjacency matrices. Importantly, these networks are more complicated than the linear chains that have been studied in the past. Further, the Cayley networks $\mathbb{Z}_2^d(l)$, for $l>1$, no longer satisfy the general columnar construction of [5], as the new generators add links connecting nodes separated by Hamming distances greater than unity.

For the above we have assumed that we have been given the static spin network of Eq. (2) that results in the Cayley network $\mathbb{Z}_2^d(l)$ and quantum information passively is transported through this network. If instead we assume a fully connected network of spins, which results in $\mathbb{Z}_2^d(l=d)$, and we have the additional ability to execute decoupling pulses on the individual spins making up this network, then we can find decoupling pulse sequences that can reduce $\mathbb{Z}_2^d(d)\rightarrow\mathbb{Z}_2^d(l)$. We expect that this protocol for perfect quantum routing by dressing the basic hypercube network with additional links may be of use in quantum computation, cryptography, and communication.

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[1] S. Bose, Phys. Rev. Lett. **91**, 207901 (2003).
[2] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, Phys. Rev. Lett. **92**, 187902 (2004).
[3] R. Raussendorf, Phys. Rev. A **72**, 052301 (2005).
[4] J. Fitzsimons and J. Twamley, Phys. Rev. Lett. **97**, 090502 (2006).
[5] M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay, and A. J. Landahl, Phys. Rev. A **71**, 032312 (2005).
[6] A. M. Childs and J. Goldstone, Phys. Rev. A **70**, 022314 (2004).
[7] A. Ambainis *et al.*, Proceedings of the 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS'07), Providence, Rhode Island, 2007, pp. 363–372, e-print arXiv:0703015.
[8] N. Biggs, *Algebraic Graph Theory*, Cambridge Tracts in Mathematics Vol. 67 (Cambridge University Press, Cambridge, UK, 1974).