Local distinguishability of orthogonal $2\otimes3$ pure states

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We present a complete characterization for the local distinguishability of orthogonal $2\otimes3$ pure states except for some special cases of three states. Interestingly, we find there is a large class of four or three states that are indistinguishable by local projective measurements and classical communication (LPCC), but can be perfectly distinguished by LOCC. That indicates the ability of LOCC for discriminating $2\otimes3$ states is strictly more powerful than that of LPCC, which is strikingly different from the case of multiqubit states. We also show that classical communication plays a crucial role for local distinguishability by constructing a class of $m\otimes n$ states which require at least $2\min\{m,n\}-2$ rounds of classical communication in order to achieve a perfect local discrimination.

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I. INTRODUCTION

The basic problem for distinguishing quantum states by local operations and classical communication (LOCC) can be formulated as follows. Suppose two spatially separated parties, say Alice and Bob, share a quantum state which is secretly chosen from a finite set of prespecified quantum states. They want to figure out the identity of the unknown state, but are only allowed to manipulate their own quantum systems and to communicate with each other using classical channels. This problem has received considerable attention and has been studied extensively. Numerous interesting results have been reported; see Refs. [1–36] for an incomplete list. Despite this exciting progress, it remains unknown how to determine the local distinguishability of a set of multipartite states.

For the convenience of the readers, here we briefly review some of these results. Walgate *et al.* showed that any two orthogonal pure states, no matter entangled or not, can always be perfectly distinguishable by LOCC [5]. Furthermore, it has been shown that the local distinguishability and the global distinguishability of two pure states have the same efficiencies [6–11]. However, the situation changes dramatically for a set of orthogonal states with three or more members, where a perfect discrimination is generally impossible. The most surprising discovery on this topic is that there exists a set of nine $3\otimes3$ orthogonal pure product states which are indistinguishable by LOCC, a phenomenon known as "nonlocality without entanglement" [2–4]. Inspired by this discovery, many researchers have devoted themselves to the local distinguishability of product states. It is now clear that any set of $2\otimes n$ orthogonal product pure states are perfectly distinguishable by LOCC, but a set of incomplete orthogonal product states which cannot be extended by adding some additional orthogonal product state (UPB) is indistinguishable by LOCC [3,4]. The problem of distinguishing a complete basis has been completely solved [23–25], but only very recently a characterization for the locally distinguishable $3\otimes3$ product states was obtained by Feng and Shi [33].

One of the main difficulties in studying local distinguishability is that there is no effective characterization of LOCC operations. In order to partially overcome this obstacle, many researchers began to employ separable operations instead of LOCC operations to study the local distinguishability. The effectiveness of this method can be roughly understood as follows. First, the class of separable operations has a rather beautiful mathematical structure. It is much easier to work with separable operations rather than LOCC operations. Second, the class of LOCC operations is a subset of the class of separable operations [2]. So one can obtain useful necessary conditions about local distinguishability by applying separable operations. Third, any separable operation can be implemented by some LOCC operation with a nonzero success probability. In other words, separable operations and LOCC operations are probabilistically equivalent. Due to these reasons, separable operations have been widely used in studying local distinguishability. We shall briefly review two kinds of results: Probabilistic discrimination and perfect discrimination. Chefles first studied the distinguishability of a set of general quantum states by probabilistic LOCC and presented a necessary and sufficient condition for the unambiguous distinguishability [22]. A simplified version of this condition when only pure states are under consideration was independently obtained by Bandyopadhyay and Walgate [28], with which it was demonstrated that any three pure states are distinguishable by stochastic LOCC. Based on these results, Duan *et al.* studied the local distinguishability of an arbitrary basis of a multipartite state space and provided a universal tight lower bound on the number of locally unambiguously distinguishable members in an arbitrary basis [29]. Walgate and Scott further showed that this lower bound plays a crucial role in deciding the generic properties such as local unambiguous distinguishability of a set of randomly chosen states [30]. Separable operations were also used to show that a certain set of states are not perfectly distinguishable by LOCC. More precisely, Nathanson showed that any $(d+1)$ maximally entangled bipartite states on $d\otimes d$ cannot

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be perfectly distinguishable by separable operations, thus also are indistinguishable by LOCC [17]. The same result was independently obtained by Owari and Hayashi using a slightly different method [18]. It is interesting that before this result Ghosh *et al.* and Fan have respectively solved the special cases of $d=2$ and $d=3$ using a rather different approach [15,16]. Watrous constructed a class of bipartite subspaces having no basis distinguishable by separable operations, thus solved an open problem concerning the environment-assisted capacity of quantum channels [26]. Hayashi *et al.* studied the relation between average entanglement degree and local distinguishability of a set of orthogonal states, and provided a very general bound on the number of states which can be locally distinguishable. In Ref. [32] we systematically studied the distinguishability of quantum states by separable operations and found a characterization for the distinguishability of quantum states by separable operations. Notably, we showed that separable operations acting on two qubits are strictly powerful than LOCC operations. A more general class of locally indistinguishable subspaces was also constructed.

All of the above work suggests that the problem of deciding the local distinguishability of a set of general quantum states is rather complicated. Interestingly, for some very simple cases such as two-qubit states, an analytical solution is possible. Ghosh *et al.* first obtained some partial results on the local distinguishability of two-qubit states using some bounds on entanglement distillation [19]. Based on an idea of Groisman and Vaidman [20], Walgate and Hardy obtained a very simple characterization for a set of $2\otimes n$ orthogonal pure states to be perfectly distinguishable by LOCC if the owner of the qubit makes the first nontrivial measurement [21]. Employing this condition, they finally settled the local distinguishability of $2\otimes 2$ states [21]. Another immediate consequence is that local projective measurements and classical communication (LPCC) is sufficient for the local distinguishability of multiqubit states [36], which greatly simplifies the local distinguishability of multiqubit states. But this is not true in general. In Ref. [3] Bennett and co-workers constructed a set of five $3\otimes 4$ pure product states which are perfectly distinguishable by LOCC but not by LPCC. Very recently a set of $3\otimes 3$ states with similar property was obtained by Cohen [31]. However, we still do not know whether the general POVM is required in order to distinguish $2\otimes 3$ states. It seems somewhat strange that the local distinguishability of $2\otimes 3$ states when the owner of the qutrit performs the first nontrivial measurement has never been touched yet since the work of Walgate and Hardy [21].

The purpose of this paper is to study the local distinguishability of $2\otimes 3$ states. We assume the dimension of Alice's system is 2, and the dimension of Bob's system is 3. Due to the result in Ref. [21], we only consider the case when Bob goes first, which means that Bob first does a nontrivial measurement on his own system. We find that for the discrimination of six states and five states, LOCC and LPCC are equally powerful, i.e., a set of six or five $2\otimes 3$ states is locally distinguishable if and only if they are distinguishable by LPCC. But for four states and three states, there exists a large class of states which can be distinguished by LOCC, but not by LPCC. Therefore, we conclude that local POVM is strictly powerful than local projective measurements even for $2\otimes 3$ system. Furthermore, we obtain a complete characterization of four $2\otimes 3$ states that are distinguishable by LOCC but not by LPCC. For three states, such a characterization is very difficult to obtain. Nevertheless, we construct a general class of three states which are distinguishable by LOCC but not by LPCC. A feasible procedure for determining the local distinguishability of three states is also presented.

We further study the effect of classical communication for discrimination. We show that in general many rounds of classical communication are necessary. We demonstrate this result by constructing a class of $m\otimes n$ orthogonal states which requires at least $2\min\{m,n\}-2$ rounds to achieve a perfect discrimination. In some sense, our result is in accordance with the recent result by Owari and Hayashi in Ref. [35], where they showed that two-way classical communication can effectively increase the local distinguishability. We would like to point out that the problem studied in Ref. [35] is quite different from ours. More precisely, Ref. [35] considers the discrimination between a pure state and a mixed state and requires that the detection of pure state can be achieved perfectly. The goal is to minimize the minimal error of detecting the mixed state. Here we only consider pure states and require each state to be identified perfectly.

The rest of the paper is organized as follows. In Sec. II we first give a characterization for the distinguishability of $2\otimes 3$ states by LPCC. Then in Sec. III and Sec. IV we present in sequel our results about the local distinguishability of six and five $2\otimes 3$ states. Sections V and VI devote to the local distinguishability of four and three states, respectively. In Sec. VII we present a nontrivial set of bipartite pure states which requires a multiround of classical communication to achieve a perfect discrimination. We conclude the paper with a brief discussion in Sec. VIII.

For simplicity, in what follows we shall write $|\alpha\rangle=|\beta\rangle$ for any two states which are different from each other only with a nonzero factor. Sometimes we simply say POVM or projective measurements instead of local POVM or local projective measurements, respectively.

II. DISTINGUISHABILITY OF 2⊗3 STATES BY LPCC

In Ref. [3] Bennett *et al.* showed that any finite set of $2\otimes n$ orthogonal product states can be perfectly distinguishable using LPCC. For $2\otimes 3$ states this interesting result has a converse as follows.

Theorem 1. A set of $2\otimes 3$ states are distinguishable by LPCC only if there is a set of orthogonal product states such that each of the given states can be written as a disjoint summation of these product states.

Let us make the above theorem more transparent. Suppose $\{|\psi_k\rangle : k=1,\ldots,n\}$ is a set of $2\otimes 3$ states. Then these states are distinguishable by LPCC if and only if there exists a set of orthogonal product states $\{|\phi_j\rangle : j=1,\ldots,m\}$ and a partition of $\{1,\ldots,m\}$, say $S_1,\ldots,S_n$, such that $|\psi_k\rangle \in \operatorname{span}\{|\phi_j\rangle : j\in S_k\}$, where $\cup_{i=1}^m S_i=\{1,\ldots,m\}$ and $S_i\cap S_j=\emptyset$ for any $i\neq j$.

Proof. If Alice goes first, then it has been proven in Ref.

[21] that a set of $2\otimes n$ states is distinguishable by LOCC and can be written as a disjoint summation of states from a set of orthogonal product states.

Suppose now Bob goes first. If Bob's measurement operators can be written as $P_1=|0\rangle\langle 0|+|1\rangle\langle 1|$ and $P_2=|2\rangle\langle 2|$, then if measurement outcome is 1, we can write Alice's measurement as $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. We construct a set of orthogonal product states as follows: $\{|\alpha\rangle_A|2\rangle_B, |\alpha^{\perp}\rangle_A|2\rangle_B, |0\rangle_A|\beta\rangle_B, |0\rangle_A|\beta^{\perp}\rangle_B, |1\rangle_A|\gamma\rangle_B, |1\rangle_A|\gamma^{\perp}\rangle_B\}$, where $|\beta\rangle$, $|\beta^{\perp}\rangle$, $|\gamma\rangle$, $|\gamma^{\perp}\rangle$ belong to span$\{|0\rangle,|1\rangle\}$. It is easy to see that if a set of states is distinguishable by the LPCC we write above, they can be rewritten as disjoint summation of states from the above set.

If Bob's measurement operators can be written as $P_1=|0\rangle\langle 0|$, $P_2=|1\rangle\langle 1|$, and $P_3=|2\rangle\langle 2|$, then we construct a set of orthogonal product states as $\{|\alpha\rangle_A|0\rangle_B, |\alpha^{\perp}\rangle_A|0\rangle_B, |\beta\rangle_A|1\rangle_B, |\beta^{\perp}\rangle_A|1\rangle_B, |\gamma\rangle_A|2\rangle_B$, and $|\gamma^{\perp}\rangle_A|2\rangle_B\}$. Obviously, if a set of states can be distinguished by the LPCC we write above, these states can be rewritten as a disjoint summation of states from the above set. ■

III. SIX STATES

Six states form a complete basis for the $2\otimes 3$ system. By a result of Horodecki *et al.* [23], we know that six states are distinguishable by LOCC only if they are product states. Conversely, by the result of Bennett *et al.* mentioned above, we conclude that any $2\otimes 3$ product basis is perfectly distinguishable by LPCC. Thus we arrive at the following.

Theorem 2. Six orthogonal $2\otimes 3$ states are perfectly distinguishable by LPCC if and only if they form a complete orthogonal product basis. Furthermore, the condition for LOCC distinguishability is the same as LPCC distinguishability.

IV. FIVE STATES

Theorem 3. Five orthogonal $2\otimes 3$ states are locally distinguishable if and only if at most one of them is entangled.

Proof. This result is a direct consequence of a more general result presented in Ref. [32]. Here we present a self-contained proof.

Suppose the nontrivial measurement performed by Bob is $\{M_m\}$. We consider rank(M_m) and sort conditions according to M_m's rank.

If rank$(M_m)=3$, then none of these states is eliminated after measurement. That implies in the next step Alice should distinguish five orthogonal states. From theorem 1 of Ref. [21], we know at most one of the five states is entangled. Because M_m is of full rank, it does not change the state's property of being entangled or separable. Thus at most one of the five original states is entangled.

If rank$(M_m)=2$, then $M_m|\beta\rangle=0$. Let B' denote the subspace orthogonal to $|\beta\rangle$, and system AB' is $2\otimes 2$. As Alice can distinguish at most four orthogonal states in AB', one state must be excluded after Bob's measurement; then it can be denoted as $|\psi_0\rangle=|\alpha\rangle_A|\beta\rangle_B$. The other four states are

$$|\psi_1\rangle = |\eta_1\rangle_{AB'} + \lambda_1|\alpha_\perp\rangle_A|\beta\rangle,$$

$$|\psi_2\rangle = |\eta_2\rangle_{AB'} + \lambda_2|\alpha_\perp\rangle_A|\beta\rangle,$$

$$|\psi_3\rangle = |\eta_3\rangle_{AB'} + \lambda_3|\alpha_\perp\rangle_A|\beta\rangle,$$

$$|\psi_4\rangle = |\eta_4\rangle_{AB'} + \lambda_4|\alpha_\perp\rangle_A|\beta\rangle. \tag{1}$$

If one of $|\eta_i\rangle$ is 0, suppose it is $|\eta_1\rangle$; then to keep orthogonality, only λ_1 is nonzero, and $|\psi_1\rangle$ is product state. Now we have three nonzero $|\eta_i\rangle$, and as $I\otimes M_m|\eta_i\rangle$ can be distinguished by Alice, directly from Ref. [21], at most one of $I\otimes M_m|\eta_i\rangle$ is an entangled state. On subsystem B', M_m is of full rank; because of full rank operator's property we used previously, at most one of the three left states $|\psi_i\rangle=|\eta_i\rangle$ is entangled. We then reach the conclusion that at most one state is entangled.

Suppose $|\eta_i\rangle\neq 0$ for each i. As Alice uses projective measurements $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ to distinguish states after Bob's measurement, the five states can be rewritten as

$$|\psi_0\rangle = |\alpha\rangle_A|\beta\rangle_B, \tag{2}$$

$$|\psi_1\rangle = |0\rangle_A|\gamma_1\rangle_B + \lambda_1|\alpha_\perp\rangle_A|\beta\rangle,$$

$$|\psi_2\rangle = |0\rangle_A|\gamma_2\rangle_B + \lambda_2|\alpha_\perp\rangle_A|\beta\rangle,$$

$$|\psi_3\rangle = |1\rangle_A|\gamma_3\rangle_B + \lambda_3|\alpha_\perp\rangle_A|\beta\rangle,$$

$$|\psi_4\rangle = |1\rangle_A|\gamma_4\rangle_B + \lambda_4|\alpha_\perp\rangle_A|\beta\rangle. \tag{3}$$

To keep the orthogonality relation $\langle\psi_k|\psi_l\rangle=0$, where $k\in\{1,2\}$ and $l\in\{3,4\}$, we have $\lambda_1=\lambda_2=0$ or $\lambda_3=\lambda_4=0$. Suppose $\lambda_3=\lambda_4=0$, λ_1 and λ_2 are not 0, and we also have $\langle\gamma_3|\gamma_4\rangle=0$.

If $|\alpha\rangle=|1\rangle$, then the five states are all product states. We suppose $|\alpha\rangle\neq|1\rangle$. Then for arbitrary $E_m=M_m^{\dagger}M_m$, the following condition is satisfied: $\langle\gamma_1|E_m|\beta\rangle=\langle\gamma_2|E_m|\beta\rangle=\langle\gamma_3|E_m|\gamma_4\rangle=0$.

We choose M_m satisfying $M_m|\beta\rangle\neq 0$. After Bob's measurement, Alice does a projective measurement $\{|0'\rangle,|1'\rangle\}$. If $\{|0'\rangle,|1'\rangle\}=\{|\alpha\rangle,|\alpha^{\perp}\rangle\}$, then as $\langle\psi_1|(|\alpha\rangle\langle\alpha|\otimes E_m)|\psi_2\rangle=0$; we have $\langle\gamma_1|E_m|\gamma_2\rangle=0$. So $\langle\psi_1|(I\otimes E_m)|\psi_2\rangle=\lambda_1\lambda_2^*\langle\beta|E_m|\beta\rangle=0$, then one of λ_1 and λ_2 is 0. If $\{|0'\rangle,|1'\rangle\}\neq\{|\alpha\rangle,|\alpha^{\perp}\rangle\}$, then after Alice's measurement, at most three states are left as Bob can now distinguish at most three states. Because $E_m|\beta\rangle\neq 0$, the first three states are not 0. So the last two states must be eliminated after Alice's measurement. Then we have $E_m|\gamma_3\rangle=0$ and $E_m|\gamma_4\rangle=0$. $|\gamma_3\rangle$ and $|\gamma_4\rangle$ form a basis of the subspace orthogonal to $|\beta\rangle$; thus $E_m=k|\beta\rangle\langle\beta|$. As $\langle\psi_1|I\otimes E_m|\psi_2\rangle=\langle\gamma_1|E_m|\gamma_2\rangle+\lambda_1\lambda_2^*\langle\beta|E_m|\beta\rangle=k\lambda_1\lambda_2^*=0$, one of λ_1 and λ_2 is 0. So one of $|\psi_1\rangle$ and $|\psi_2\rangle$ is a product state. Notice that we already have three product states: $|\psi_0\rangle$, $|\psi_3\rangle$, and $|\psi_4\rangle$; we now have four product states.

If rank$(M_m)=1$ for each m, then $E_m=M_m^{\dagger}M_m=\lambda_m|\beta\rangle\langle\beta|$ for some $\lambda_m>0$. Let B' denote the subspace orthogonal to $|\beta\rangle$. Then five states can be rewritten as

$$|\alpha\rangle_A|\beta\rangle_B + |\eta_0\rangle_{AB'},$$

$$|\alpha^{\perp}\rangle_A|\beta\rangle_B + |\eta_1\rangle_{AB'},$$

$$|\eta_2\rangle_{AB'},|\eta_3\rangle_{AB'},|\eta_4\rangle_{AB'}, \tag{4}$$

where $\langle\eta_i|\eta_j\rangle=0$.

The system AB' is $2\otimes 2$, so at most four states can be orthogonal to each other; then one of $|\eta_0\rangle_{AB'}$ and $|\eta_1\rangle_{AB'}$ should vanish. Therefore, one of $|\psi_0\rangle$ and $|\psi_1\rangle$ is a product state, and can be written as $|\alpha\rangle_A|\beta\rangle_B$ corresponding to $E_m=\lambda_m|\beta\rangle\langle\beta|$. The number of measurement operators is at least 3 in order to satisfy $\Sigma E_m=I$. If the number is exactly 3, then the measurement is actually a projective measurement. By the result in Sec. II, at most one entangled state exists. We only need to consider the case when the number is larger than 3. In this case we have at least four different measurement operators and each of them corresponds to a different product state. So there must be at least four product states.

From the discussion above, we conclude that at most one entangled state exists in the set of five orthogonal states which are distinguishable by LOCC.

Now we turn to show that any five orthogonal $2\otimes 3$ pure states containing at most one entangled state can be locally distinguished. By Ref. [3] we know that any set of orthogonal $2\otimes 3$ product states can be extended by a complete orthogonal product basis. Thus the entangled state (if it exists) is actually a superposition of two orthogonal product states which together with the other four orthogonal product states form an orthonormal product basis. We have known that an orthonormal $2\otimes 3$ product basis can be distinguished by LOCC. It is clear that the same protocol can also be used to distinguish the original five states with the slight difference that there are two measurement outcomes corresponding to the entangled state. ■

From the above proof it is obvious that five states are perfectly distinguished by LOCC if and only if they are perfectly distinguished by LPCC.

V. FOUR STATES

Now we consider the LOCC distinguishability of four states. We have the following key theorem.

Theorem 4. Four orthogonal $2\otimes 3$ states are perfectly distinguishable by LOCC only if at least two of them are product states.

Proof. We consider rank of measurement operators performed by Bob. There are three different cases.

Case 1. One measurement operator has rank 3. After Bob's measurement, no state is eliminated. So Alice has to distinguish four orthogonal states, which is possible only when at least two states are product states [21]. As a full rank measurement operator does not change the state's property of being entangled or separable, at least two original states are product states.

Case 2. One measurement operator M_1 has rank 2. Let us assume $M_1|2\rangle_B=0$. There are three subcases we need to consider.

Case 2.1. Two states can be written as $|\alpha\rangle|2\rangle$ and $|\beta\rangle|2\rangle$. Then there are already two product states.

Case 2.2. Only one state can be written as $|\alpha\rangle|2\rangle$. Then after Bob's measurement with outcome 1, three states are left. As Alice's measurement can only be a projective measurement like $\{|0\rangle,|1\rangle\}$, we can rewrite three postmeasurement states as follows:

$$|0\rangle_A|\xi_1\rangle_B + |\eta_1\rangle_A|2\rangle_B,$$

$$|1\rangle_A|\xi_2\rangle_B + |\eta_2\rangle_A|2\rangle_B,$$

$$|0\rangle_A|\xi_3\rangle_B + |1\rangle_A|\xi_4\rangle_B + |\eta_3\rangle_A|2\rangle_B.$$

To keep orthogonality, we have $\langle\eta_1|\eta_2\rangle=\langle\alpha|\eta_2\rangle=\langle\alpha|\eta_1\rangle=0$. As the dimension of Alice's system is 2, from the equation above, we have one of $|\eta_1\rangle$ and $|\eta_2\rangle$ is 0. Then there are two product states.

Case 2.3. No state can be written as $|\alpha\rangle|2\rangle$. The four original states must be written as

$$|0\rangle_A|\alpha_1\rangle_B + |\eta_1\rangle_A|2\rangle_B,$$

$$|0\rangle_A|\alpha_2\rangle_B + |\eta_2\rangle_A|2\rangle_B,$$

$$|1\rangle_A|\alpha_3\rangle_B + |\eta_3\rangle_A|2\rangle_B,$$

$$|1\rangle_A|\alpha_4\rangle_B + |\eta_4\rangle_A|2\rangle_B.$$

We choose another measurement operator M_2 satisfying $M_2|2\rangle\neq 0$ which always exists. If the condition for M_2 can be sorted into the above cases 2.1 and 2.2, then we reach the conclusion that two states are product states. So we only have to prove the case that $I\otimes M_2|\psi_i\rangle\neq 0$ for each i. From Ref. [21] we know that the local distinguishability implies that all postmeasurement states corresponding outcome 2 should be product states.

Suppose no $|\eta_i\rangle\neq 0$ is product state, then only under the condition that $M_2|\alpha_i\rangle=\lambda_i M_2|2\rangle$ could $I\otimes M_2|\psi_i\rangle$ be a product state. The states after measurement can be written as $I\otimes M_2|\psi_i\rangle=(\lambda_i|0\rangle+|\eta_i\rangle)M_2|2\rangle$ or $(\lambda_i|1\rangle+|\eta_i\rangle)M_2|2\rangle$. Notice that at most one of $|\eta_i\rangle$ is 0 which means there is at most one product state. If one of $|\eta_i\rangle$ is 0, then the state is already a product state. To keep orthogonality, one of the remaining three $\lambda_i|0\rangle+|\eta_i\rangle$ or $\lambda_i|1\rangle+|\eta_i\rangle$ is 0, which means the state is also a product state. We then have two product states. If none of $|\eta_i\rangle$ is 0, then to keep orthogonality, two $\lambda_i|0\rangle+|\eta_i\rangle$ or $\lambda_i|1\rangle+|\eta_i\rangle$ are 0, and the two states are product states.

Case 3. Each measurement operator has rank 1, and can be written as $E_i=|e_i\rangle\langle e_i|$ (unnormalized). Let $|\psi_i\rangle=|0\rangle_A|\alpha_i\rangle+|1\rangle_A|\beta_i\rangle$. After the measurement, two states must be eliminated to keep orthogonality. This means that for each $|e_i\rangle$, there are two states $|\psi_{i_1}\rangle$ and $|\psi_{i_2}\rangle$ satisfying

$$(I\otimes|e_i\rangle\langle e_i|)|\psi_{i_k}\rangle=0,\quad k=1,2,$$

where i can take at least three different values as there are at least three measurement operators. So we have at least six orthogonal equations. Then there are two states $|\psi_i\rangle$ such that both of them have two orthogonality equations, i.e., each of them is orthogonal to two $|e_k\rangle$. We can write these orthogonality equations explicitly as follows: $\langle\alpha_i|e_{i1}\rangle=\langle\alpha_i|e_{i2}\rangle=\langle\beta_i|e_{i1}\rangle=\langle\beta_i|e_{i2}\rangle=0$. As $|e_{i1}\rangle$ and $|e_{i2}\rangle$ are linearly independent, $|\alpha_i\rangle=|\beta_i\rangle$. It follows that two states should be product states. ■

But different from the condition of five or six states, we find two classes of four states which can only be distinguished by LOCC but not by projective measurements. The result suggests that LOCC are more powerful than LPCC. We list our results as two following theorems and each theorem discusses one class of states. We then prove that these two classes consist of all states which can be distinguished by LOCC but not by LPCC. Note that in the following we label four states by $|\psi_k\rangle$ with $k=1,2,3,4$.

Theorem 5. The following four orthogonal $2\otimes 3$ states can be distinguished by LOCC but not by LPCC.

$$|0\rangle_A|0\rangle_B, \quad |1\rangle_A|\alpha\rangle_B,$$

$$|0\rangle_A(a_1|1\rangle + b_1|2\rangle)_B + |1\rangle_A(c_1|\alpha^\perp\rangle + d_1|2\rangle)_B,$$

$$|0\rangle_A(a_2|1\rangle + b_2|2\rangle)_B + |1\rangle_A(c_2|\alpha^\perp\rangle + d_2|2\rangle)_B, \tag{5}$$

where $a_1a_2^*+c_1c_2^*=b_1b_2^*+d_1d_2^*=0$, $|\alpha\rangle$ and $|\alpha^\perp\rangle$ belong to span$\{|0\rangle,|1\rangle\}$, $|\alpha\rangle\neq|0\rangle$, and $k=-a_1a_2^*/b_1b_2^*=-c_1c_2^*/d_1d_2^*$ is a real number and satisfies $0<k<1$.

Proof. First, to prove these states can be distinguished by LOCC, we give a set of Bob's measurement operators:

$$M_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\sqrt{k}\end{pmatrix}, \quad M_2=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\sqrt{1-k}\end{pmatrix}.$$

If the measurement outcome is 1, then four postmeasurement states would be

$$|0\rangle_A|0\rangle_B, \quad |1\rangle_A|\alpha\rangle_B,$$

$$|0\rangle_A(a_1|1\rangle + \sqrt{k}b_1|2\rangle)_B + |1\rangle_A(c_1|\alpha^\perp\rangle + \sqrt{k}d_1|2\rangle)_B,$$

$$|0\rangle_A(a_2|1\rangle + \sqrt{k}b_2|2\rangle)_B + |1\rangle_A(c_2|\alpha^\perp\rangle + \sqrt{k}d_2|2\rangle)_B. \tag{6}$$

The above four states then can be distinguished by Alice with a projective measurement $\{|0\rangle,|1\rangle\}$.

If the measurement outcome is 2, then two left states are

$$\sqrt{1-k}(b_1|0\rangle + d_1|1\rangle)_A|2\rangle_B,$$

$$\sqrt{1-k}(b_2|0\rangle + d_2|0\rangle)_A|2\rangle_B. \tag{7}$$

We can verify that the above two states are orthogonal, and thus can be perfectly distinguished. As a result, the original four states can be perfectly distinguished by LOCC.

Next we shall show that the above four states cannot be distinguished by LPCC. Suppose Bob goes first. Since $\Sigma P_m=I$, there is a rank 1 projective measurement operator which can be written as $P_1=|\theta\rangle\langle\theta|$. If $\langle\theta|0\rangle\neq 0$, then $(I\otimes|\theta\rangle\langle\theta|)|\psi_1\rangle=\langle\theta|0\rangle|0\rangle|\theta\rangle$. To keep orthogonality between $|\psi_1\rangle$, $|\psi_3\rangle$, and $|\psi_4\rangle$, $|\theta\rangle$ should be orthogonal to $a_1|1\rangle+b_1|2\rangle$ and $a_2|1\rangle+b_2|2\rangle$. The above two states are linearly independent because if $a_1|1\rangle+b_1|2\rangle=\lambda(a_2|1\rangle+b_2|2\rangle)$, then $k=-a_1a_2^*/b_1b_2^*=-a_1a_1^*/b_1b_1^*<0$. Then $\langle\theta|1\rangle=\langle\theta|2\rangle=0$, $|\theta\rangle=|0\rangle$. However, the projector $|0\rangle\langle 0|$ cannot keep orthogonality, because we can prove that orthogonality requires $k=1$. Thus the assumption of $\langle\theta|0\rangle\neq 0$ is incorrect. So we should have $\langle\theta|0\rangle=0$.

Similarly, we can prove $\langle\theta|\alpha\rangle=0$. As $|\alpha\rangle\neq|0\rangle$, $|\theta\rangle$ is orthogonal to span$\{|0\rangle,|1\rangle\}$, $P_1=|\theta\rangle\langle\theta|=|2\rangle\langle 2|$. The other projective measurement operator can only have rank 2 and should be $P_2=|0\rangle\langle 0|+|1\rangle\langle 1|$. Unfortunately, by a similar argument we can show that $P_2|\psi_i\rangle$ cannot be distinguished by Alice. So the original four states cannot be distinguished by projective measurements if Bob goes first.

If instead of Bob going first, Alice goes first, then after Alice's measurement at most three states are left. In order to eliminate one state, Alice's measurement operators must be $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. But these operators cannot keep orthogonality between the four states. So the original four states cannot be distinguished by LOCC if Alice goes first. Thus we finish our proof. ■

An explicit example is as follows:

$$|0\rangle_A|0\rangle_B, \quad |1\rangle_A|1\rangle_B,$$

$$|0\rangle_A(|1\rangle + |2\rangle)_B + |1\rangle_A(|0\rangle - 2|2\rangle)_B,$$

$$|0\rangle_A(|1\rangle - 2|2\rangle)_B - |1\rangle_A(|1\rangle + |2\rangle)_B. \tag{8}$$

The measurement operators performed by Bob are

$$M_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\sqrt{1/2}\end{pmatrix}, \quad M_2=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\sqrt{1/2}\end{pmatrix}.$$

Another different class of states is as follows.

Theorem 6. The following four orthogonal states can be distinguished by LOCC but not by LPCC:

$$|0\rangle_A|0\rangle_B, \quad |\alpha\rangle_A|1\rangle_B,$$

$$a_1|1\rangle_A|0\rangle_B + b_1|\alpha^\perp\rangle_A|1\rangle_B + c_1|1\rangle_A|2\rangle_B,$$

$$a_2|1\rangle_A|0\rangle_B + b_2|\alpha^\perp\rangle_A|1\rangle_B + c_2|\alpha^\perp\rangle_A|2\rangle_B, \tag{9}$$

where $a_1a_2^*+b_1b_2^*+c_1c_2^*\langle\alpha^\perp|1\rangle=0$, and $k=-a_1a_2^*/c_1c_2^*\langle\alpha^\perp|1\rangle$ is a real number which satisfies $0<k<1$. $|\alpha\rangle\neq|0\rangle$ and $|1\rangle$.

Proof. Consider the following general measurement:

$$M_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&\sqrt{k}\end{pmatrix}, \quad M_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&\sqrt{1-k}\end{pmatrix}.$$

If the measurement outcome is 1, then after the measurement, three left states are

$$|0\rangle_A|0\rangle_B,$$

$$a_1|1\rangle_A|0\rangle_B + c_1\sqrt{k}|1\rangle_A|2\rangle_B,$$

$$a_2|1\rangle_A|0\rangle_B + c_2\sqrt{k}|\alpha^\perp\rangle_A|2\rangle_B. \quad (10)$$

Alice can distinguish the states using projective measurements $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. If measurement result is 2, then three left states after the measurement are

$$|\alpha\rangle_A|1\rangle_B,$$

$$b_1|\alpha^\perp\rangle_A|1\rangle_B + \sqrt{1-k}c_1|1\rangle_A|2\rangle_B,$$

$$b_2|\alpha^\perp\rangle_A|1\rangle_B + \sqrt{1-k}c_2|\alpha^\perp\rangle_A|2\rangle_B. \quad (11)$$

Alice can distinguish the three states using $|\alpha\rangle\langle\alpha|$ and $|\alpha^\perp\rangle\langle\alpha^\perp|$.

The next part is to prove that the four states cannot be distinguished by LPCC. As $\Sigma P_m = I$, there is $P_1 = |\theta\rangle\langle\theta|$. If $\langle\theta|0\rangle \neq 0$, then as $\langle\alpha|0\rangle \neq 0$, to keep orthogonality between $|\psi_1\rangle$ and $|\psi_2\rangle$ we have $\langle\theta|1\rangle = 0$. Because $\langle\alpha^\perp|0\rangle \neq 0$, to keep orthogonality between $|\psi_1\rangle$ and $|\psi_4\rangle$ we also have $\langle\theta|2\rangle = 0$; therefore $|\theta\rangle = |0\rangle$. But $|0\rangle\langle 0|$ cannot distinguish the four states, so $\langle\theta|0\rangle$ must be 0.

Using the same method, we can also first assume $\langle\theta|1\rangle \neq 0$, then we prove that $|\theta\rangle\langle\theta|$ cannot distinguish states, so $\langle\theta|1\rangle = 0$. Therefore, $|\theta\rangle = |2\rangle$. But $\langle\psi_3|(I\otimes|2\rangle\langle 2|)|\psi_4\rangle \neq 0$, so the four states cannot be distinguished by projective measurements if Bob goes first.

Suppose Alice goes first. After Alice does any operator $|\theta\rangle\langle\theta|$ there are at most three states left. As $|\psi_3\rangle$ and $|\psi_4\rangle$ are entangled states, $(|\theta\rangle\langle\theta|\otimes I)|\psi_{3(4)}\rangle \neq 0$, so one of $(|\theta\rangle\langle\theta|\otimes I)|\psi_{1(2)}\rangle = 0$. $|\theta\rangle$ must be $|1\rangle$ or $|\alpha^\perp\rangle$. But $|1\rangle\langle 1|$ and $|\alpha^\perp\rangle\langle\alpha^\perp|$ cannot keep orthogonality. So the four states cannot be distinguished if Alice goes first. Hence we finish the proof. ∎

We also give an explicit example:

$$|0\rangle_A|0\rangle_B, \quad \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)_A|1\rangle_B,$$

$$-\frac{1}{2}|1\rangle_A|0\rangle_B + \left(\frac{|0\rangle-|1\rangle}{2\sqrt{2}}\right)_A|1\rangle_B + |1\rangle_A|2\rangle_B,$$

$$\frac{1}{\sqrt{2}}|1\rangle_A|0\rangle_B - \left(\frac{|0\rangle-|1\rangle}{2}\right)_A|1\rangle_B - \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)_A|2\rangle_B.$$

The measurement performed by Bob is given by

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{1/2} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{1/2} \end{pmatrix}.$$

Interestingly, the above two classes of states completely characterize the local distinguishability of four $2\otimes 3$ states.

Theorem 7. Any four $2\otimes 3$ orthogonal states can be distinguished by LOCC but not by LPCC if and only if they can be written as one of the forms in the above two theorems.

Proof. We have proved in lemma 2 that two of the four states should be product states if they are distinguishable by LOCC.

Case 1. If the two product states can be written as $|0\rangle_A|0\rangle_B$ and $|1\rangle_A|\eta_0\rangle_B$, where $|\eta_0\rangle$ and $|\eta_0^\perp\rangle$ belong to span$\{|0\rangle,|1\rangle\}$. Then the other two states can be written as

$$|0\rangle_A|\alpha_1\rangle_B + |1\rangle_A|\eta_1\rangle_B,$$

$$|0\rangle_A|\alpha_2\rangle_B + |1\rangle_A|\eta_2\rangle_B, \quad (12)$$

where $|\alpha_1\rangle$ and $|\alpha_2\rangle$ belong to span$\{|1\rangle,|2\rangle\}$, and $|\eta_1\rangle$ and $|\eta_2\rangle$ belong to span$\{|\eta_0^\perp\rangle,|2\rangle\}$.

We assume that $|\alpha_1\rangle \neq |\alpha_2\rangle$ and $|\eta_1\rangle \neq |\eta_2\rangle$ and $|\eta_0\rangle \neq |0\rangle$. Other cases such as $|\alpha_1\rangle = \lambda|\alpha_2\rangle$ or $|\eta_1\rangle = \lambda|\eta_2\rangle$ or $|\eta_0\rangle = |0\rangle$ will be discussed later. To keep orthogonality after measurement, $\langle 0|E_m|\alpha_1\rangle = \langle 0|E_m|\alpha_2\rangle = 0$, as $|\alpha_1\rangle \neq |\alpha_2\rangle$, we have $\langle 0|E_m|1\rangle = \langle 0|E_m|2\rangle = 0$. For the same reason, $\langle\eta_0|E_m|\eta_0^\perp\rangle = \langle\eta_0|E_m|2\rangle = 0$, as $|\eta_0\rangle \neq |0\rangle$, $\langle 1|E_m|2\rangle = 0$. We obtain that E_m is diagonal under the bases $\{|0\rangle,|1\rangle,|2\rangle\}$, $E_m = \mathrm{diag}(\lambda_0,\lambda_1,\lambda_2)$. We rewrite $|\psi_3\rangle$ and $|\psi_4\rangle$ as

$$|0\rangle_A(a_1|1\rangle + a_2|2\rangle)_B + |1\rangle_A(a_3|0\rangle + a_4|1\rangle + a_5|2\rangle)_B,$$

$$|0\rangle_A(b_1|1\rangle + b_2|2\rangle)_B + |1\rangle_A(b_3|0\rangle + b_4|1\rangle + b_5|2\rangle)_B.$$

To keep orthogonality, we should have $\langle\psi_3|I\otimes E_m|\psi_4\rangle = 0$, which is equivalent to

$$a_3b_3^*\lambda_0 + (a_1b_1^* + a_4b_4^*)\lambda_1 + (a_2b_2^* + a_5b_5^*)\lambda_2 = 0.$$

There are only two linearly independent solutions to the above equation. Suppose $E_1 = \mathrm{diag}(\lambda_0,\lambda_1,\lambda_2)$ and $E_2 = \mathrm{diag}(\lambda_0',\lambda_1',\lambda_2')$ are two independent solutions. If we have another operator E_3, it must be written as $E_3 = aE_1 + bE_2$, so we can use only E_1 and E_2 to distinguish the states instead of using three or more operators. Therefore, there are only two measurement operators $E_1 = \mathrm{diag}(\lambda_0,\lambda_1,\lambda_2)$ and $E_2 = \mathrm{diag}(1-\lambda_0,1-\lambda_1,1-\lambda_2)$.

If $\lambda_0 \neq 0$, $E_1|0\rangle \neq 0$. After Alice's measurement $|\theta\rangle\langle\theta|$, $\langle\psi_1|(|\theta\rangle\langle\theta|\otimes E_1)|\psi_3\rangle = \lambda_0\langle 0|\theta\rangle\langle\theta|1\rangle\langle 0|\eta_1\rangle = 0$. If $|\theta\rangle \neq |0\rangle$ or $|1\rangle$, then as $|\eta_1\rangle$ belongs to span$\{|\eta_0^\perp\rangle,|2\rangle\}$, $|\eta_1\rangle = |2\rangle$. For the same reason $|\eta_2\rangle = |\eta_1\rangle = |2\rangle$, but we have assumed in the beginning that $|\eta_2\rangle \neq |\eta_1\rangle$. So $|\theta\rangle = |0\rangle$ or $|1\rangle$ and Alice's measurement operators are $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. Similarly, if $\lambda_1 \neq 0$, then Alice's measurement operators must also be $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$.

From $\langle\psi_3|(|0\rangle\langle 0|\otimes E_m)|\psi_4\rangle = 0$, and $\langle\psi_3|1\rangle\langle 1|\otimes E_m|\psi_4\rangle = 0$, we have $a_1b_1^*\lambda_2 + a_2b_2^*\lambda_3 = 0$ and $a_3b_3^*\lambda_1 + a_4b_4^*\lambda_2 + a_5b_5^*\lambda_3 = 0$. If $1-\lambda_1 \neq 0$ or $1-\lambda_2 \neq 0$, we also have $a_1b_1^*(1-\lambda_2) + a_2b_2^*(1-\lambda_3) = 0$ and $a_3b_3^*(1-\lambda_1) + a_4b_4^*(1-\lambda_2) + a_5b_5^*(1-\lambda_3) = 0$. Then from those equations above, $a_1b_1^* + a_2b_2^* = 0$ and $a_3b_3^* + a_4b_4^* + a_5b_5^* = 0$ stand; therefore, the four states can be distinguished by Alice first doing measurements $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. As we have assumed these four states cannot be distinguished by projective measurements, we have either $\lambda_1 = \lambda_2 = 0$ or $1-\lambda_1 = 1-\lambda_2 = 0$. So the POVM consists of $E_1 = \mathrm{diag}(1,1,k)$ and $E_2 = \mathrm{diag}(0,0,1-k)$ which is just the case in theorem 4.

Here we will discuss other conditions we mentioned in the beginning. First, if $|\eta_0\rangle = |0\rangle$, then $|0\rangle\langle 0|$ can distinguish $|\psi_1\rangle$ and $|\psi_2\rangle$. As $|\alpha_i\rangle$ and $|\eta_i\rangle$ belong to span$\{|1\rangle,|2\rangle\}$, $|1\rangle\langle 1| + |2\rangle\langle 2|$ can distinguish $|\psi_3\rangle$ and $|\psi_4\rangle$. Therefore, the four states can be distinguished by projective measurements.

Secondly, if $|\eta_2\rangle=\lambda|\eta_1\rangle$, we choose M_1 which satisfies $M_1|\eta_1\rangle\neq 0$. Then Alice's measurement cannot be $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, because $\langle\psi_3|(|1\rangle\langle 1|\otimes E_1)|\psi_4\rangle=\lambda\langle\eta_1|E_1|\eta_1\rangle\neq 0$. Then to keep orthogonality after measurements $|0'\rangle\langle 0'|\otimes M_1$ and $|1'\rangle\langle 1'|\otimes M_1$, we have $\langle\psi_1|(|0'\rangle\langle 0'|\otimes E_1)|\psi_2\rangle=\langle 0|0'\rangle\langle 0'|1\rangle\langle 0|E_1|\eta_0\rangle=0$, so $E_1|0\rangle\perp|\eta_0\rangle$. Similarly, we have $E_1|0\rangle\perp\{|\eta_0\rangle,|\alpha_1\rangle,|\alpha_2\rangle,|\eta_1\rangle\}$ and $E_1|\eta_0\rangle\perp\{|0\rangle,|\alpha_1\rangle,|\alpha_2\rangle,|\eta_1\rangle\}$.

As $|\eta_0\rangle=a_1|0\rangle+a_2|1\rangle$, $|\alpha_i\rangle=b_1|1\rangle+b_2|2\rangle$ and $|\eta_1\rangle=c_1|\eta_0^{\perp}\rangle+c_2|2\rangle$, at most one of the next two equations holds: $|\alpha_i\rangle=\lambda_i|\eta_0\rangle+\mu_i|\eta_1\rangle$ and $|\alpha_i\rangle=\lambda_i|0\rangle+\mu_i|\eta_1\rangle$.

If both of the above equations don't hold, then dimension of each set is 3. So $E_1|0\rangle=E_1|1\rangle=0$, $E_1=|2\rangle\langle 2|$, and the four states can be distinguishable by Bob's projective measurements $|0\rangle\langle 0|+|1\rangle\langle 1|$ and $|2\rangle\langle 2|$.

Without loss of generality, suppose that the second one does not hold, then we have $E_1|\eta_0\rangle=0$, and because $|\alpha_i\rangle=\lambda_i|\eta_0\rangle+\mu_i|\eta_1\rangle$ the four states can be rewritten as

$$|0\rangle|0\rangle,\ |1\rangle|\eta_0\rangle,$$

$$\lambda_1|0\rangle|\eta_0\rangle+|\beta_1\rangle|\eta_1\rangle,$$

$$\lambda_2|0\rangle|\eta_0\rangle+|\beta_2\rangle|\eta_1\rangle. \qquad (13)$$

$\langle\psi_3|(I\otimes E_1)|\psi_4\rangle=\lambda_1\lambda_2\langle\eta_0|E_1|\eta_0\rangle+\langle\beta_1|\beta_2\rangle\langle\eta_1|E_1|\eta_1\rangle=0$, so $\langle\beta_1|\beta_2\rangle=0$, then $\langle\psi_3|\psi_4\rangle=\lambda_1\lambda_2=0$. One of λ_1 and λ_2 is 0; then the four states can be distinguished by projective measurements. The condition $|\alpha_2\rangle=\lambda|\alpha_1\rangle$ can be discussed similarly.

Case 2. If two product states can be written as $|0\rangle_A|0\rangle_B$ and $|\alpha\rangle_A|1\rangle_B$, then the other two states are

$$a_1|1\rangle_A|0\rangle_B+b_1|\alpha^{\perp}\rangle_A|1\rangle_B+|\theta_1\rangle_A|2\rangle_B,$$

$$a_2|1\rangle_A|0\rangle_B+b_2|\alpha^{\perp}\rangle_A|1\rangle_B+|\theta_2\rangle_A|2\rangle_B. \qquad (14)$$

As the condition that $|\alpha\rangle=|1\rangle$ can be counted into case 1, we suppose here $|\alpha\rangle\neq|1\rangle$. If one of $|\theta_i\rangle$ is 0, then the four states can be distinguished by projective measurements $|2\rangle\langle 2|$ and $|0\rangle\langle 0|+|1\rangle\langle 1|$. So we suppose none of $|\theta_i\rangle$ is 0. We also assume that at least one of a_i or b_i is not 0, because otherwise the four states can be distinguished by projective measurements.

To keep orthogonality between $|\psi_1\rangle$ and $|\psi_2\rangle$ after Bob's measurement, $\langle 0|E_m|1\rangle=0$. And as $\langle\psi_1|I\otimes E_m|\psi_3\rangle=\langle 0|\theta_1\rangle\langle 0|E_m|2\rangle=0$ and $\langle\psi_1|I\otimes E_m|\psi_4\rangle=\langle 0|\theta_2\rangle\langle 0|E_m|2\rangle=0$. If $|\theta_1\rangle$ or $|\theta_2\rangle$ is not $|1\rangle$, then $\langle 0|E_m|2\rangle=0$. Similarly, if $|\theta_1\rangle$ or $|\theta_2\rangle$ is not $|\alpha^{\perp}\rangle$, then $\langle 1|E_m|2\rangle=0$. So E_m is diagonal, $E_m=(\lambda_1,\lambda_2,\lambda_3)$.

We also suppose $|\alpha\rangle\neq|0\rangle$. The conditions such as $|\theta_1\rangle=|\theta_2\rangle=|1\rangle$ or $|\alpha^{\perp}\rangle$, and $|\alpha\rangle=|0\rangle$ will be discussed later. We choose M_1 satisfying $M_1|0\rangle\neq 0$, and denote Alice's measurement operators as $|0'\rangle\langle 0'|$ and $|1'\rangle\langle 1'|$. As we suppose one of a_i is not 0, without losing generality, $a_1\neq 0$, then $\langle\psi_1|(|0'\rangle\langle 0'|\otimes E_1)|\psi_3\rangle=\langle 0|0'\rangle\langle 0'|1\rangle\langle 0|E_1|0\rangle=0$. $\langle 0|E_1|0\rangle\neq 0$, so either $\langle 0|0'\rangle=0$ or $\langle 0'|1\rangle=0$, then Alice's measurements should be $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$.

Similarly, we consider distinguishability between $|\psi_2\rangle$, $|\psi_3\rangle$, and $|\psi_4\rangle$. If $M_1|1\rangle$ is also not 0, then Alice's measurements should be $|\alpha\rangle\langle\alpha|$ and $|\alpha^{\perp}\rangle\langle\alpha^{\perp}|$. Notice that the two sets $\{|0\rangle\langle 0|,|1\rangle\langle 1|\}$ and $\{|\alpha\rangle\langle\alpha|,|\alpha^{\perp}\rangle\langle\alpha^{\perp}|\}$ are different as we suppose $|\alpha\rangle$ is not equal to $|0\rangle$ or $|1\rangle$, Alice cannot distinguish the four states after Bob's measurement. Therefore, only one of $M_1|0\rangle$ and $M_1|1\rangle$ is not 0. As E_m is diagonal, there are at most two linear independent solutions of $(\lambda_1,\lambda_2,\lambda_3)$ which results from similar discussion as in case 1. The two measurements can be written as: $E_1=\mathrm{diag}(1,0,k)$ and $E_2=\mathrm{diag}(0,1,1-k)$. If the result is 1, then Alice's measurements should be $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. $\langle\psi_3|(|0\rangle\langle 0|\otimes E_1)|\psi_4\rangle=k\langle\theta_1|0\rangle\langle\theta_2|0\rangle=0$, so one of $|\theta_i\rangle$ is $|1\rangle$. For the same reason, the other is $|\alpha^{\perp}\rangle$. It is the case in theorem 5.

We discuss other conditions here. First, if $|\theta_1\rangle=|\theta_2\rangle=|1\rangle$, then $|\psi_3\rangle$ and $|\psi_4\rangle$ are

$$|1\rangle_A(a_1|0\rangle+c_1|2\rangle)_B+b_1|\alpha^{\perp}\rangle_A|1\rangle_B,$$

$$|1\rangle_A(a_2|0\rangle+c_2|2\rangle)_B+b_2|\alpha^{\perp}\rangle_A|1\rangle_B. \qquad (15)$$

If $|\alpha\rangle=|0\rangle$, then the four states are all product states, and can be distinguished by projective measurements. We then suppose $|\alpha\rangle\neq|0\rangle$. To keep orthogonality between $I\otimes M_m|\psi_1\rangle$ and $I\otimes M_m|\psi_2\rangle$, $\langle 0|E_m|1\rangle=0$. $\langle\psi_2|(I\otimes E_m)|\psi_3\rangle=c_1\langle\alpha|1\rangle\langle 1|E_m|2\rangle=0$. As at least one of c_i is not 0, we have $\langle 1|E_m|2\rangle=0$.

We choose M_1 satisfying $M_1|1\rangle\neq 0$. After Bob's measurement, Alice should distinguish four states $I\otimes M_1|\psi_i\rangle$. Suppose one of Alice's measurement operators is $|0'\rangle\langle 0'|$; then $\langle\psi_2|(|0'\rangle\langle 0'|\otimes E_1)|\psi_3\rangle=\langle\alpha|0'\rangle\langle 0'|\alpha^{\perp}\rangle^{*}b_1\langle 1|E_1|1\rangle=0$. The equation is also satisfied for b_2. As we suppose one of b_i is not 0, $\langle 0'|\alpha\rangle=0$ or $\langle 0'|\alpha^{\perp}\rangle=0$. So Alice's measurement should be $|\alpha\rangle\langle\alpha|$ and $|\alpha^{\perp}\rangle\langle\alpha^{\perp}|$. As $\langle 1|E_1|0\rangle=\langle 1|E_1|2\rangle=0$, $\langle\psi_3|(|\alpha\rangle\langle\alpha|\otimes E_1)|\psi_4\rangle=\langle 1|\alpha\rangle\langle 1|\alpha\rangle^{*}(a_1\langle 0|+c_1\langle 2|)E_1(a_2|0\rangle+c_2|2\rangle)=0$, therefore $(a_1\langle 0|+c_1\langle 2|)E_1(a_2|0\rangle+c_2|2\rangle)=0$. It results in $\langle\psi_3|(|\alpha^{\perp}\rangle\langle\alpha^{\perp}|\otimes E_1)|\psi_4\rangle=\langle 1|\alpha^{\perp}\rangle\langle 1|\alpha^{\perp}\rangle^{*}(a_1\langle 0|+c_1\langle 2|)E_1(a_2|0\rangle+c_2|2\rangle)+b_1b_2^{*}\langle 1|E_1|1\rangle=b_1b_2^{*}\langle 1|E_1|1\rangle=0$; so one of b_i is 0. Then the four states can be distinguished by Bob's measurement operators $|1\rangle\langle 1|$ and $|0\rangle\langle 0|+|2\rangle\langle 2|$. Similarly, the condition that $|\theta_1\rangle=|\theta_2\rangle=|\alpha^{\perp}\rangle$ can be discussed using the above method.

Secondly, we discuss the condition that $|\alpha\rangle=|0\rangle$ while one of $|\theta_i\rangle$ is not $|1\rangle$. The four states are

$$|0\rangle_A|0\rangle_B,\quad |0\rangle_A|1\rangle_B,$$

$$|1\rangle_A(a_1|0\rangle+b_1|1\rangle)_B+|\theta_1\rangle_A|2\rangle_B,$$

$$|1\rangle_A(a_2|0\rangle+b_2|1\rangle)_B+|\theta_2\rangle_A|2\rangle_B. \qquad (16)$$

If one of $|\theta_i\rangle=|1\rangle$, we will have three product states, then these four states can be distinguished by Alice first doing measurement $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, so we suppose none of $|\theta_i\rangle$ is equal to $|1\rangle$. From orthogonality, we can get that E_m is diagonal. We choose M_1 satisfying $M_1|0\rangle\neq 0$ or $M_1|1\rangle\neq 0$ then, as we proved above, Alice's measurement should be $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. So $\langle\psi_3|(|0\rangle\langle 0|\otimes E_1)|\psi_4\rangle=\langle\theta_1|0\rangle\langle\theta_2|0\rangle\langle 2|E_1|2\rangle$

$=0$, $M_1|2\rangle=0$. As a result, if $M_1|0\rangle\neq 0$ or $M_1|1\rangle\neq 0$, then $M_1|2\rangle=0$. One measurement operator must be $|2\rangle\langle 2|$ as $\Sigma_m E_m=I$; then we get $\langle\theta_1|\theta_2\rangle=0$. These four states can be distinguished by Bob's measurement $|2\rangle\langle 2|$ and $|0\rangle\langle 0|+|1\rangle\langle 1|$. ■

VI. THREE STATES

We can easily construct a class of three states that are distinguishable by LOCC but not by LPCC as follows.

Theorem 8. Three orthogonal $2\otimes 3$ states $|\psi_i\rangle=|0\rangle_A|\eta_i\rangle_B+|1\rangle_A|\xi_i\rangle_B$, which have the following forms are distinguishable by LOCC but not by LPCC:

$$|0\rangle_A|0\rangle_B,$$

$$|0\rangle_A(a_1|1\rangle+a_2|2\rangle)_B+|1\rangle_A(a_3|0\rangle+a_4|1\rangle+a_5|2\rangle)_B,$$

$$|0\rangle_A(b_1|1\rangle+b_2|2\rangle)_B+|1\rangle_A(b_3|0\rangle+b_4|1\rangle+b_5|2\rangle)_B,$$

where a_i and b_i satisfy $a_1b_1^*+a_2b_2^*+a_3b_3^*+a_4b_4^*+a_5b_5^*=0$, $\alpha a_1b_1^*+\beta a_2b_2^*=0$, $a_3b_3^*+\alpha a_4b_4^*+\beta a_5b_5^*=0$, $\langle\eta_2|\eta_3\rangle\neq 0$, $\langle\xi_2|\xi_3\rangle\neq 0$, $\langle\eta_2|\eta_3\rangle+\langle\xi_2|\eta_2\rangle\langle\eta_2|\xi_3\rangle\neq 0$, $\langle\eta_2|\eta_3\rangle+\langle\xi_2|\eta_3\rangle\times\langle\eta_3|\xi_3\rangle\neq 0$, $|\eta_2\rangle\neq|\eta_3\rangle$, $a_3b_3^*\neq 0$, and $0<\alpha,\beta<1$.

Proof. Consider the following two-outcome measurement:

$$M_1=\begin{pmatrix}1&0&0\\0&\sqrt{\alpha}&0\\0&0&\sqrt{\beta}\end{pmatrix},\quad M_2=\begin{pmatrix}0&0&0\\0&\sqrt{1-\alpha}&0\\0&0&\sqrt{1-\beta}\end{pmatrix}.$$

If the measurement outcome is 1, then the postmeasurement states are

$$|0\rangle|0\rangle,$$

$$a_3|1\rangle|0\rangle+(a_1|0\rangle+a_4|1\rangle)\sqrt{\alpha}|1\rangle+(a_2|0\rangle+a_5|1\rangle)\sqrt{\beta}|2\rangle,$$

$$b_3|1\rangle|0\rangle+(b_1|0\rangle+b_4|1\rangle)\sqrt{\alpha}|1\rangle+(b_2|0\rangle+b_5|1\rangle)\sqrt{\beta}|2\rangle. \tag{17}$$

Due to the relationship given above, these three (unnormalized) states $I\otimes M_1|\psi_i\rangle$ are mutually orthogonal and can be distinguished if Alice performs a measurement $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$.

If the measurement result is 2, then there are only two orthogonal states $I\otimes M_2|\psi_2\rangle$ and $I\otimes M_2|\psi_3\rangle$ left, and $\langle\psi_2|I\otimes E_2|\psi_3\rangle=0$. So the two states can be distinguished by LOCC.

The next part of the proof is to prove the three states cannot be distinguished by projective measurements. Let $P_1=|\theta\rangle\langle\theta|$. We assume $\langle\theta|0\rangle\neq 0$, then, to keep orthogonality between the three states, one state should be eliminated if the measurement result is 1 as dimension of Alice's part is 2. Without losing generality, we can suppose $I\otimes P_1|\psi_3\rangle=0$, then $\langle\theta|\eta_3\rangle=\langle\theta|\xi_3\rangle=0$. From $\langle\psi_1|I\otimes P_1|\psi_2\rangle=0$, we have $\langle\theta|\eta_2\rangle=0$. The conditions in the theorem indicate that $|\eta_2\rangle$, $|\eta_3\rangle$, $|\xi_3\rangle$ are linear independent; therefore $|\theta\rangle$ does not exist. Then $\langle\theta|0\rangle$ must be equal to 0.

The left projective measurement is $P_2=|0\rangle\langle 0|+|\theta^\perp\rangle\langle\theta^\perp|$, where $|\theta^\perp\rangle$ belongs to span$\{|1\rangle,|2\rangle\}$. Notice that the necessary condition for Alice to distinguish three states is at most one state is entangled, then one of $I\otimes(|0\rangle\langle 0|+|\theta^\perp\rangle\langle\theta^\perp|)|\psi_{2(3)}\rangle$ must be product state. As $P_2|\eta_i\rangle\neq P_2|\xi_i\rangle$ and $P_2|\xi_i\rangle\neq 0$, we have $P_2|\eta_i\rangle=0$ if the state is product state. It indicates that one of $|\eta_2\rangle$ or $|\eta_3\rangle$must be orthogonal to $|\theta^\perp\rangle$. Suppose $|\theta^\perp\rangle$ is orthogonal to $|\eta_2\rangle$; then $|\eta_2\rangle=|\theta\rangle$. Because the condition $\langle\eta_2|\eta_3\rangle+\langle\xi_2|\eta_2\rangle\langle\eta_2|\xi_3\rangle\neq 0$ is satisfied, $P_1=|\theta\rangle\langle\theta|=|\eta_2\rangle\langle\eta_2|$ cannot keep orthogonality between $|\psi_2\rangle$ and $|\psi_3\rangle$. Thus, if Bob goes first, these states cannot be distinguished by LPCC.

On the other hand, suppose Alice goes first with measurement $\{|0'\rangle,|1'\rangle\}$. As $\langle\psi_1|(|0'\rangle\langle 0'|\otimes I|\psi_2\rangle=a_3\langle 0|0'\rangle\langle 0'|1\rangle=0$, Alice's measurement must be $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. Because $\langle\eta_2|\eta_3\rangle\neq 0$, $|0\rangle\langle 0|$ cannot keep orthogonality. Therefore, these three states cannot be distinguished by LPCC. ■

We give a specific example of three states which have the form in the theorem:

$$|0\rangle_A|0\rangle_B,$$

$$|0\rangle_A(3|0\rangle+3|2\rangle)_B+|0\rangle_A(|0\rangle+3|1\rangle-2|2\rangle)_B,$$

$$|0\rangle_A(3|0\rangle-2|2\rangle)_B+|0\rangle_A(2|0\rangle-1|1\rangle+|2\rangle)_B. \tag{18}$$

The measurement performed by Bob is as follows:

$$M_1=\begin{pmatrix}1&0&0\\0&\sqrt{1/3}&0\\0&0&\sqrt{1/2}\end{pmatrix},\quad M_2=\begin{pmatrix}0&0&0\\0&\sqrt{2/3}&0\\0&0&\sqrt{1/2}\end{pmatrix}.$$

It is easy to prove that the above three states can be distinguished by the above general measurement but not by projective measurements.

For three states, to determine whether they can be distinguished by LOCC is much complicated. We will give a protocol to determine whether three given orthogonal states can be distinguished.

First, the three states $|\psi_i\rangle$ are denoted as $|0\rangle_A|\alpha_i\rangle_B+|1\rangle_A|\beta_i\rangle_B$. After Bob's measurement, states become $I\otimes M_m|\psi_i\rangle$. Taking the condition for Alice to distinguish three states into consideration, the three states after Bob's measurement can be written as $|0^*\rangle_A|\xi_i\rangle_B+|1^*\rangle_A|\theta_i\rangle_B$, where $|\xi_i\rangle$ and $|\theta_i\rangle$ are two sets of orthogonal states of Bob's system; $|0^*\rangle$ and $|1^*\rangle$ are two specific bases of Alice's. In spite of coefficients, we have $\Sigma|\xi_i\rangle\langle\xi_i|=I$ and $\Sigma|\theta_i\rangle\langle\theta_i|=I$.

Suppose $|0^*\rangle=a|0\rangle+b|1\rangle$ and $|1^*\rangle=-b^*|0\rangle+a^*|1\rangle$. Then we have $|0^*\rangle\langle 0^*|\otimes M_m|\psi_i\rangle=|0^*\rangle_A M_m(a|\alpha_i\rangle+b|\beta_i\rangle)_B=|0^*\rangle_A|\xi_i\rangle_B$ and $|1^*\rangle\langle 1^*|\otimes M_m|\psi_i\rangle=|1^*\rangle_A M_m(-b^*|\alpha_i\rangle+a^*|\beta_i\rangle)_B=|1^*\rangle_A|\theta_i\rangle_B$.

Let $|\phi_i\rangle$ denote $a|\alpha_i\rangle+b|\beta_i\rangle$, then we can construct another set of states $|\eta_i\rangle$ satisfying $\langle\eta_i|\phi_j\rangle=0$ for any $j\neq i$. Because $|\xi_i\rangle$ is a set of orthogonal states, $\langle\xi_i|\xi_j\rangle=\langle\xi_i|M_m|\phi_j\rangle=0$. Comparing to the definition of $|\eta_i\rangle$, we have $|\eta_i\rangle=M_m|\xi_i\rangle$. So we can choose positive numbers λ_i, to have the following equation satisfied: $\Sigma\lambda_i|\eta_i\rangle\langle\eta_i|=\Sigma M_m^\dagger|\xi_i\rangle\langle\xi_i|M_m=M_m^\dagger M_m=E_m$.

If we let $|\varphi_i\rangle = -b^*|\xi_i\rangle + a^*|\theta_i\rangle$ then, using the same method, we can find $\langle \mu_i|\varphi_j\rangle = 0$, for any $j \neq i$. We can also choose proper positive numbers ν_i to have the following equation satisfied: $\Sigma \nu_i|\mu_i\rangle\langle\mu_i| = E_m$.

We finally have the equation

$$\sum \nu_i|\mu_i\rangle\langle\mu_i| = \sum \lambda_i|\eta_i\rangle\langle\eta_i| = E_m.$$

There are eight independent variables a, b, λ_i, ν_i and nine equations. Getting value of the variables which satisfy the above equation, we can construct a set of POVM to distinguish the three given states. From the equation, we can see that it is much more difficult than four states' condition to judge whether the three states can be distinguished by LOCC. Actually we cannot provide an analytical characterization. Nevertheless, we can still get some results qualitatively.

If the equation is satisfied for any a and b, we can adjust a and b to make E_m satisfy $\Sigma_m E_m = I$. If the equation is satisfied for a certain value a_0 and b_0, then we only have $E_0 = I$. Therefore, Bob can only do a trivial operation on his system. Now we only need to judge whether these states can be distinguished if Alice goes first, which is much easier. If there is no solution to the equation, then these three states cannot be locally distinguished.

VII. NONTRIVIAL EXAMPLE REQUIRING MULTIROUND CLASSICAL COMMUNICATION

Now we turn to discuss the role of classical communication in local discrimination. We find a set of $m \otimes n$ states needs at least $2\min\{m,n\} - 2$ rounds to be distinguished using LOCC.

First, suppose $m = n$, where m is the dimension of the first system, Alice's system, and n is the dimension of the second system, Bob's system. We construct a set of states as follows:

$$\begin{array}{lllll}
|0\rangle|\eta_{00}\rangle + & |\alpha_{00}\rangle|0\rangle + & |1\rangle|\eta_{10}\rangle + & \cdots + & |n-1\rangle|n-2\rangle, \\
|0\rangle|\eta_{01}\rangle, & |\alpha_{01}\rangle|0\rangle, & |1\rangle|\eta_{11}\rangle, & \cdots, & |n-1\rangle|n-1\rangle, \\
|0\rangle|\eta_{02}\rangle, & |\alpha_{02}\rangle|0\rangle, & |1\rangle|\eta_{12}\rangle, & \cdots, & \\
\vdots & \vdots & \vdots & \cdots, & \\
|0\rangle|\eta_{0n-1}\rangle, & |\alpha_{0n-2}\rangle|0\rangle, & |1\rangle|\eta_{1n-2}\rangle, & \cdots, &
\end{array}$$

where $\{|\eta_{ki}\rangle, 0 \le i \le n-k-1\}$ is an orthonormal basis for $\text{span}\{|k\rangle, \ldots, |n-1\rangle\}$ and $\{|\alpha_{li}\rangle, 0 \le i \le n-l-2\}$ is an orthonormal basis for $\text{span}\{|l+1\rangle, \ldots, |n-1\rangle\}$. $\langle \eta_{k1}|\eta_{l1}\rangle \neq 0$, $\langle \alpha_{k1}|\alpha_{l1}\rangle \neq 0$, and $\langle \eta_{k0}|i\rangle \neq 0$ for $k \le i \le n-1$, $\langle \alpha_{l0}|j\rangle \neq 0$ for $l+1 \le j \le n-1$. The total number of the states is $n^2 - 2n + 3$.

Theorem 9. The above $n^2 - 2n + 3$ states need at least $2n - 2$ rounds classical communication to be distinguishable by LOCC.

Proof. The key idea is to prove that measurement operators should be projective measurements. Suppose Alice goes first, and let E_m denote Alice's POVM operator with outcome m. As $\langle \eta_{01}|\eta_{k1}\rangle \neq 0$ and the orthogonality between $|0\rangle_A|\eta_{01}\rangle$ and $|k\rangle_A|\eta_{k1}\rangle$ should be kept after the measurement, we have $\langle 0|E_m|k\rangle = 0$; similarly, $\langle j|E_m|k\rangle = 0$. Therefore, E_m is diagonal and $E_m = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$.

To keep orthogonality of $|\psi_0\rangle$ and $|\alpha_{0i}\rangle_A|0\rangle_B$, E_m should also be diagonal under the bases $\{|\alpha_{0i}\rangle, |0\rangle\}$, then $E_m = \lambda'_0|\alpha_{00}\rangle\langle\alpha_{00}| + \lambda'_1|\alpha_{01}\rangle\langle\alpha_{01}| + \cdots + \lambda'_{n-2}|\alpha_{0n-2}\rangle\langle\alpha_{0n-2}| + \lambda'_{n-1}|0\rangle\langle 0|$. From the restriction in the theorem, we have $\langle \alpha_{00}|j\rangle \neq 0$, for any $j \neq 0$. Therefore, $\langle \alpha_{00}|E_m|j\rangle = \lambda_j\langle\alpha_{00}|j\rangle = \lambda'_0\langle\alpha_{00}|j\rangle$, $\lambda_j = \lambda'_0$, so $E_m = \text{diag}(\lambda_0, \lambda'_0, \ldots, \lambda'_0)$.

If λ'_0 and λ_0 are both not 0, then after Alice's measurement, Bob should do a nontrivial operation on his own system according to Alice's result. We denote F_n as Bob's operator. As we discussed above, we can conclude that F_n is diagonal on bases $\{|0\rangle, |1\rangle, \ldots, |n-1\rangle\}$. To keep orthogonality of $|\psi_0\rangle$ and $|0\rangle_A|\eta_{0j}\rangle_B$, we can also rewrite $F_n = \mu'_0|\eta_{00}\rangle\langle\eta_{00}| + \mu'_1|\eta_{01}\rangle\langle\eta_{01}| + \cdots + \mu'_{n-1}|\eta_{0n-1}\rangle\langle\eta_{0n-1}|$. Following the steps above, as $\langle \eta_{00}|j\rangle \neq 0$, we have $\mu_j = \mu'_0$ for arbitrary j. Thus $F_n = \mu'_0 I$ is a trivial operator. Finally, either $\lambda_0 = 0$ or $\lambda'_0 = 0$.

Notice that this result also suggests that these states cannot be distinguished if Bob goes first. As we can see the process as Alice first does a diagonal operator on her system, $\lambda_0 = \lambda'_0 = 1$. As they are both not 0, we have proved in the above paragraph that, after Alice's measurement, these states cannot be distinguished.

We go back to Alice's first nontrivial measurement. Due to the above result, Alice's measurement only has two measurement operators: $E_1 = \text{diag}(1, 0, \ldots, 0)$ and $E_2 = \text{diag}(0, 1, \ldots, 1)$. If the measurement outcome is 1, Bob only needs to do projective measurements to distinguish the left states. If the measurement outcome is 2, the system is then $(n-1) \otimes n$.

It is then Bob's turn to do measurement. Following the method we used above, we can similarly prove that Bob's measurement must be $E_1 = \text{diag}(1, 0, \ldots, 0)$ and $E_2 = \text{diag}(0, 1, \ldots, 1)$. By induction, we find the number of rounds needed for distinguishing is $2n - 2$. Hence we complete the proof. ∎

In general case, $m \neq n$; we can suppose $m < n$, then to distinguish the set of states we give in the theorem $2m - 2$ rounds are needed. We can also construct a set of states which requires $2m - 1$ rounds to achieve a perfect discrimination. An explicit construction is as follows:

$$\begin{array}{lllll}
|\alpha_{00}\rangle|0\rangle + & |0\rangle|\eta_{00}\rangle + & |\alpha_{10}\rangle|1\rangle + & \cdots + & |m-1\rangle|m-2\rangle, \\
|\alpha_{01}\rangle|0\rangle, & |0\rangle|\eta_{01}\rangle, & |\alpha_{11}\rangle|1\rangle, & \cdots, & |m-1\rangle|m-1\rangle, \\
|\alpha_{02}\rangle|0\rangle, & |1\rangle|\eta_{02}\rangle, & |\alpha_{12}\rangle|1\rangle, & \cdots, & \\
\vdots & \vdots & \vdots & \cdots, & \\
|\alpha_{0m-1}\rangle|0\rangle, & |1\rangle|\eta_{0n-2}\rangle, & |\alpha_{1m-2}\rangle|1\rangle, & \cdots, &
\end{array}$$

where $\{|\eta_{ki}\rangle, 0 \le i \le n-k-2\}$ is an orthonormal basis for $\text{span}\{|k+1\rangle, \ldots, |n-1\rangle\}$ and $\{|\alpha_{li}\rangle, 0 \le i \le m-l-1\}$ is an orthonormal basis for $\text{span}\{|l\rangle, \ldots, |n-1\rangle\}$. $\langle \eta_{k1}|\eta_{l1}\rangle \neq 0$, $\langle \alpha_{k1}|\alpha_{l1}\rangle \neq 0$, and $\langle \eta_{k0}|i\rangle \neq 0$ for $k+1 \le i \le n-1$, $\langle \alpha_{l0}|j\rangle \neq 0$ for $l \le j \le m-1$.

The proof of the example above is almost the same as the previous one.

VIII. CONCLUSION

We have studied the local distinguishablity of $2\otimes 3$ states when the owner of the qutrit performs the first nontrivial measurement. We surprisingly find that for certain four or three states we need to perform the general local measurement in order to achieve a perfect discrimination; only LPCC is not sufficient. We have almost completely characterized the local distinguishability of $2\otimes 3$ states except for some special cases when only three states are under consideration. It would be of great interest to extend these results to $2\otimes n$ states where $n>3$.

We further construct a special set of $m\otimes n$ states which require at least $2\min\{m,n\}-2$ rounds of classical communication to finish the discrimination. Our result indicates that classical communication plays a crucial role in local discrimination. An interesting open problem is to construct a set of states which may require more rounds to achieve a perfect discrimination.

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