Methods for producing decoherence-free states and noiseless subsystems using photonic qutrits

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(Received 14 September 2007; published 14 January 2008)

We outline a proposal for a method of preparing an encoded two-state system (logical qubit) that is immune to collective noise acting on the Hilbert space of the states supporting it. The logical qubit is comprised of three photonic three-state systems (qutrits) and is generated by the process of spontaneous parametric down conversion. The states are constructed using linear optical elements along with three down-conversion sources, and are deemed successful by the simultaneous detection of six events. We also show how to select a maximally entangled state of two qutrits by similar methods. For this maximally entangled state we describe conditions for the state to be decoherence-free which do not correspond to collective errors.

DOI: 10.1103/PhysRevA.77.012314 PACS number(s): 03.67.Pp, 03.67.Lx, 42.50.Dv

I. INTRODUCTION

Quantum information and quantum technologies have been under active investigation for their practical benefits for quite some time. However, only fairly recently have researchers begun to investigate more thoroughly the properties and benefits of using higher dimensional Hilbert spaces for these purposes. Higher dimensional systems (in analogy with the term qubit for two state systems, d-state systems are hereafter referred to as qudits) have properties which are quite different from their two-state counterparts which could be useful for quantum information processing. For example, we note that d-state systems can be more entangled than qubits [1–3] and can share a larger fraction of their entanglement [4]. We also note that the monogamy rule is different, but not well understood for qudits [5]. These properties, as well as the larger dimension alone, could aid many quantum information processing tasks, including quantum key distribution [6–9], quantum bit commitment [10,11], quantum computing [12–15], and quantum games. They also seem to be required, in some form, for the solution to a version of the Byzantine agreement problem [16].

As the theoretical benefits become more well-known, proof-of-principle experiments will help in our understanding of the behavior of qudits for quantum information processing as well as a better understanding of quantum mechanics itself. In this regard, we note several interesting experiments using qudits. In the context of quantum computing nuclear magnetic resonance techniques have been used to process information encoded in single-qutrit systems [17]. Qudits have also been used in quantum cryptography to improve reliable detection of eavesdroppers [18,19], and to demonstrate the tossing of quantum coins [20].

In large part, the differences between qubits and higher dimensional systems stems from the difference in the representation theory of the group which governs their closed system evolution. For a d-state system this group can be taken to be SU(d) if we neglect an overall phase. Fundamental differences include entanglement properties as well as positivity conditions. (It is known that the two are related, see [21].) Such conditions indicate important aspects of various entanglements between systems and environments [22,23]. This could be vital for ensuring reliable quantum information processing such as error correction for open systems which are initially coupled with their environment in some way [24]. It is also clear that these differences have implications for cryptography, shared reference frames [25–30], decoherence-free subspaces (DFSs), and noiseless subsystems (NSs) [31–37] (see also [38,39] for reviews), through the difference in selection rules for these systems.

In this paper, we aim to provide a method for producing, albeit indirectly, entanglement between three qutrits. One particular type of entangled state we present forms a noiseless state which is protected against noises which are not collective and another is a NS protected against collective errors. This subsystem is the smallest subsystem of three qudits which can be formed which is immune to collective noise on the set of qudit states. For simplicity, we use qutrits ($d=3$) though in principle any d could be used. Here we propose an experimental realization of a noiseless subsystem of three qudits. Our experiment uses down converted photons and linear optical elements to form our NS, the realization of which relies on the simultaneous detections of six photons. This provides a proof-of-principle experiment for the formation and manipulation of quantum optical entangled qutrits.

Section II describes the logically encoded states which protects information from collective errors. Section III describes the use of the polarization of photons to encode qutrit states. We then describe, in Sec. IV, methods of selecting these DFS and NS states using experimental arrangements consisting of nonlinear crystals, mirrors, wave plates, beam splitters, and detectors. We conclude with a discussion in Sec. V.

II. ENCODING LOGICAL STATES

In this section we discuss the encoding of logical decoherence-free states and noiseless subsystems using stan-

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dard computational basis states. The corresponding physical states are described in the following section.

A. Encoding into a noiseless subsystem

In [37] it was shown that the Hilbert space of three physical qutrits can be used to support a single decoherence-free qubit. Each logical state of this decoherence-free qubit is given by a superposition of eight states labeled $\psi_i^{8,j}$, where the superscript j is a degeneracy label used to distinguish the two logical values, i.e., $j=0,1$. The constant superscript 8 reflects the dimension of the irreducible representation of SU(3), and the subscript i labels a particular basis state in the representation. Using this notation the general form of the logical zero state is given by

$$|0_L\rangle = \sum_i \alpha_i \psi_i^{8,0}. \tag{2.1}$$

Similarly, the most general logical one state is given by

$$|1_L\rangle = \sum_i \beta_i \psi_i^{8,1}. \tag{2.2}$$

The initialization itself is arbitrary as described in Ref. [40]. According to the theory of noiseless subsystems, these sets of states $\psi_i^{8,0}$ ($\psi_i^{8,1}$) will mix with each other, but not with $\psi_i^{8,1}$ ($\psi_i^{8,0}$) under collective noise. As long as this condition holds, the information encoded in $|\psi_L\rangle = a|0_L\rangle + b|1_L\rangle$ will be protected. The simplest form of logical zero results when all but a single expansion coefficient are set to zero. If we let $\alpha_{i'}$ remain, logical zero takes the form

$$|0_L\rangle = \alpha_{i'} \psi_{i'}^{8,0}. \tag{2.3}$$

Suppose that $i'=3$, then, after normalizing this state ($\alpha_3=1$), logical zero becomes

$$|0_L\rangle = \psi_3^{8,0}. \tag{2.4}$$

The complete set of eight basis states of logical zero $\psi_i^{8,0}$ were given explicitly in Ref. [37] in terms of five quantum numbers. After translating these states into a computational basis of the three-level system, where the three orthogonal states are described by the kets $|0\rangle$, $|1\rangle$, and $|2\rangle$, the expression for our example can be taken to be

$$|0_L\rangle = (|011\rangle - |101\rangle)/\sqrt{2}, \tag{2.5}$$

where we have used, and will continue to use, the shorthand notation $|ABC\rangle$ for the tensor product of the three qutrit states $|A\rangle \otimes |B\rangle \otimes |C\rangle$. In a similar fashion the logical one state can be observed to take a simple form by choosing to let all but one of the expansion coefficients β_i vanish. If we choose the nonzero coefficient to be β_8, then, after normalization, the logical one state is given explicitly by

$$|1_L\rangle = (-|021\rangle + |120\rangle - |201\rangle + |210\rangle)/2. \tag{2.6}$$

For initialization of the logical qubit state, we may create an arbitrary superposition of these two logical basis states. After undergoing collective decoherence effects, the state $|0_L\rangle$ ($|1_L\rangle$) will be of the form Eq. (2.1) [Eq. (2.2)].

B. Encoding maximally entangled states

It was noted and discussed in [37] that the two inequivalent fundamental representations of SU(d), $d\geq 3$ can have a fundamentally different impact on the decoherence-free subspaces and noiseless subsystems which they encode. The simplest example exhibiting such a difference is the combination of two qutrits of different types. In the case that one of the two inequivalent irreducible representations (irreps) is of one type and the other of the other type, a singlet state may be formed. In the case that they are both the same, a singlet is not readily available. This is due to the fact that the two different representations are conjugate to each other and thus transform in "opposite" ways.

More specifically, in the notation of [37], a singlet state of two qutrits can be represented as

$$|\Phi_s\rangle = \frac{1}{\sqrt{3}}(|0\bar{0}\rangle + |1\bar{1}\rangle + |2\bar{2}\rangle). \tag{2.7}$$

In this case, the transformation properties are such that one of the qutrits will transform according to the conjugate representation of the other. Parametrizations of these two different transformations were given as $D^{(0,1)}$ and $D^{(1,0)}$ in Ref. [41].

Alternatively, the barred states could be comprised of two unbarred states. That is, they may arise from an entangled state of two unbarred representations. In this case, the state would appear as the totally antisymmetric state of three qutrits,

$$|\Phi_s\rangle = \frac{1}{\sqrt{6}}(|012\rangle + |120\rangle + |201\rangle - |102\rangle - |021\rangle - |210\rangle), \tag{2.8}$$

and the state is invariant under collective transformations, that is, transformations which act the same on each of the three qutrit states. (It is interesting to note that this particular state of three qutrits has cast doubt on the monogamy relation for qudits [5].)

In a given experiment which produces a two qutrit state, the transformation properties may not be important. It could be that we only want to produce a particular state in the Hilbert space of two qutrits. Here, our objective is to describe experiments which enable the selection and exploration of qutrit states. We therefore describe a method of selecting two qutrit states in Sec. IV E before discussing the transformation properties.

Both the logical qubit as well as the singlet state have now been encoded in terms of the computational basis states of three qutrits. This encoding is, however, quite generic. As mentioned above, these three orthogonal states in which a qutrit can be described are simply referred to as $|0\rangle$, $|1\rangle$, and $|2\rangle$. In the next section we will establish a connection between these generic labels and the polarization states of a pair of photons.

III. QUTRIT ENCODING USING PHOTON POLARIZATION

The preparation of three-level optical quantum systems has been demonstrated in Ref. [42] using the polarization states of down-converted photon pairs. There, incident photons with frequency ν entering a nonlinear β-barium borate (BBO) crystal spontaneously split into two entangled photons each of which having a frequency of roughly $\nu/2$ (conservation of energy ensures that the frequencies of the down-converted photons sum to the frequency of the pump photon, selection of pairs with common frequency can be accomplished by placing narrow-band frequency filters outside of the crystal) and linearly polarized either along the same axis or along orthogonal axes depending on how the crystal is cut. Two orthogonal polarization modes can be constructed using the former polarization state, corresponding to type-I phase matching conditions, by a physical rotation of the crystal by 90°. The latter polarization state of the down-converted biphotons results from passage through a type-II phase matched crystal.

Twin photons of equal wavelength produced in a type-I crystal have collinear polarization vectors and thus propagate through the crystal with equal group velocities and emerge from the crystal on a cone due to the conservation of transverse momentum. Spatial mode selection of a pair may be obtained by placing optical fibers or small apertures in a screen on regions of the cone that are on opposite sides of the pump beam. In type-II down-conversion photon pairs are orthogonally polarized and experience different refractive indices inside the crystal. Their respective group velocities are not equal and they emerge on two separate cones, one cone for each of the two orthogonal polarizations. By adjusting the angle between the crystal optic axis and the pump beam the authors in Ref. [43] demonstrated how the cones can be made to overlap, thereby producing entangled Bell-type states along the two directions of cone intersection. Along these directions the light can be described by the state

$$|\Psi\rangle = (|HV\rangle + e^{i\alpha}|VH\rangle)/\sqrt{2}, \tag{3.1}$$

where $|H\rangle$ and $|V\rangle$ indicate horizontal and vertical polarization, respectively. The relative phase α, arising from the crystal birefringence, can be arbitrarily chosen by an appropriate crystal rotation or by using an additional birefringent phase shifter [43]. All four Bell states were realized as special cases of this general state, in particular, the states

$$|\psi^{+}\rangle = (|HV\rangle + |VH\rangle)/\sqrt{2}, \tag{3.2}$$

and

$$|\phi^{+}\rangle = (|HH\rangle + |VV\rangle)/\sqrt{2}, \tag{3.3}$$

relevant for our purposes here, were obtained by setting α equal to zero [for both Eqs. (3.3) and (3.2)] and placing a half-wave plate in one photon path angled to rotate horizontal polarization to vertical and vice versa [for Eq. (3.3)]. An alternative approach to the creation of the same entangled states was demonstrated in Ref. [44] by replacing the single type-II crystal with two adjacent type-I crystals oriented at 90° with respect to each other.

Suppose now, for simplicity, that we choose the set ($|HH\rangle$, $|VH\rangle$, $|VV\rangle$) as a basis for the qutrit states. In order to connect these basis states with the previously labeled basis states $|0\rangle$, $|1\rangle$, and $|2\rangle$ let us make the following definitions [45]:

$$|0\rangle \equiv |VV\rangle,$$

$$|1\rangle \equiv |VH\rangle,$$

$$|2\rangle \equiv |HH\rangle. \tag{3.4}$$

These relations can then be used to encode the logical zero and one states of the decoherence-free qubit:

$$|0_L\rangle = (|V_1V_2V_3H_4V_5H_6\rangle - |V_1H_2V_3V_4V_5H_6\rangle)/\sqrt{2},$$

$$|1_L\rangle = (-|V_1V_2H_3H_4V_5H_6\rangle + |V_1H_2H_3H_4V_5V_6\rangle - |H_1H_2V_3V_4V_5H_6\rangle + |H_1H_2V_3H_4V_5V_6\rangle)/2. \tag{3.5}$$

Each of the three qutrits have now been encoded in terms of the polarization states of entangled photons produced via down conversion. The numbers attached to the H's and V's serve as spatial mode labels and indicate a specific photon having a polarization of the type to which it is attached, for example, the state $|H_1H_2\rangle$ corresponds to a qutrit state represented by two horizontally polarized photons labeled photon 1 and photon 2, etc.

Now that a physical encoding has been established, the next step is to produce the superpositions given by Eqs. (3.5).

IV. DFS STATE PREPARATION

Polarizing beam splitters (PBSs), wave plates, and photodetectors have been widely used for the preparation of quantum states [46–48] and in the implementation of quantum gates [49–52]. Before discussing the details of our proposal, we first review the basic idea, given in Ref. [48], and the extension [47] on which it rests. During our discussion of the experiments we will assume that each of the nonlinear crystals being used simultaneously emits one pair of photons and that our detectors are highly efficient. We will also assume that we can eliminate which-way information through, for example, the insertion of appropriate narrow-band filters as discussed in [48,46]. These assumptions will simplify our argument.

A. Selection of a four-particle Greenberger-Horne-Zeilinger state

Consider the arrangement in Fig. 1. Ultraviolet light is pumped through two separate parametric down-conversion (PDC) sources (e.g., BBO, BiBO crystals) which, upon successful down conversion, emit pairs of photons in the Bell state Eq. (3.3), see [53]. The beam splitter is made to transmit horizontally polarized light while reflecting light polarized along the vertical axis. Therefore, the photons reaching detector $D2$ are those from path 2 (path 3) having exited the

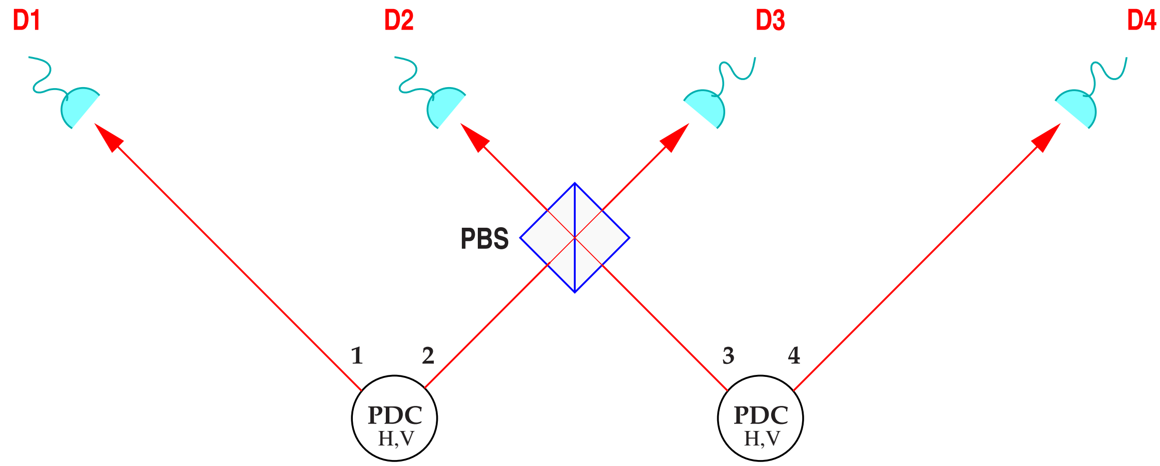

FIG. 1. (Color online) Four-photon polarization-entanglement source [48].

PBS with a vertical (horizontal) polarization. Similarly, photons reaching detector $D3$ arrive with a vertical (horizontal) polarization after entering the PBS via path 3 (path 2).

With a fourfold coincidence detection, one can infer that the polarization state of the four-photon system being detected is either $|H_1H_2H_3H_4\rangle$ or $|V_1V_2V_3V_4\rangle$. The reason is this; in order for detectors $D2$ and $D3$ to fire at the same time one of two photons being detected must have come from the left crystal via path 2 and the other from the right crystal via path 3, and as mentioned above, if photons coming from paths 2 and 3 have orthogonal polarizations they will both arrive at the same detector thereby ruling out a fourfold coincidence detection. If detectors $D2$ and $D3$ fire simultaneously both photons being detected are projected into the same polarization state which then implies that the photons arriving at detectors $D1$ and $D4$ have that same polarization since pairs created in these type-II crystals are emitted in the state Eq. (3.3). Given our assumptions of path indistinguishability [46,48], the state $(1/\sqrt{2})(|H_1H_2H_3H_4\rangle+|V_1V_2V_3V_4\rangle)$.

The optical arrangement just discussed was extended in [47] to include an additional nonlinear crystal is produced. Using three crystals, along with the action of a Hadamard gate, the authors just mentioned were able to produce a six-photon "cluster" state. We will briefly discuss their method in order to clarify the details of our proposal.

B. Selection of a six-particle Greenberger-Horne-Zeilinger state

The setup shown in Fig. 2 is an extension of the previous arrangement which includes an additional nonlinear crystal along with a Hadamard gate (a wave plate set to rotate the polarization of a photon by 45°) placed in path 4. It is not difficult then to convince oneself that in the absence of the Hadamard gate, a sixfold coincidence detection projects this system into a state that can be described by the six-photon Greenberger-Horne-Zeilinger (GHZ) superposition given by $(1/\sqrt{2})(|H_1H_2H_3H_4H_5H_6\rangle+|V_1V_2V_3V_4V_5V_6\rangle)$. In fact, any n-photon GHZ state $(n=2,4,6,\ldots)$ of the form $(1/\sqrt{2})\times(|H_1H_2\ldots H_n\rangle+|V_1V_2\ldots V_n\rangle)$ can be selected using the same type of extension along with an n-fold coincidence detection.

The situation changes when a Hadamard gate $\mathbf{H}$ is placed in path 4. The action of this gate on the states $|H\rangle$ and $|V\rangle$ is such that

$$\mathbf{H}|H\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), \tag{4.1}$$

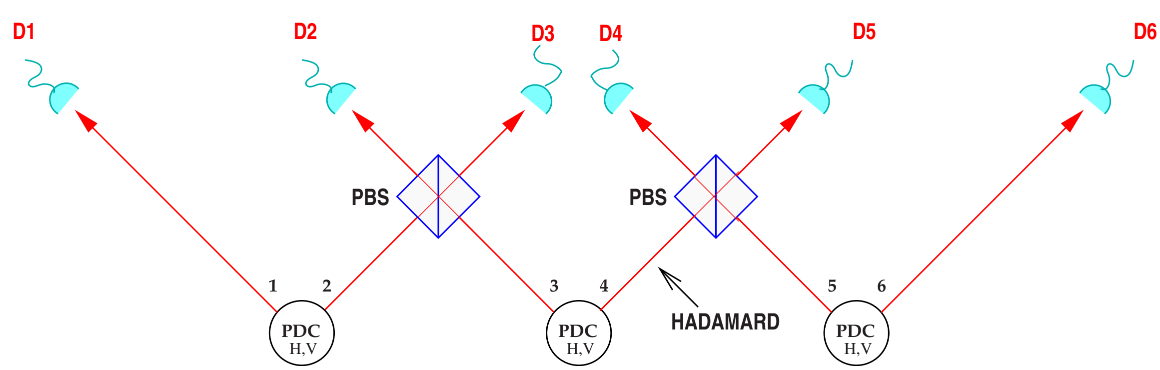

FIG. 2. (Color online) Six-photon polarization-entanglement source.

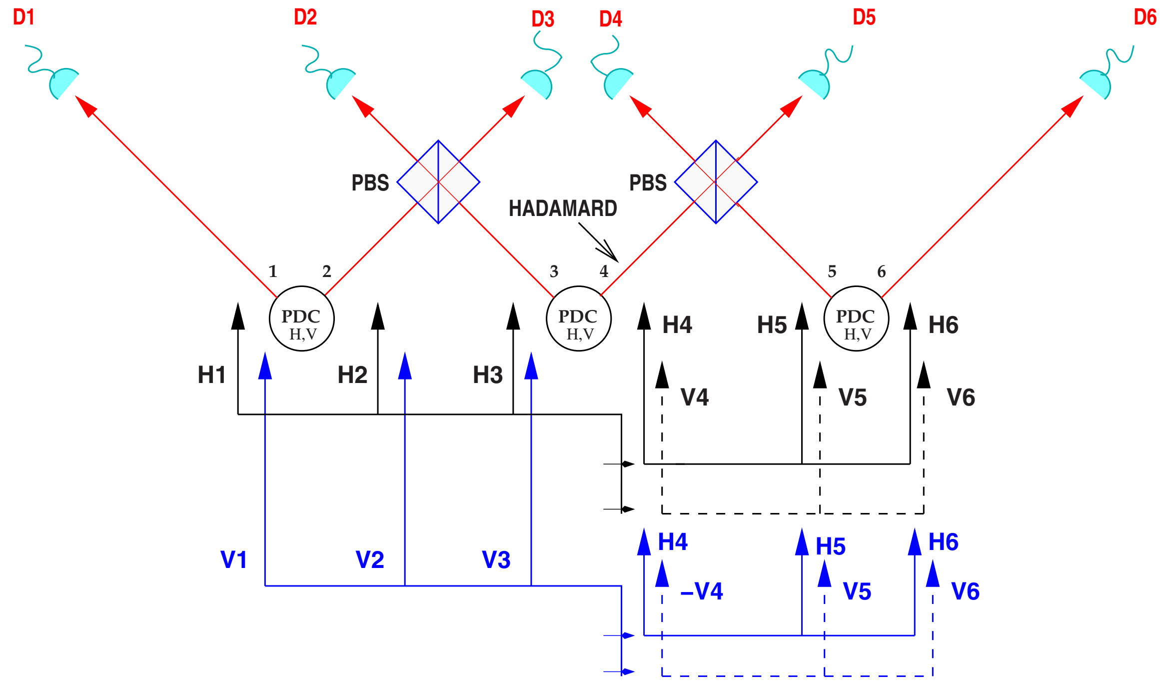

FIG. 3. (Color online) Photon states compatible with six simultaneous detections.

$$\mathbf{H}|V\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle). \tag{4.2}$$

The inclusion of this gate leads to even more possible states that are compatible with a sixfold coincidence detection. These new states are shown in Fig. 3 using the lines below the experimental setup. The uppermost line corresponds to the left crystal emitting a photon pair which is subsequently detected at $D1$ and $D3$. In order for $D2$ to fire simultaneously along with the others it must then detect the presence of a photon emitted into path 3 that exited the PBS horizontally polarized. The Hadamard gate in path 4 rotates the polarization of the photon passing through it so that the chances of it being detected at either $D4$ or $D5$ are equally likely. This Hadamard rotation splits the upper line into two equally probable events that are compatible with a sixfold coincidence detection. These possibilities can be seen by following the two lines, one solid and one dashed, that split off from the original.

The case where the left crystal emits a pair which is subsequently detected at $D1$ and $D2$ is represented by the lower set of lines (blue lines) in Fig. 3. All four of these possibilities could trigger a sixfold detection, and since these photon paths are indistinguishable such a detection postselects the six-photon "cluster" state

$$|C_6\rangle = \tfrac{1}{2}(|H_1H_2H_3H_4H_5H_6\rangle + |H_1H_2H_3V_4V_5V_6\rangle + |V_1V_2V_3H_4H_5H_6\rangle - |V_1V_2V_3V_4V_5V_6\rangle) \tag{4.3}$$

realized in Ref. [47]. With these examples in mind we proceed with the preparation of a logical decoherence-free qubit encoded in terms of the six-photon GHZ states given in Eq. (3.5).

C. Physical preparation of logical zero

In Fig. 4 we present a pictorial illustration of a setup which can be used to prepare the decoherence-free logical zero state given in Eq. (3.5). The arrangement is similar to that used in Ref. [47] in that its photon supply depends on three down-conversion sources and in the use of polarization beam splitters to direct orthogonally polarized photons down separate paths. Also, successful state preparation again rests on the simultaneous detection of six photons, a pair from each of the three different crystals. The three crystals, PDC's 1, 2, and 3, are each fed ultraviolet light by the same source and emit photons in the Bell state given by Eq. (3.2). For convenience, let us refer to the photon emitted into the particular path i as photon i, so, for example, the photon emitted by PDC2 into path 3 will be called photon 3, etc. Now, occasionally these independent down-conversion processes occur in the three crystals at the appropriate times needed for six simultaneous detections. If all optical path lengths are equal, a sixfold coincidence detection requires nearly simultaneous down conversion to take place in each of the three crystals. Note that we may assume that all optical paths are equal since if they were not, the path lengths can be adjusted by placing mirrors in some or all of the paths and varying $\Delta d1, \Delta d2, \ldots, \Delta d6$.

Assuming these adjustments have been made correctly, and that no down-converted photons have been lost, there will be nearly simultaneous recordings from $D1$, $D2$, $D3$, and $D6$ due to the respective detections of photons 3, 1, 5, and 6. This results because there are no other paths for them to take (note the opaque block serving as a photon trap placed between the PBSs adjacent to $D2$ and $D3$). This combined four photon system is thus detected in the polarization state

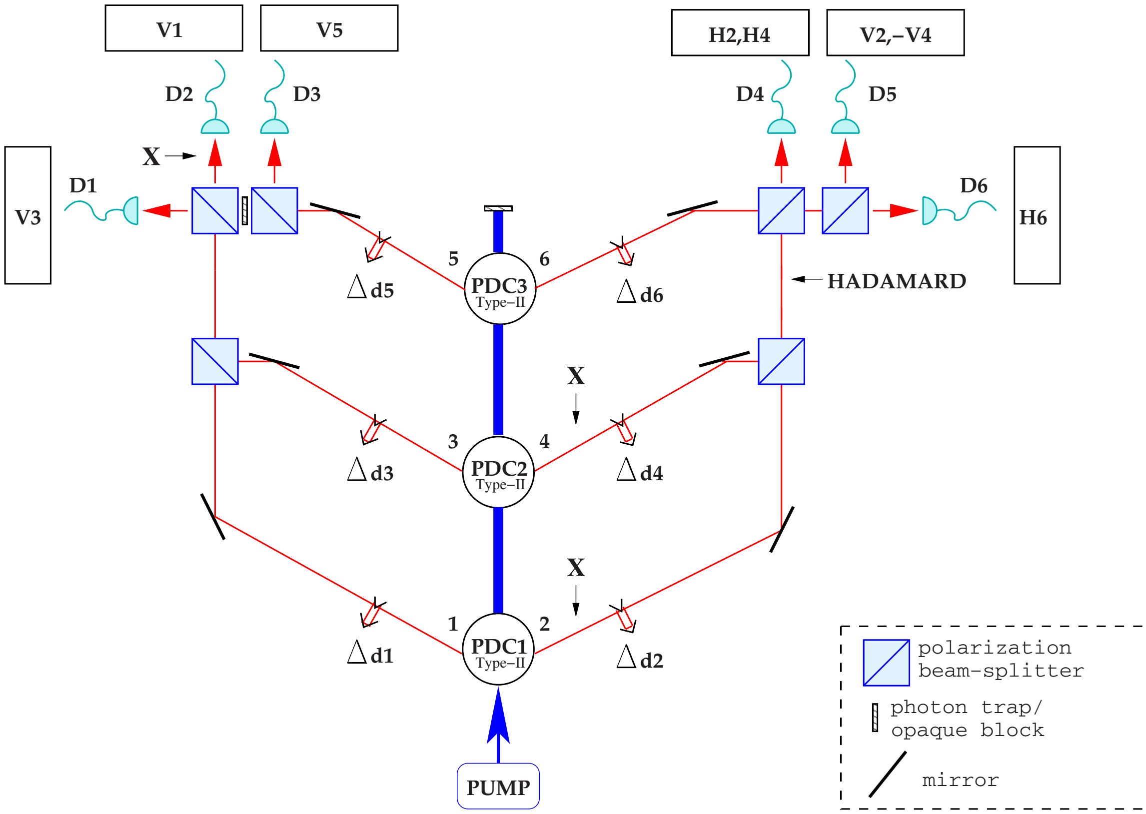

FIG. 4. (Color online) Schematic of a setup that can be used for the preparation of a decoherence-free logical zero state. Boxes adjacent to detectors $D1$, $D2, \ldots, D6$ indicate possible photon states being detected in the event of six simultaneous detections. The X gates ($X \equiv \sigma_x$ is a wave plate set to rotate the polarization by 90°) transform $|H\rangle$ into $|V\rangle$ and vice versa. The Hadamard gate transforms states according to Eqs. (4.1) and (4.2).

$$|\Psi_{1,3,5,6}\rangle = |V_1V_3V_5H_6\rangle. \tag{4.4}$$

On the other hand, if photons 2 and 4 arrive at either $D4$ or $D5$ and both of these detectors fire simultaneously it is impossible to know, based only on the knowledge of these two detection events, which one of them arrived in the horizontal state and which one arrived in the vertical state. This limited knowledge implies that the state of photons 2 and 4 is

$$|\Psi_{2,4}\rangle = \frac{1}{\sqrt{2}}(|V_2H_4\rangle - |H_2V_4\rangle). \tag{4.5}$$

In the event of six simultaneous detections one can infer that the decoherence-free logical zero state $|0_L\rangle = (|V_1V_2V_3H_4V_5H_6\rangle - |V_1H_2V_3V_4V_5H_6\rangle)/\sqrt{2}$ of Eq. (3.5) has been successfully prepared (see the Appendix for more details). With a few small modifications, the setup just described can also be used for the preparation of the logical one state in Eq. (3.5). We present that arrangement next and discuss the events necessary for a sixfold coincidence detection.

D. Physical preparation of logical one

By moving the Hadamard, removing one of the X gates, and replacing two of the polarized beam splitters with unpolarized (ordinary) beam splitters the setup shown in Fig. 4 can be modified to prepare a decoherence-free logical one state. This arrangement is shown below in Fig. 5. On the left side of this figure we see that if detectors $D1$, $D2$, and $D3$ all fire at once the combined photon 1, 3, and 5 system was detected in the state

$$|\Psi_{1,3,5}\rangle = \frac{1}{\sqrt{2}}(|V_1H_3V_5\rangle + |H_1V_3V_5\rangle). \tag{4.6}$$

Consider the case where photons 1, 3, and 5 are detected in the state $|V_1H_3V_5\rangle$ corresponding to the first term in Eq. (4.6). In this case one may infer that photon 2 was emitted in a horizontally polarized state and photon 4 in a vertical state since there are no polarization rotating devices in which photons 1 and 3 encounter. This being the case, after a common rotation by separate X gates, photons 2 and 4 exit the first PBS via different output paths. Now photon 4, being horizontally polarized, exits through the top path where it is detected at $D4$. Photon 2, being vertically polarized, exits the PBS via the right side and then passes through a Hadamard gate rotating its state in accordance with Eq. (4.2). After passing through the next PBS photon 2 continues on its way propagating toward detectors $D5$ and $D6$. If it happens to trigger $D5$ photon 6 will then be detected at $D6$ in a horizontal state in the event of six simultaneous detections. The six-photon state corresponding to this set of events is given by

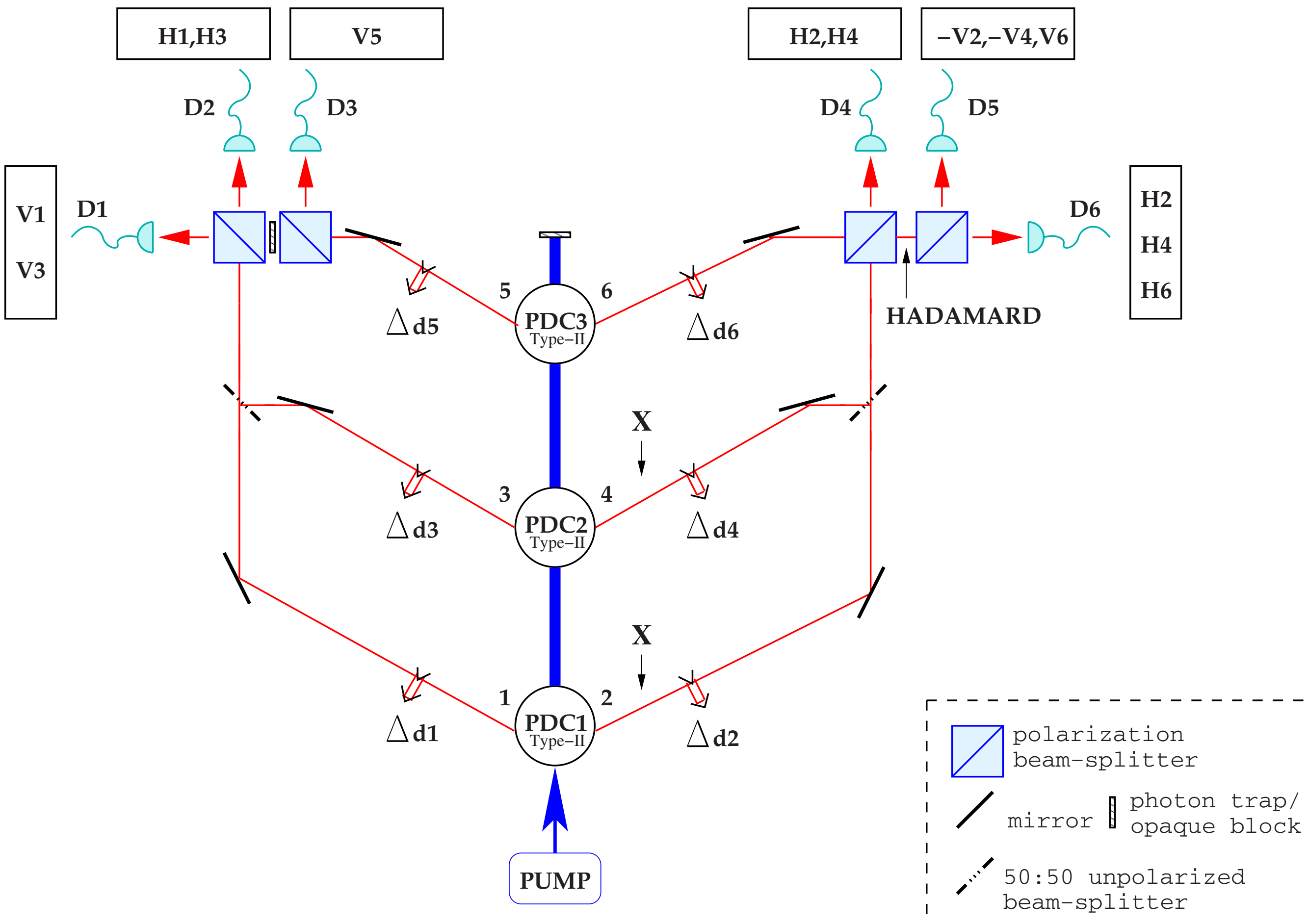

FIG. 5. (Color online) Schematic of a setup that can be used for the preparation of a decoherence-free logical one state.

$$|\Psi^{(1)}_{1,2,3,4,5,6}\rangle = -|V_1V_2H_3H_4V_5H_6\rangle, \tag{4.7}$$

which is indeed one of the terms forming a logical one. If photon 2 is instead detected at $D6$ in a horizontal state, a sixfold coincidence detection requires photon 6 to be detected at $D5$ in a vertical state yielding the second possibility

$$|\Psi^{(2)}_{1,2,3,4,5,6}\rangle = |V_1H_2H_3H_4V_5V_6\rangle. \tag{4.8}$$

Again, this state can be seen to be one of the terms forming logical one. Note that these are the only two states that can trigger six simultaneous detections when photons 1, 3, and 5 are described by $|V_1H_3V_5\rangle$. However, if photons 1, 3, and 5 are in the state forming the second term in Eq. (4.6), namely $|H_1V_3V_5\rangle$, two more possibilities arise. The same type of argument shows that these states are given by

$$|\Psi^{(3)}_{1,2,3,4,5,6}\rangle = -|H_1H_2V_3V_4V_5H_6\rangle, \tag{4.9}$$

and

$$|\Psi^{(4)}_{1,2,3,4,5,6}\rangle = |H_1H_2V_3H_4V_5V_6\rangle. \tag{4.10}$$

These are the only four states that are compatible with a sixfold coincidence detection. The setup just described can therefore be used to prepare the four-term coherent superposition representing the decoherence-free logical one state.

E. Preparation of a maximally entangled two-qutrit state

As discussed in [37], a maximally entangled state of two qutrits can take one of two forms, one given by Eq. (2.7), and one by

$$|\Phi_m\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle). \tag{4.11}$$

The difference in the barred and unbarred states, as a practical matter, is the way in which each transforms. The state given in Eq. (2.7) is decoherence-free while $|\Phi_m\rangle$ is not. This is due to the difference in transformation properties. The transformation matrices will not be repeated here, but were given in Ref. [41] in a particular parametrization.

In terms of the definitions above for $|0\rangle$, $|1\rangle$, and $|2\rangle$, a maximally entangled state takes the form

$$|\psi_{\max}\rangle = \frac{1}{\sqrt{3}}(|V_1V_2V_3V_4\rangle + |V_1H_2V_3H_4\rangle + |H_1H_2H_3H_4\rangle). \tag{4.12}$$

This particular state may be postselected based on a fourfold coincidence detection using the optical arrangement shown in Fig. 6. Photons emitted into paths 2 and 4 are both probabilistically transmitted through identical gates, shown in

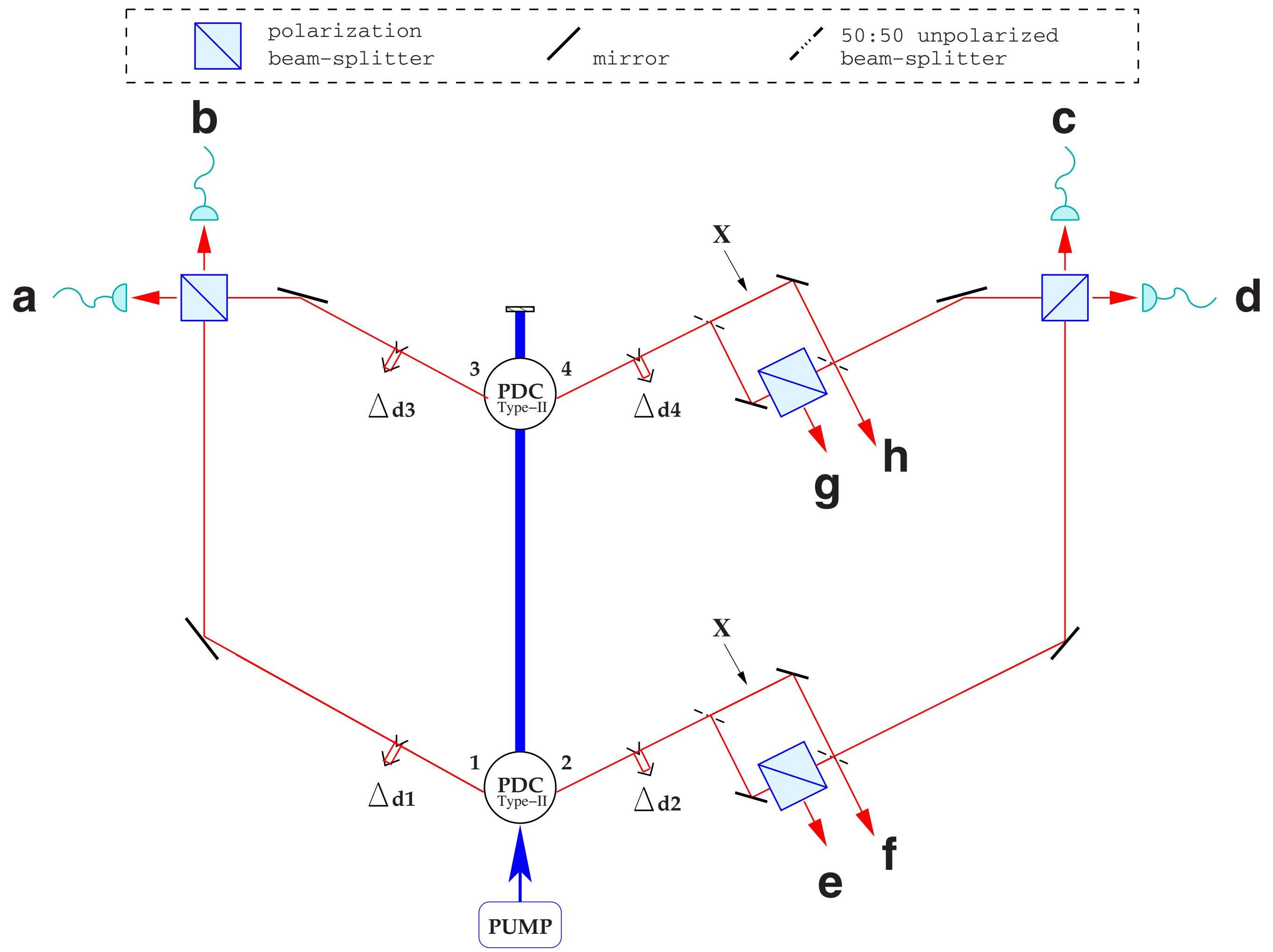

FIG. 6. (Color online) Optical arrangement that may be used for the preparation of a maximally entangled two-qutrit state.

Fig. 7, before being detected by either c or d. Since photon pairs are emitted in accordance with Eq. (3.2) the only three possible states that can produce four simultaneous detections at a, b, c, and d are those forming the superposition of Eq. (4.12). The transmission coefficients of the gates (Fig. 7) on the right for states $|H\rangle$ and $|V\rangle$ ensure that the probabilities of each term forming this superposition are equal (see the Appendix for more details).

For this state to be decoherence-free, we would require, as mentioned in Sec. II B, each qutrit to transform differently, and in some sense they must transform in an opposite fashion. Clearly if they transform in the same way, the state $|\Phi_m\rangle$

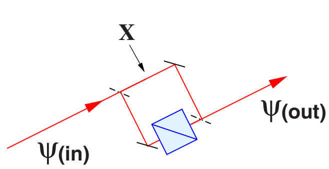

FIG. 7. (Color online) Gate used in the preparation of a maximally entangled two-qutrit state. For $|\psi(\text{in})\rangle = |H\rangle$ the gate outputs the state $|\psi(\text{out})\rangle = \frac{1}{2}(|H\rangle + |V\rangle + \sqrt{2}|\text{vacuum}\rangle)$. For $|\psi(\text{in})\rangle = |V\rangle$ the corresponding output is $|\psi(\text{out})\rangle = \frac{1}{2}(|H\rangle + \sqrt{3}|\text{vacuum}\rangle)$.

in Eq. (4.11) is changed. Take for example the transformation

$$|j\rangle \to \exp(-i\alpha_j)|j\rangle,$$

where α_j is different for each $j(=0,1,2)$, for example $\alpha_j = \pi j/10$. In this case the new state is different from the original even after an overall phase is extracted and even though each has been transformed in the same way. This is not a collective decoherence-free state. On the other hand, consider the simultaneous transformation

$$|j\rangle \to \exp(-i\alpha_j)|j\rangle,$$

for the basis states of the first and

$$|j\rangle \to \exp(+i\alpha_j)|j\rangle,$$

for the basis states of the second qutrit. In this case $|\Phi_m\rangle$ is unchanged from the original. In general, the first qutrit must transform according to the conjugate transformation of the second qutrit. This is the sense in which the representations need to be "opposite" of each other. The most general such transformations are given by $D^{(0,1)}$ and $D^{(1,0)}$ in Ref. [41].

Many decoherence-free, or noiseless subsystems which have been described theoretically, and which have been produced in the laboratory, have been those which are immune to collective errors. This can be explained, in part, by the simple structure of such states as explained in Ref. [37]. However, here we have provided a particular example of a state which is decoherence-free under the simultaneous, but *not* collective transformation of two states.

V. CONCLUSION

We have presented a proposal for the physical preparation of a single decoherence-free qubit embedded in the product space of three physical qutrits. Two similar setups have been described to complete this task; one for the preparation of the logical zero state, and the other for the preparation of the logical one state. They are similar up to the inclusion (or exclusion) of an X gate, the location of a Hadamard gate, and the replacement of two polarized (unpolarized) beam splitters with two unpolarized (polarized) beam splitters. Distinguishing between logical zero and logical one after being prepared in this way could prove to be a formidable task since the basic constituents of these postselected states arriving at a particular detector (assumed here to be a quantum nondemolition device) all carry the same polarization and are by necessity indistinguishable. Decoding these states and implementing the derived logical gates compatible with a three-qutrit NS undergoing collective noise [54,55] are subjects for future research.

We have also presented an experimental arrangement for the production of a maximally entangled state of two qutrits, which when transformed properly, exhibits decoherence-free effects. This will enable the determination of the experimental difference between inequivalent representations of qutrit states. Furthermore this experiment provides an example of a state which would remain unchanged under the simultaneous, but not identical, transformations of its constituents.

A possible downside to our proposed experiments is that they produce the desired states only in the case of fourfold (for the singlet) and sixfold (for the logical states) coincidence detections, which might be quite infrequent. The authors in Ref. [47] have realized such sixfold coincidence detections but the rate at which they occurred remains unclear. If this hurdle is not too high the proposed proof-of-principal experiment could be extended for practical use by using quantum nondemolition (QND) measurements at each of the six outputs. In Ref. [56] a proposal for such QND devices was given that uses only beam splitters and photodetectors along with auxiliary photons. Another concern is due to the fact that a n-pair emission in a single crystal is nearly as probable as n down conversions taking place in separate crystals [57–59]. If we restrict our attention for the moment to the case of only three simultaneous down-conversion processes occurring, we see that the arrangement used to prepare the logical zero state only yields sixfold detections in the event of single-pair emissions in each of the crystals. On the other hand, the arrangement used to prepare the logical one state can produce a sixfold detection when a double pair is produced in one of the crystals while no pairs are produced in another. This apparent problem vanishes if we recognize the fact that the numbers attached to specific photons serve merely as labels indicating which particular spatial mode into which a particular photon was emitted. If two down conversions happen to take place in PDC1, for example, while no photons are converted in PDC2, the arrangement can still postselect the appropriate superposition based on six detections as long as we label photons properly. This same reasoning applies to the arrangement used to prepare the maximally entangled state in the case of double emissions in one of the two crystals and none in the other. If we extend our analysis to situations where more than three down conversions take place simultaneously we cannot address the problem by simple label considerations since then there might be more than six photons arriving at the detectors. If the detectors were number resolving this would not be an issue since we could postselect states based on the arrival and detection of six and only six photons at the appropriate detectors. Although the probability of witnessing higher numbers of photon pairs created simultaneously dramatically decreases with increasing pairs, these problems could be eliminated in the future if improvements in entangled photon sources such as those presented in Ref. [60] are made.

Path indistinguishability is also essential for the preparation of quantum superpositions. When a single pump photon decays into a pair of daughters they emerge highly correlated in frequency due to the conservation of energy, and their "spontaneous" emission clearly signifies a highly temporal correlation as well. These correlations, along with others, could in principal be exploited to determine which photon pair arrived at which detectors. In order to account for these possible exploitations the authors in Ref. [48] suggested placing narrow-band filters, centered at half of the pump frequency, in the photon paths. This would not only minimize the frequency differences of detected photons, but also reduce the temporal correlations since there is an uncertainty in the time in which it takes to pass through one of them. If the uncertainty in the amount of time in which it takes to pass through a filter was high enough to establish an intrinsic uncertainty in the arrival times of photons belonging to a pair yet still remained compatible with the resolution time of a detector so that six nearly simultaneous detections could occur, the condition of path indistinguishability could be realistically met.

We believe these proposed experiments are able to be performed using readily available technologies and will allow the exploration of new types of qutrit states—decoherence-free subspaces and noiseless subsystems comprised of qutrit states.

Note added in proof. Recently, several authors have presented related experiments which help show the feasibility of our proposed experiments [61,62].

ACKNOWLEDGMENTS

We gratefully acknowledge Hwang Lee for several helpful comments. This material is based upon work supported by the National Science Foundation under Grant No. 0545798 to M.S.B.

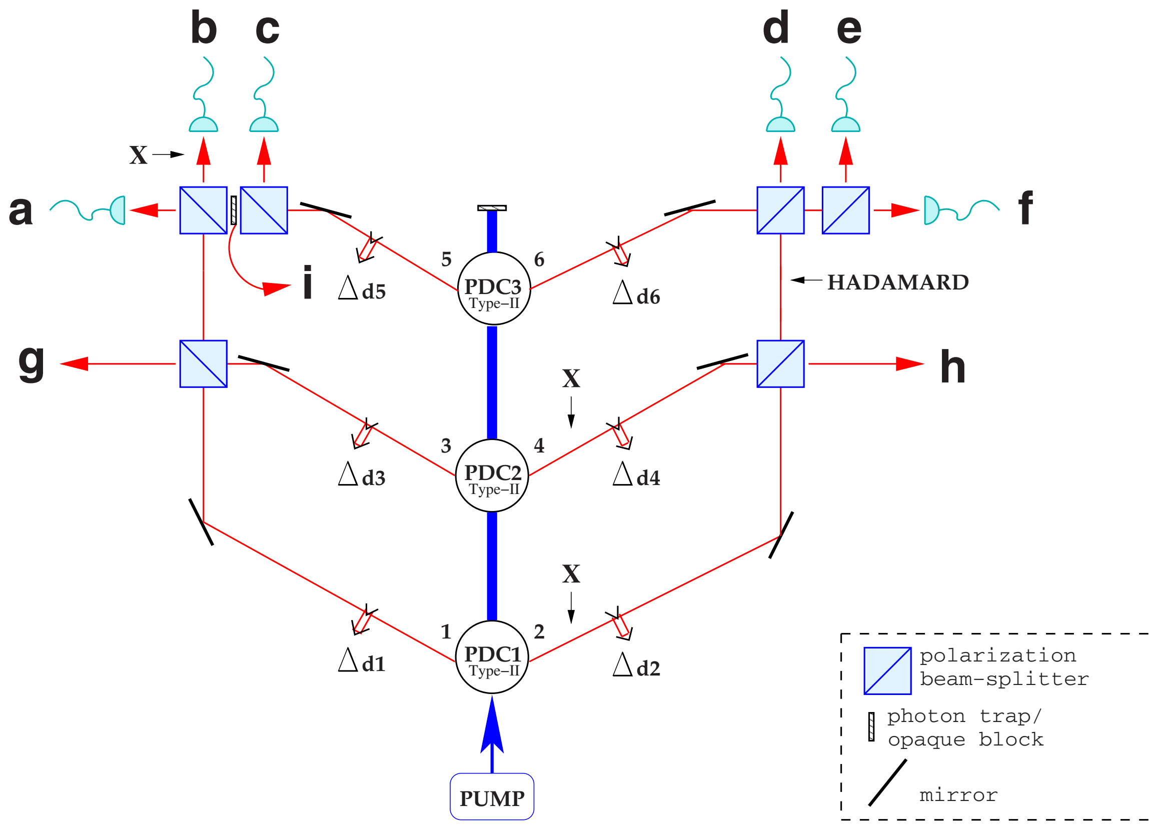

FIG. 8. (Color online) Optical arrangement that may be used for the preparation of a decoherence-free logical zero state.

APPENDIX: DETAILED CALCULATIONS OF MODE TRANSFORMATIONS

The states postselected by coincidence detections at the appropriate outputs of the optical arrangements shown in this paper may be calculated in terms of the transformations of the down-converted modes. If we let $\hat{p}^{\dagger}_{H,i}$, and $\hat{p}^{\dagger}_{V,i}$ respectively represent the creation operators for the horizontal and vertical polarization modes of the ith photon ($i=1,2,3,4,5,6$) then a successful down-conversion in PDC1 may be described by the operator $\hat{O}^{\dagger}_{1,2}$, where $\hat{O}^{\dagger}_{1,2}$ is given by

$$\hat{O}^{\dagger}_{1,2} = (\hat{p}^{\dagger}_{H,1}\hat{p}^{\dagger}_{V,2} + \hat{p}^{\dagger}_{V,1}\hat{p}^{\dagger}_{H,2})/\sqrt{2}. \tag{A1}$$

Similarly, a successful down conversion in PDC2 (PDC3) can be described by the creation operators $\hat{O}^{\dagger}_{3,4}$ ($\hat{O}^{\dagger}_{5,6}$), where $\hat{O}^{\dagger}_{3,4}$ ($\hat{O}^{\dagger}_{5,6}$) are given by

$$\hat{O}^{\dagger}_{3,4} = (\hat{p}^{\dagger}_{H,3}\hat{p}^{\dagger}_{V,4} + \hat{p}^{\dagger}_{V,3}\hat{p}^{\dagger}_{H,4})/\sqrt{2} \tag{A2}$$

and

$$\hat{O}^{\dagger}_{5,6} = (\hat{p}^{\dagger}_{H,5}\hat{p}^{\dagger}_{V,6} + \hat{p}^{\dagger}_{V,5}\hat{p}^{\dagger}_{H,6})/\sqrt{2}. \tag{A3}$$

The creation operator corresponding to the event of two independent down-conversion processes occurring simultaneously (one in PDC1 and another in PDC2) is thus given by

$$\begin{aligned}\hat{O}^{\dagger}_{1,2}\hat{O}^{\dagger}_{3,4} &= (\hat{p}^{\dagger}_{H,1}\hat{p}^{\dagger}_{V,2} + \hat{p}^{\dagger}_{V,1}\hat{p}^{\dagger}_{H,2})(\hat{p}^{\dagger}_{H,3}\hat{p}^{\dagger}_{V,4} + \hat{p}^{\dagger}_{V,3}\hat{p}^{\dagger}_{H,4})/2 \\ &= (\hat{p}^{\dagger}_{H,1}\hat{p}^{\dagger}_{V,2}\hat{p}^{\dagger}_{H,3}\hat{p}^{\dagger}_{V,4} + \hat{p}^{\dagger}_{H,1}\hat{p}^{\dagger}_{V,2}\hat{p}^{\dagger}_{V,3}\hat{p}^{\dagger}_{H,4} \\ &\quad + \hat{p}^{\dagger}_{V,1}\hat{p}^{\dagger}_{H,2}\hat{p}^{\dagger}_{H,3}\hat{p}^{\dagger}_{V,4} + \hat{p}^{\dagger}_{V,1}\hat{p}^{\dagger}_{H,2}\hat{p}^{\dagger}_{V,3}\hat{p}^{\dagger}_{H,4})/2, \end{aligned} \tag{A4}$$

while the corresponding operator for three down-conversions occurring (one in PDC1, one in PDC2, and one in PDC3) is given by

$$\begin{aligned}\hat{O}^{\dagger}_{1,2}\hat{O}^{\dagger}_{3,4}\hat{O}^{\dagger}_{5,6} &= (\hat{p}^{\dagger}_{H,1}\hat{p}^{\dagger}_{V,2} + \hat{p}^{\dagger}_{V,1}\hat{p}^{\dagger}_{H,2})(\hat{p}^{\dagger}_{H,3}\hat{p}^{\dagger}_{V,4} + \hat{p}^{\dagger}_{V,3}\hat{p}^{\dagger}_{H,4})(\hat{p}^{\dagger}_{H,5}\hat{p}^{\dagger}_{V,6} + \hat{p}^{\dagger}_{V,5}\hat{p}^{\dagger}_{H,6})/\sqrt{8} = (\hat{p}^{\dagger}_{H,1}\hat{p}^{\dagger}_{V,2}\hat{p}^{\dagger}_{H,3}\hat{p}^{\dagger}_{V,4}\hat{p}^{\dagger}_{H,5}\hat{p}^{\dagger}_{V,6} \\ &\quad + \hat{p}^{\dagger}_{H,1}\hat{p}^{\dagger}_{V,2}\hat{p}^{\dagger}_{H,3}\hat{p}^{\dagger}_{V,4}\hat{p}^{\dagger}_{V,5}\hat{p}^{\dagger}_{H,6} + \hat{p}^{\dagger}_{H,1}\hat{p}^{\dagger}_{V,2}\hat{p}^{\dagger}_{V,3}\hat{p}^{\dagger}_{H,4}\hat{p}^{\dagger}_{H,5}\hat{p}^{\dagger}_{V,6} + \hat{p}^{\dagger}_{H,1}\hat{p}^{\dagger}_{V,2}\hat{p}^{\dagger}_{V,3}\hat{p}^{\dagger}_{H,4}\hat{p}^{\dagger}_{V,5}\hat{p}^{\dagger}_{H,6} + \hat{p}^{\dagger}_{V,1}\hat{p}^{\dagger}_{H,2}\hat{p}^{\dagger}_{H,3}\hat{p}^{\dagger}_{V,4}\hat{p}^{\dagger}_{H,5}\hat{p}^{\dagger}_{V,6} \\ &\quad + \hat{p}^{\dagger}_{V,1}\hat{p}^{\dagger}_{H,2}\hat{p}^{\dagger}_{H,3}\hat{p}^{\dagger}_{V,4}\hat{p}^{\dagger}_{V,5}\hat{p}^{\dagger}_{H,6} + \hat{p}^{\dagger}_{V,1}\hat{p}^{\dagger}_{H,2}\hat{p}^{\dagger}_{V,3}\hat{p}^{\dagger}_{H,4}\hat{p}^{\dagger}_{H,5}\hat{p}^{\dagger}_{V,6} + \hat{p}^{\dagger}_{V,1}\hat{p}^{\dagger}_{H,2}\hat{p}^{\dagger}_{V,3}\hat{p}^{\dagger}_{H,4}\hat{p}^{\dagger}_{V,5}\hat{p}^{\dagger}_{H,6})\sqrt{8}. \end{aligned} \tag{A5}$$

If we now label the optical modes at the detector outputs of Fig. 4 by $\hat{a}^\dagger$, $\hat{b}^\dagger$, $\hat{c}^\dagger$, $\hat{d}^\dagger$, $\hat{e}^\dagger$, and $\hat{f}^\dagger$ and label the other three outputs by $\hat{g}^\dagger$, $\hat{h}^\dagger$, and $\hat{i}^\dagger$ the setup used to prepare the logical zero state can be seen in Fig. 8 to transform the down-converted modes according to the following relations:

$$\hat{p}^\dagger_{H,1} \to \hat{b}^\dagger_{V,1},\ \hat{p}^\dagger_{V,1} \to \hat{g}^\dagger_{V,1},$$

$$\hat{p}^\dagger_{H,2} \to \hat{h}^\dagger_{V,2},\ \hat{p}^\dagger_{V,2} \to (\hat{d}^\dagger_{H,2} + \hat{e}^\dagger_{V,2})/\sqrt{2},$$

$$\hat{p}^\dagger_{H,3} \to \hat{g}^\dagger_{H,3},\ \hat{p}^\dagger_{V,3} \to \hat{a}^\dagger_{V,3},$$

$$\hat{p}^\dagger_{H,4} \to (\hat{d}^\dagger_{H,4} - \hat{e}^\dagger_{V,4})/\sqrt{2},\ \hat{p}^\dagger_{V,4} \to \hat{h}^\dagger_{H,4},$$

$$\hat{p}^\dagger_{H,5} \to \hat{i}^\dagger_{H,5},\ \hat{p}^\dagger_{V,5} \to \hat{c}^\dagger_{V,5},$$

$$\hat{p}^\dagger_{H,6} \to \hat{f}^\dagger_{H,6},\ \hat{p}^\dagger_{V,6} \to \hat{d}^\dagger_{V,6}. \tag{A6}$$

Using these transformation relations the simultaneous down-conversion operator in Eq. (A5) can then be calculated to transform according to the equation below. If detectors are placed at the outputs of the optical modes $\hat{a}^\dagger$, $\hat{b}^\dagger$, $\hat{c}^\dagger$, $\hat{d}^\dagger$, $\hat{e}^\dagger$, and $\hat{f}^\dagger$ and each register a detection simultaneously the state postselected in this way is a superposition of the states corresponding to the terms in Eq. (A7) containing all six of these modes. The two terms satisfying this requirement are $-\hat{b}^\dagger_{V,1}\hat{d}^\dagger_{H,2}\hat{a}^\dagger_{V,3}\hat{e}^\dagger_{V,4}\hat{c}^\dagger_{V,5}\hat{f}^\dagger_{H,6}$ and $\hat{b}^\dagger_{V,1}\hat{e}^\dagger_{V,2}\hat{a}^\dagger_{V,3}\hat{d}^\dagger_{H,4}\hat{c}^\dagger_{V,5}\hat{f}^\dagger_{H,6}$ (both scaled by the same coefficient) which yield the logical zero state given by Eq. (3.5):

$$\begin{aligned}\hat{O}^\dagger_{1,2}\hat{O}^\dagger_{3,4}\hat{O}^\dagger_{5,6} \to \Bigg[&\hat{b}^\dagger_{V,1}(\hat{d}^\dagger_{H,2} + \hat{e}^\dagger_{V,2})\left(\frac{1}{\sqrt{2}}\right)\hat{g}^\dagger_{H,3}\hat{h}^\dagger_{H,4}\hat{i}^\dagger_{H,5}\hat{d}^\dagger_{V,6} + \hat{b}^\dagger_{V,1}(\hat{d}^\dagger_{H,2} + \hat{e}^\dagger_{V,2})\left(\frac{1}{\sqrt{2}}\right)\hat{g}^\dagger_{H,3}\hat{h}^\dagger_{H,4}\hat{c}^\dagger_{V,5}\hat{f}^\dagger_{H,6} + \hat{b}^\dagger_{V,1}(\hat{d}^\dagger_{H,2} + \hat{e}^\dagger_{V,2})\left(\frac{1}{\sqrt{2}}\right)\hat{a}^\dagger_{V,3}(\hat{d}^\dagger_{H,4} \\ &- \hat{e}^\dagger_{V,4})\left(\frac{1}{\sqrt{2}}\right)\hat{i}^\dagger_{H,5}\hat{d}^\dagger_{V,6} + \hat{b}^\dagger_{V,1}(\hat{d}^\dagger_{H,2} + \hat{e}^\dagger_{V,2})\left(\frac{1}{\sqrt{2}}\right)\hat{a}^\dagger_{V,3}(\hat{d}^\dagger_{H,4} - \hat{e}^\dagger_{V,4})\left(\frac{1}{\sqrt{2}}\right)\hat{c}^\dagger_{V,5}\hat{f}^\dagger_{H,6} + \hat{g}^\dagger_{V,1}\hat{h}^\dagger_{V,2}\hat{g}^\dagger_{H,3}\hat{h}^\dagger_{H,4}\hat{i}^\dagger_{H,5}\hat{d}^\dagger_{V,6} \\ &+ \hat{g}^\dagger_{V,1}\hat{h}^\dagger_{V,2}\hat{g}^\dagger_{H,3}\hat{h}^\dagger_{H,4}\hat{c}^\dagger_{V,5}\hat{f}^\dagger_{H,6} + \hat{g}^\dagger_{V,1}\hat{h}^\dagger_{V,2}\hat{a}^\dagger_{V,3}(\hat{d}^\dagger_{H,4} - \hat{e}^\dagger_{V,4})\left(\frac{1}{\sqrt{2}}\right)\hat{i}^\dagger_{H,5}\hat{d}^\dagger_{V,6} + \hat{g}^\dagger_{V,1}\hat{h}^\dagger_{V,2}\hat{a}^\dagger_{V,3}(\hat{d}^\dagger_{H,4} - \hat{e}^\dagger_{V,4})\left(\frac{1}{\sqrt{2}}\right)\hat{c}^\dagger_{V,5}\hat{f}^\dagger_{H,6}\Bigg]\left(\frac{1}{\sqrt{8}}\right).\end{aligned} \tag{A7}$$

The setup shown in Fig. 9, used to prepare the logical one state, transforms the down-converted modes according to the following relations (ignoring the unimportant phase shifts in modes $\hat{g}^\dagger$ and $\hat{h}^\dagger$), Eq. (A8)

$$\hat{p}^\dagger_{H,1} \to (\hat{g}^\dagger_{H,1} + \hat{b}^\dagger_{H,1})\frac{1}{\sqrt{2}}, \quad \hat{p}^\dagger_{V,1} \to (\hat{g}^\dagger_{V,1} + \hat{a}^\dagger_{V,1})\frac{1}{\sqrt{2}},$$

$$\hat{p}^\dagger_{H,2} \to (\sqrt{2}\hat{h}^\dagger_{V,2} + \hat{f}^\dagger_{H,2} - \hat{e}^\dagger_{V,2})\tfrac{1}{2},$$

$$\hat{p}^\dagger_{V,2} \to (\hat{h}^\dagger_{H,2} + \hat{d}^\dagger_{H,2})\frac{1}{\sqrt{2}}, \quad \hat{p}^\dagger_{H,3} \to (\hat{g}^\dagger_{H,3} + \hat{b}^\dagger_{H,3})\frac{1}{\sqrt{2}},$$

$$\hat{p}^\dagger_{V,3} \to (\hat{g}^\dagger_{V,3} + \hat{a}^\dagger_{V,3})\frac{1}{\sqrt{2}},$$

$$\hat{p}^\dagger_{H,4} \to (\sqrt{2}\hat{h}^\dagger_{V,4} + \hat{f}^\dagger_{H,4} - \hat{e}^\dagger_{V,4})\frac{1}{2},\ \hat{p}^\dagger_{V,4} \to (\hat{h}^\dagger_{H,4} + \hat{d}^\dagger_{H,4})\frac{1}{\sqrt{2}},$$

$$\hat{p}^\dagger_{H,5} \to \hat{i}^\dagger_{H,5},$$

$$\hat{p}^\dagger_{V,5} \to \hat{c}^\dagger_{V,5}, \quad \hat{p}^\dagger_{H,6} \to (\hat{f}^\dagger_{H,6} + \hat{e}^\dagger_{V,6})\frac{1}{\sqrt{2}}, \quad \hat{p}^\dagger_{V,6} \to \hat{d}^\dagger_{V,6}. \tag{A8}$$

These relations are used below to calculate the transformation of Eq. (A5). Again, the terms containing each of the modes $\hat{a}^\dagger$, $\hat{b}^\dagger$, $\hat{c}^\dagger$, $\hat{d}^\dagger$, $\hat{e}^\dagger$, and $\hat{f}^\dagger$ are compatible with a sixfold coincidence detection. Expanding Eq. (A9) reveals the four terms $\hat{b}^\dagger_{H,1}\hat{d}^\dagger_{H,2}\hat{a}^\dagger_{V,3}\hat{f}^\dagger_{H,4}\hat{c}^\dagger_{V,5}\hat{e}^\dagger_{V,6}$, $-\hat{b}^\dagger_{H,1}\hat{d}^\dagger_{H,2}\hat{a}^\dagger_{V,3}\hat{e}^\dagger_{V,4}\hat{c}^\dagger_{V,5}\hat{f}^\dagger_{H,6}$, $\hat{a}^\dagger_{V,1}\hat{f}^\dagger_{H,2}\hat{b}^\dagger_{H,3}\hat{d}^\dagger_{H,4}\hat{c}^\dagger_{V,5}\hat{e}^\dagger_{V,6}$, and $-\hat{a}^\dagger_{V,1}\hat{e}^\dagger_{V,2}\hat{b}^\dagger_{H,3}\hat{d}^\dagger_{H,4}\hat{c}^\dagger_{V,5}\hat{f}^\dagger_{H,6}$ (all scaled by the same coefficient) which meet this requirement. These terms correspond to the logical one state of Eq. (3.5):

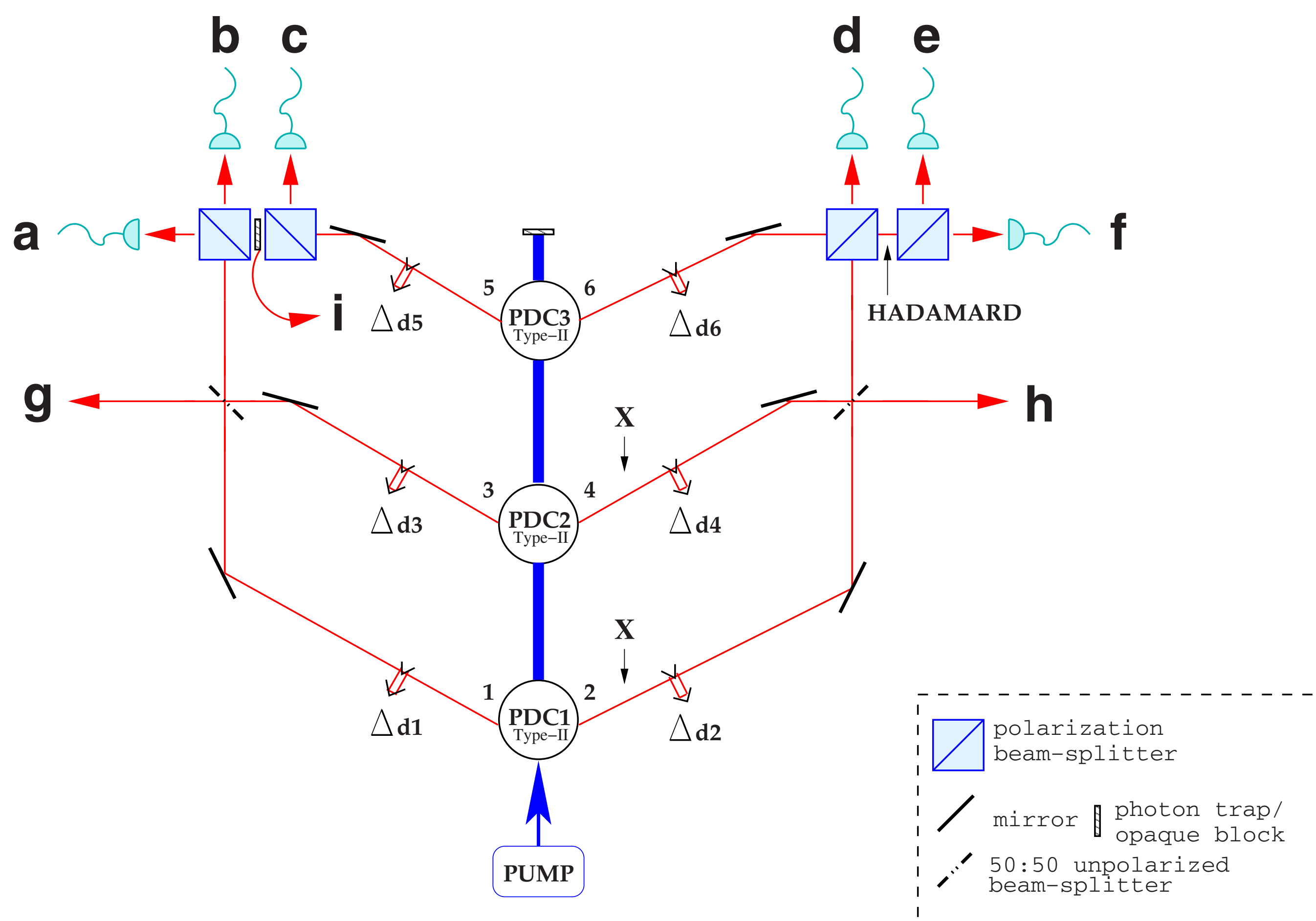

FIG. 9. (Color online) Optical arrangement that may be used for the preparation of a decoherence-free logical one state.

$$
\begin{aligned}
\hat{O}^\dagger_{1,2}\hat{O}^\dagger_{3,4}\hat{O}^\dagger_{5,6} \to \Bigg[& (\hat{g}^\dagger_{H,1}+\hat{b}^\dagger_{H,1})\left(\frac{1}{\sqrt{2}}\right)(\hat{h}^\dagger_{H,2}+\hat{d}^\dagger_{H,2})\left(\frac{1}{\sqrt{2}}\right)(\hat{g}^\dagger_{H,3}+\hat{b}^\dagger_{H,3})\left(\frac{1}{\sqrt{2}}\right)+(\hat{h}^\dagger_{H,4}+\hat{d}^\dagger_{H,4})\left(\frac{1}{\sqrt{2}}\right)\hat{i}^\dagger_{H,5}\hat{d}^\dagger_{V,6}+(\hat{g}^\dagger_{H,1}+\hat{b}^\dagger_{H,1})\left(\frac{1}{\sqrt{2}}\right)\\
&\times(\hat{h}^\dagger_{H,2}+\hat{d}^\dagger_{H,2})\left(\frac{1}{\sqrt{2}}\right)(\hat{g}^\dagger_{H,3}+\hat{b}^\dagger_{H,3})\left(\frac{1}{\sqrt{2}}\right)+(\hat{h}^\dagger_{H,4}+\hat{d}^\dagger_{H,4})\left(\frac{1}{\sqrt{2}}\right)\hat{c}^\dagger_{V,5}(\hat{f}^\dagger_{H,6}+\hat{e}^\dagger_{V,6})\left(\frac{1}{\sqrt{2}}\right)+(\hat{g}^\dagger_{H,1}+\hat{b}^\dagger_{H,1})\left(\frac{1}{\sqrt{2}}\right)(\hat{h}^\dagger_{H,2}\\
&+\hat{d}^\dagger_{H,2})\left(\frac{1}{\sqrt{2}}\right)(\hat{g}^\dagger_{V,3}+\hat{a}^\dagger_{V,3})\left(\frac{1}{\sqrt{2}}\right)+(\sqrt{2}\hat{h}^\dagger_{V,4}+\hat{f}^\dagger_{H,4}-\hat{e}^\dagger_{V,4})\left(\frac{1}{2}\right)\hat{i}^\dagger_{H,5}\hat{d}^\dagger_{V,6}+(\hat{g}^\dagger_{H,1}+\hat{b}^\dagger_{H,1})\left(\frac{1}{\sqrt{2}}\right)(\hat{h}^\dagger_{H,2}+\hat{d}^\dagger_{H,2})\left(\frac{1}{\sqrt{2}}\right)\\
&\times(\hat{g}^\dagger_{V,3}+\hat{a}^\dagger_{V,3})\left(\frac{1}{\sqrt{2}}\right)+(\sqrt{2}\hat{h}^\dagger_{V,4}+\hat{f}^\dagger_{H,4}-\hat{e}^\dagger_{V,4})\left(\frac{1}{2}\right)\hat{c}^\dagger_{V,5}(\hat{f}^\dagger_{H,6}+\hat{e}^\dagger_{V,6})\left(\frac{1}{\sqrt{2}}\right)+(\hat{g}^\dagger_{V,1}+\hat{a}^\dagger_{V,1})\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}\hat{h}^\dagger_{V,2}+\hat{f}^\dagger_{H,2}\\
&-\hat{e}^\dagger_{V,2})\left(\frac{1}{2}\right)+(\hat{g}^\dagger_{H,3}+\hat{b}^\dagger_{H,3})\left(\frac{1}{\sqrt{2}}\right)(\hat{h}^\dagger_{H,4}+\hat{d}^\dagger_{H,4})\left(\frac{1}{\sqrt{2}}\right)\hat{i}^\dagger_{H,5}\hat{d}^\dagger_{V,6}+(\hat{g}^\dagger_{V,1}+\hat{a}^\dagger_{V,1})\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}\hat{h}^\dagger_{V,2}+\hat{f}^\dagger_{H,2}-\hat{e}^\dagger_{V,2})\left(\frac{1}{2}\right)\\
&+(\hat{g}^\dagger_{H,3}+\hat{b}^\dagger_{H,3})\left(\frac{1}{\sqrt{2}}\right)(\hat{h}^\dagger_{H,4}+\hat{d}^\dagger_{H,4})\left(\frac{1}{\sqrt{2}}\right)\hat{c}^\dagger_{V,5}(\hat{f}^\dagger_{H,6}+\hat{e}^\dagger_{V,6})\left(\frac{1}{\sqrt{2}}\right)+(\hat{g}^\dagger_{V,1}+\hat{a}^\dagger_{V,1})\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}\hat{h}^\dagger_{V,2}+\hat{f}^\dagger_{H,2}-\hat{e}^\dagger_{V,2})\left(\frac{1}{2}\right)\\
&+(\hat{g}^\dagger_{V,3}+\hat{a}^\dagger_{V,3})\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}\hat{h}^\dagger_{V,4}+\hat{f}^\dagger_{H,4}-\hat{e}^\dagger_{V,4})\left(\frac{1}{2}\right)\hat{i}^\dagger_{H,5}\hat{d}^\dagger_{V,6}+(\hat{g}^\dagger_{V,1}+\hat{a}^\dagger_{V,1})\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}\hat{h}^\dagger_{V,2}+\hat{f}^\dagger_{H,2}-\hat{e}^\dagger_{V,2})\left(\frac{1}{2}\right)+(\hat{g}^\dagger_{V,3}\\
&+\hat{a}^\dagger_{V,3})\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}\hat{h}^\dagger_{V,4}+\hat{f}^\dagger_{H,4}-\hat{e}^\dagger_{V,4})\left(\frac{1}{2}\right)\hat{c}^\dagger_{V,5}(\hat{f}^\dagger_{H,6}+\hat{e}^\dagger_{V,6})\left(\frac{1}{\sqrt{2}}\right)\Bigg]\left(\frac{1}{\sqrt{8}}\right). \qquad \text{(A9)}
\end{aligned}
$$

In Fig. 6 we see that the mode transformations corresponding to the maximally entangled state arrangement are given by

$$\hat{p}^{\dagger}_{H,1} \to \hat{b}^{\dagger}_{H,1}, \quad \hat{p}^{\dagger}_{V,1} \to \hat{a}^{\dagger}_{V,1}, \quad \hat{p}^{\dagger}_{H,2} \to (\hat{f}^{\dagger}_{V,2} + \hat{d}^{\dagger}_{V,2} + \hat{f}^{\dagger}_{H,2} + \hat{c}^{\dagger}_{H,2})/2,$$

$$\hat{p}^{\dagger}_{V,2} \to (\hat{f}^{\dagger}_{H,2} + \hat{c}^{\dagger}_{H,2} + \sqrt{2}\hat{e}^{\dagger}_{V,2})/2, \quad \hat{p}^{\dagger}_{H,3} \to \hat{a}^{\dagger}_{H,3}, \quad \hat{p}^{\dagger}_{V,3} \to \hat{b}^{\dagger}_{V,3},$$

$$\hat{p}^{\dagger}_{H,4} \to (\hat{h}^{\dagger}_{V,4} + \hat{c}^{\dagger}_{V,4} + \hat{h}^{\dagger}_{H,4} + \hat{d}^{\dagger}_{H,4})/2, \tag{A10}$$

$$\hat{p}^{\dagger}_{V,4} \to (\hat{h}^{\dagger}_{H,4} + \hat{d}^{\dagger}_{H,4} + \sqrt{2}\hat{g}^{\dagger}_{V,4})/2.$$

Using these transformation relations the simultaneous down-conversion operator in Eq. (A4) can then be calculated to transform according to Eq. (A11) below.

If detectors are placed at the outputs of the optical modes $\hat{a}^{\dagger}$, $\hat{b}^{\dagger}$, $\hat{c}^{\dagger}$, and $\hat{d}^{\dagger}$ and each register a detection simultaneously the state postselected in this way is a superposition of the states corresponding to the terms in Eq. (A11) containing all four of these modes. The three terms satisfying this requirement are $\hat{b}^{\dagger}_{H,1}\hat{c}^{\dagger}_{H,2}\hat{a}^{\dagger}_{H,3}\hat{d}^{\dagger}_{H,4}$, $\hat{a}^{\dagger}_{V,1}\hat{d}^{\dagger}_{V,2}\hat{b}^{\dagger}_{V,3}\hat{c}^{\dagger}_{V,4}$, and $\hat{a}^{\dagger}_{V,1}\hat{c}^{\dagger}_{H,2}\hat{b}^{\dagger}_{V,3}\hat{d}^{\dagger}_{H,4}$ (all scaled by the same coefficient) which yield the state given by Eq. (4.12).

$$\begin{aligned}\hat{O}^{\dagger}_{1,2}\hat{O}^{\dagger}_{3,4} \to \Big[\left(\tfrac{1}{4}\right)\hat{b}^{\dagger}_{H,1}(\hat{f}^{\dagger}_{H,2} + \hat{c}^{\dagger}_{H,2} + \sqrt{2}\hat{e}^{\dagger}_{V,2})\hat{a}^{\dagger}_{H,3}(\hat{h}^{\dagger}_{H,4} + \hat{d}^{\dagger}_{H,4} + \sqrt{2}\hat{g}^{\dagger}_{V,4}) + \left(\tfrac{1}{4}\right)\Big[\hat{b}^{\dagger}_{H,1}(\hat{f}^{\dagger}_{H,2} + \hat{c}^{\dagger}_{H,2} + \sqrt{2}\hat{e}^{\dagger}_{V,2})\hat{b}^{\dagger}_{V,3} \\ \times(\hat{h}^{\dagger}_{V,4} + \hat{c}^{\dagger}_{V,4} + \hat{h}^{\dagger}_{H,4} + \hat{d}^{\dagger}_{H,4})\Big] + \left(\tfrac{1}{4}\right)\hat{a}^{\dagger}_{V,1}(\hat{f}^{\dagger}_{V,2} + \hat{d}^{\dagger}_{V,2} + \hat{f}^{\dagger}_{H,2} + \hat{c}^{\dagger}_{H,2})\hat{a}^{\dagger}_{H,3}(\hat{h}^{\dagger}_{H,4} + \hat{d}^{\dagger}_{H,4} + \sqrt{2}\hat{g}^{\dagger}_{V,4}) \\ + \left(\tfrac{1}{4}\right)\hat{a}^{\dagger}_{V,1}(\hat{f}^{\dagger}_{V,2} + \hat{d}^{\dagger}_{V,2} + \hat{f}^{\dagger}_{H,2} + \hat{c}^{\dagger}_{H,2})\hat{b}^{\dagger}_{V,3}(\hat{h}^{\dagger}_{V,4} + \hat{c}^{\dagger}_{V,4} + \hat{h}^{\dagger}_{H,4} + \hat{d}^{\dagger}_{H,4})\Big]\left(\tfrac{1}{2}\right).\end{aligned} \tag{A11}$$

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