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Entanglement measures and approximate quantum error correction

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It is shown that if the loss of entanglement along a quantum channel is sufficiently small, then approximate quantum error correction is possible, thereby generalizing what happens for coherent information. Explicit bounds are obtained for the entanglement of formation and the distillable entanglement, and their validity naturally extends to other bipartite entanglement measures in between. Robustness of derived criteria is analyzed and their tightness compared. Finally, as a by-product, we prove a bound quantifying how large the gap between the entanglement of formation and distillable entanglement can be for any given finite dimensional bipartite system, thus providing a sufficient condition for distillability in term of the entanglement of formation.

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I. INTRODUCTION

The possibility of performing quantum error correction obviously lies behind and justifies the vast efforts made up to now in order to develop quantum computation techniques, since it allows fault-tolerant computation [1] even when quantum systems—in fact extremely sensitive to noise—are considered as the basic carriers of information. Besides well-known algebraic conditions for *exact* quantum error correction, which directly lead to algebraic quantum error correcting codes (for a thorough presentation of quantum error correction theory and a detailed account about the enormous literature about it, see, e.g., Refs. [3,4]), an information-theoretical approach to quantum error correction [5–7] can shed some light on the dynamical processes which underlie quantum noise, offering at the same time the opportunity to better understand the conditions under which *approximate* quantum error correction is feasible [8]. In the present paper, we will be working within the latter scenario.

Approximate quantum error correction is not just a theoretical issue: in fact, in all practical implementations the experimenter can only rely upon some confidence level—exact processes exist as abstract mathematical concepts only. Then, conditions for approximate quantum error correction can provide useful ways to test the reliability of a real apparatus. In Ref. [8], Schumacher and Westmoreland proved that an adequate information-theoretical quantity to consider is the coherent information: the loss of coherent information along a quantum noisy channel is small if and only if the quantum noisy channel can be approximately corrected. In a subsequent paper [9], the same authors provided another criterion, this time for exact quantum error correction: the loss of entanglement (of formation) is null if and only if the channel can be exactly corrected. They left open the question whether the loss of entanglement provides not only a condition for exact correction, but also a condition for approximate correction. In this paper we will show that this is actually the case, extending our analysis to different entanglement measures, thereby proving that many inequivalent ways to quantify entanglement lead in fact to analogous conditions for approximate quantum error correction. We will moreover obtain, as a by-product, an inequality directly relating the entanglement of formation with the distillable entanglement present in a general bipartite mixed quantum state. Such inequality makes rigorous the intuition that the gap between the entanglement of formation and distillable entanglement, which is known to exist generically large for general mixed quantum states [10], cannot be *completely* arbitrary, in the sense that, given a finite dimensional bipartite state, whenever the entanglement of formation is "sufficiently close" to its maximum value, then also the distillable entanglement has to be "correspondingly large." (The concepts of "sufficiently close" and "correspondingly large," clearly depending on the dimensions of the subsystems, will be quantitatively defined below.)

The paper is organized as follows. In Sec. II we recall some basic notions about quantum channels and their purification into the unitary evolution of a larger closed system. In Sec. III we present some known information-theoretical conditions for exact as well as approximate quantum error correction. In Sec. IV we review a useful monogamy relation satisfied by quantum and classical correlations in a tripartite pure quantum state. Such a relation will be exploited in Sec. V to show that to have a small loss of entanglement of formation is equivalent to having small classical correlations between the reference system and the environment. This simple observation will lead us to the main result stated as Theorem 1. Section VI extends the same analysis to other entanglement measures. In particular, it is shown that for certain entanglement measures it is possible to derive the same result as for the entanglement of formation, but in a simpler way, moreover greatly improving the tightness of the bound. This second result, independent of the previous one, is stated as Theorem 2. Section VII stresses two remarks by comparing the two theorems obtained so far. The first remark shows that they can be combined to explicitly obtain the above-mentioned inequality, regarding the gap between entanglement of formation and distillable entanglement for a general bipartite mixed state. The second remark proposes a possible connection between different bipartite entanglement

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measures, used here to derive different criteria for approximate quantum error correction, and correspondingly induced topologies on the set of quantum channels. A brief summary (Sec. VIII) concludes the paper.

II. TRIPARTITE PURIFICATION OF CHANNELS

Let us consider an input quantum system Q whose state is described by the density matrix ρ^Q defined on the (finite dimensional) input Hilbert space $\mathcal{H}^Q$. A *channel*, mapping states on $\mathcal{H}^Q$ [that is, the set of nonnegative, trace-one operators on $\mathcal{H}^Q$, briefly denoted as $\mathfrak{S}(\mathcal{H}^Q)$] to states on $\mathcal{H}^{Q'}$, can be represented as a completely positive trace-preserving (CP-TP) linear map $\mathcal{E}:\mathfrak{S}(\mathcal{H}^Q)\rightarrow\mathfrak{S}(\mathcal{H}^{Q'})$. We will use the notation $\rho^{Q'}:=\mathcal{E}(\rho^Q)$. It is a well-known fact that channels can be written in their so-called Kraus form [11], that is

$$\mathcal{E}(\rho^Q)=\sum_m E_m\rho^Q E_m^\dagger,\quad \forall\ \rho^Q,$$

where the Kraus operators E_m satisfy the normalization condition $\sum_m E_m^\dagger E_m=\mathbb{1}^Q$.

Besides the above-mentioned abstract definition, we can give a different description of channels, by exploiting a powerful representation theorem, a direct consequence of Stinespring theorem [12], which states that all channels can be realized by means of a suitable unitary interaction U^{QE} of the input system Q with an *ancilla* E (initialized in a fixed pure state $|0^E\rangle\in\mathcal{H}^E$), followed by a trace over the ancillary degrees of freedom, in formula

$$\mathcal{E}(\rho^Q)=\mathrm{Tr}_{E'}[U^{QE}(\rho^Q\otimes|0\rangle\langle 0|^E)(U^{QE})^\dagger].$$

(We put a prime also on E because in general the output ancilla system could be different from the input one.) Such a purification of the channel can always be realized, without loss of generality, with $\dim\mathcal{H}^{E'}\leqslant\dim\mathcal{H}^Q\times\dim\mathcal{H}^{Q'}$ and it is unique up to local isometries on $\mathcal{H}^{E'}$. Since in the following we will consider entropic quantities, such an isometric freedom is completely innocuous.

It is now convenient to introduce a third reference system R, which purifies ρ^Q as

$$\Psi^{RQ}:=|\Psi\rangle\langle\Psi|^{RQ}\quad\text{such that }\mathrm{Tr}_R[\Psi^{RQ}]=\rho^Q.$$

As before, this purification also is unique up to local isometries on $\mathcal{H}^R$, so that $S(\rho^Q)=S(\rho^R)$, where $\rho^R=\mathrm{Tr}_Q[\Psi^{RQ}]$ and $S(\sigma):=-\mathrm{Tr}[\sigma\log_2\sigma]$ is the von Neumann entropy of the state σ. We can always choose, without loss of generality, the reference to be isomorphic to the input, so that $\dim\mathcal{H}^R=\dim\mathcal{H}^Q$. The reference system R goes untouched through the interaction U^{QE}, in such a way that the *global* state after the system-environment interaction is pure and given by

$$|\Psi^{R'Q'E'}\rangle:=(\mathbb{1}^R\otimes U^{QE})|\Psi^{RQ}\rangle\otimes|0^E\rangle.\tag{1}$$

(As before, we put a prime on R, even if it does not change, just to recall that we are considering the reference system *after* the unitary interaction.) Since we closed the whole system, we will be able to play with entropic quantities exploiting useful identities such as

$$I^{R':Q'}(\rho^{R'Q'})+I^{R':E'}(\rho^{R'E'})=2S(\rho^{R'})=2S(\rho^Q),\tag{2}$$

where $I^{A:B}(\sigma^{AB}):=S(\sigma^A)+S(\sigma^B)-S(\sigma^{AB})$ is the quantum mutual information [13,14] between A and B when the global state is σ^{AB}, and $\rho^{R'Q'}$, etc. are the reduced states calculated from the global tripartite pure state $|\Psi^{R'Q'E'}\rangle$ in Eq. (1).

III. KNOWN CONDITIONS FOR CHANNEL CORRECTION

How well does a channel $\mathcal{E}$ preserve quantum information? That is, how well does it preserve the entanglement that an unknown input state shares with other systems? A way to give a quantitative answer to this question is to introduce the *entanglement fidelity*, that is a nonnegative quantity, depending on the channel $\mathcal{E}$ (we now suppose that the output space coincides with the input one) and on the input state ρ^Q, defined as [15]

$$F(\rho^Q,\mathcal{E}):=\langle\Psi^{RQ}|(\mathrm{id}\otimes\mathcal{E})(\Psi^{RQ})|\Psi^{RQ}\rangle,$$

where Ψ^{RQ} is a purification of ρ^Q as before. It can be proved that $F(\rho^Q,\mathcal{E})$ does not depend on the particular purification Ψ^{RQ} of ρ^Q, and it is an intrinsic property of the channel, given the input state. If $F(\rho^Q,\mathcal{E})$ is close to unity, then the channel $\mathcal{E}$ acts almost similar to the identity channel id on the support of ρ^Q, that is, every state in the support of ρ^Q is faithfully transmitted by $\mathcal{E}$, along with its eventual entanglement with other quantum systems.

Another quantity which tells how much a given channel preserves coherence is given by the *coherent information* $I_c(\rho^Q,\mathcal{E})$, defined as [5,16]

$$I_c(\rho^Q,\mathcal{E}):=S(\rho^{Q'})-S(\rho^{R'Q'})\leqslant S(\rho^Q),$$

where, consistently with the notation introduced in the previous section, $\rho^{Q'}:=\mathcal{E}(\rho^Q)$ and $\rho^{R'Q'}:=(\mathrm{id}\otimes\mathcal{E})\Psi^{RQ}$. The coherent information can be negative and it plays a fundamental role in quantifying the rate at which a channel can reliably transmit quantum information [16–18].

Between entanglement fidelity and coherent information there exists a close relation [8] which states that, given an input state ρ^Q and a channel $\mathcal{E}:\mathfrak{S}(\mathcal{H}^Q)\rightarrow\mathfrak{S}(\mathcal{H}^{Q'})$, there exists a channel $\mathcal{R}:\mathfrak{S}(\mathcal{H}^{Q'})\rightarrow\mathfrak{S}(\mathcal{H}^Q)$ such that

$$F(\rho^Q,\mathcal{R}\circ\mathcal{E})\geqslant 1-\sqrt{2[S(\rho^Q)-I_c(\rho^Q,\mathcal{E})]}.\tag{3}$$

In other words, if the coherent information is close to the input entropy, then the channel can be approximately corrected [19]. Most important, also the converse statement is true, in the sense that a sort of quantum Fano inequality holds [15,20]

$$S(\rho^Q)-I_c(\rho^Q,\mathcal{E})\leqslant h(1-F(\rho^Q,\mathcal{R}\circ\mathcal{E})),\tag{4}$$

for all channels $\mathcal{R}$, where $h(x)$ is an appropriate positive, concave (and hence continuous), monotonically increasing function such that $\lim_{x\rightarrow 0}h(x)=0$. In particular, for $x\leqslant 1/2$, we can take $h(x):=4x\log_2(d/x)$, where $d:=\dim\mathcal{H}^Q$ [15,20]. In other words, if a channel $\mathcal{R}$ happens to approximately correct the channel $\mathcal{E}$, then $I_c(\rho^Q,\mathcal{E})$ has to be correspondingly close to the input entropy. Notice that Eqs. (3) and (4) are nothing but entropic formulations of the fact that approximate correction is possible if and only if the joint reference-ancilla output state $\rho^{R'E'}$ is close to being factorized, that is, $\rho^{R'E'}\approx\rho^{R'}\otimes\rho^{E'}$ (about this point, see also Ref. [21]). In fact,

$$S(\rho^Q)-I_c(\rho^Q,\mathcal{E})=I^{R':E'}(\rho^{R'E'})=D(\rho^{R'E'}\|\rho^{R'}\otimes\rho^{E'}),$$

where $D(\rho\|\sigma):=\mathrm{Tr}[\rho\log_2\rho-\rho\log_2\sigma]$ is the *quantum relative entropy* and can be understood as a kind of distance between states.

From Eqs. (3) and (4), it is an immediate corollary that *perfect* correction (on the support of ρ^Q) is possible if and only if [5]

$$I_c(\rho^Q,\mathcal{E})=S(\rho^Q).$$

However, coherent information is not the only quantity which enjoys such a property. By introducing the *entanglement of formation*, defined for a bipartite mixed state σ^{AB} as [22]

$$E_f(\sigma^{AB}):=\min_{\{p_i,|\phi_i^{AB}\rangle\}_i:\Sigma_i p_i\phi_i^{AB}=\sigma^{AB}}\sum_i p_iE(\phi_i^{AB}),$$

where the minimum is taken over all possible pure state ensemble decomposition of σ^{AB} as $\sigma^{AB}=\Sigma_ip_i\phi_i^{AB}$ and $E(\phi^{AB}):=S(\mathrm{Tr}_B[\phi^{AB}])$ is the *entanglement* of the pure bipartite state ϕ^{AB}, in Ref. [9] it is proved that *perfect* correction (on the support of ρ^Q) is possible if and only if

$$E_f(\rho^{R'Q'})=S(\rho^Q).$$

The "only if" part is not surprising, since it is known that (for an elementary proof, see Sec. IV below)

$$I_c(\rho^Q,\mathcal{E})\leqslant E_f(\rho^{R'Q'}),\tag{5}$$

and the above relation can hold *strictly* (in fact, coherent information can easily be negative). Hence we immediately obtain the analogous of Eq. (4),

$$S(\rho^Q)-E_f(\rho^{R'Q'})\leqslant h(1-F(\rho^Q,\mathcal{R}\circ\mathcal{E})),\tag{6}$$

that is, the existence of an approximately correcting channel $\mathcal{R}$ implies that the entanglement of formation of $\rho^{R'Q'}$ is close to $S(\rho^Q)$ [23].

In Ref. [9] the question was left open whether the converse statement is also true, namely if the entanglement of formation of $\rho^{R'Q'}$ is a *robust* measure of the correctability of a channel. Before answering (affirmatively) this question, we have to go back to the unitary realization of channels and give an alternative interpretation of the entanglement of formation.

IV. CLASSICAL, QUANTUM, AND TOTAL CORRELATIONS

The entanglement of formation $E_f(\sigma^{AB})$ is a well-behaved measure of the quantum correlations existing between two quantum systems A and B described by the joint state σ^{AB}. On the other hand, the quantum mutual information $I^{A:B}(\sigma^{AB})$ measures the *total* correlations, quantum as well as classical, that a bipartite quantum system exhibits [24]. Notice that both entanglement of formation and quantum mutual information are by construction symmetric under the exchange of A and B.

On the contrary, the quantity measuring the amount of *classical* correlations in a bipartite quantum state loses such a symmetry, and a logical direction of classical correlations seems to naturally emerge. Such a quantity, proposed in Ref. [25], is defined as

$$C^{B\to A}(\sigma^{AB}):=\max_{\{P_i^B\}_i}\left[S(\sigma^A)-\sum_i p_iS\left(\frac{\mathrm{Tr}_B[\sigma^{AB}(\mathbb{1}^A\otimes P_i^B)]}{p_i}\right)\right],$$

where the maximum is taken over all possible positive-operator-valued measures (POVMs) $\{P_i^B\}_i$ (that is, $P_i^B>0$ for all i, and $\Sigma_iP_i^B=\mathbb{1}^B$) on the subsystem B and $p_i:=\mathrm{Tr}[\sigma^BP_i^B]$. Such a measure is asymmetric, since in general $C^{B\to A}(\sigma^{AB})\neq C^{A\to B}(\sigma^{AB})$, and it is closely related to the assisted classical capacity of quantum channels [26].

In Ref. [27] it is proved that for a tripartite pure state $|\phi^{ABC}\rangle$ the relation $C^{B\to A}(\sigma^{AB})+E_f(\sigma^{AC})=S(\sigma^A)$ holds, where σ^{AB} etc. are the reduced states of $|\phi^{ABC}\rangle$. In the case of a channel, given the global state $|\Psi^{R'Q'E'}\rangle$ in Eq. (1), we correspondingly have

$$C^{E'\to R'}(\rho^{R'E'})+E_f(\rho^{R'Q'})=S(\rho^Q).\tag{7}$$

We are now able to easily prove Eq. (5). In fact, since $I_c(\rho^Q,\mathcal{E})=S(\rho^Q)-I^{R':E'}(\rho^{R'E'})$, and from Eq. (7), thanks to the monotonicity of quantum relative entropy under the action of channels, namely $D(\rho\|\sigma)\geqslant D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma))$, $\forall\ (\rho,\sigma,\mathcal{E})$, we have that

$$C^{E'\to R'}(\rho^{R'E'})\leqslant I^{R':E'}(\rho^{R'E'}),\tag{8}$$

which in turn directly implies

$$I_c(\rho^Q,\mathcal{E})\leqslant E_f(\rho^{R'Q'}).$$

V. ENTANGLEMENT OF FORMATION AND APPROXIMATE CHANNEL CORRECTION

In this section we will present the main result, that is, the loss of entanglement of formation is small if and only if the channel can be approximately corrected. We saw before that approximate correction is possible if and only if the joint reference-ancilla output state $\rho^{R'E'}$ is almost factorized [8]. We would then like to say that the loss of entanglement of formation is small if and only if $\rho^{R'E'}$ is almost factorized.

The "if" part has already been written in the form of Eq. (8). In fact, if $\rho^{R'E'}\approx\rho^{R'}\otimes\rho^{E'}$, then $S(\rho^{R'}\otimes\rho^{E'})\approx S(\rho^{R'E'})$ thanks to Fannes' continuity property, which implies that $I^{R':E'}(\rho^{R'E'})\approx 0$, and, in turn, that $C^{E'\to R'}(\rho^{R'E'})\approx 0$, or, equivalently, that $E_f(\rho^{R'Q'})\approx S(\rho^Q)$ [see Eq. (7)].

To prove the "only if" part is a little trickier. We exploit the existence, proved in Ref. [28] for every (finite) dimension of the Hilbert space, of (rank-one) *informationally complete measurements*, that are POVMs whose elements form a basis for the operator space. In other words, there always exists a POVM $\{P_i\}_i$ such that $\mathrm{Tr}[XP_i]=0$ for all i if and only if $X=0$. Notice that this is the generalization of the usual concept of quantum state tomography. Informationally complete POVMs have a (generally nonunique) dual set $\{\tilde{P}_i\}_i$ such that the following reconstruction formula holds

$$\sum_i \mathrm{Tr}[XP_i]\tilde{P}_i = X, \quad \forall \; X. \tag{9}$$

Notice that the dual operators $\tilde{P}_i$ are generally nonpositive, but can always be chosen Hermitian [29]. We are now in position to write the following chain of inequalities ($\|X\|_1 := \mathrm{Tr}|X|$ denotes the trace norm):

$$\begin{aligned}\|\rho^{R'E'} - \rho^{R'} \otimes \rho^{E'}\|_1^2 &= \left\| \sum_i p_i (\rho_i^{R'} \otimes \tilde{P}_i^{E'} - \rho^{R'} \otimes \tilde{P}_i^{E'}) \right\|_1^2 \\ &\leqslant \sum_i p_i \|(\rho_i^{R'} - \rho^{R'}) \otimes \tilde{P}_i^{E'}\|_1^2 \\ &\leqslant K \sum_i p_i \|\rho_i^{R'} - \rho^{R'}\|_1^2 \\ &\leqslant 2K \sum_i p_i D(\rho_i^{R'} \| \rho^{R'}) \\ &\leqslant 2K C^{E' \to R'}(\rho^{R'E'}) \\ &= 2K[S(\rho^Q) - E_f(\rho^{R'Q'})]. \end{aligned} \tag{10}$$

Let us explain one by one all the passages in the above equation.

(i) In the first line we applied identity (9) to the subsystem E', where $\{P_i^{E'}\}_i$ is an informationally complete POVM and $\{\tilde{P}_i^{E'}\}_i$ its dual frame, and defined $p_i := \mathrm{Tr}[\rho^{E'} P_i^{E'}]$ and $\rho_i^{R'} := \mathrm{Tr}_{E'}[\rho^{R'E'}(\mathbb{1}^{R'} \otimes P_i^{E'})]/p_i$.

(ii) In the second line we used the convexity of the function $x \mapsto x^2$.

(iii) In the third line we defined $K := \max_i \|\tilde{P}_i^{E'}\|_1^2$, which is finite because we are considering finite-dimensional Hilbert spaces.

(iv) In the fourth line we used Pinsker inequality [30], that is, $\|\rho - \sigma\|_1^2 \leqslant 2D(\rho \| \sigma)$.

(v) In the fifth line we simply used the fact that $C^{E' \to R'}(\rho^{R'E'})$ is defined as a *maximum* over all possible measurements on E'.

(vi) In the last line we used Eq. (7).

Summarizing, we obtained that whenever $C^{E' \to R'}(\rho^{R'E'}) \to 0$, or, equivalently, $E_f(\rho^{R'Q'}) \to S(\rho^Q)$, then $\|\rho^{R'E'} - \rho^{R'} \otimes \rho^{E'}\|_1^2 \to 0$ correspondingly, which in turn implies the existence of an approximately correcting channel $\mathcal{R}$ [8]. Notice that, as a trivial corollary, we obtain that $C^{B \to A}(\sigma^{AB})=0$ if and only if $\sigma^{AB} = \sigma^A \otimes \sigma^B$.

In the sequence of inequalities in Eq. (10), the most unpleasant feature is the size of the constant K. In fact, it is clearly independent of the channel and the input state, however, we did not investigate how it depends on the dimensions of the input and output Hilbert spaces $\mathcal{H}^Q$ and $\mathcal{H}^{Q'}$. We can give a rough upper bound on K by considering the (continuous outcome) informationally complete POVM $\{P_g\}_{g \in \mathbb{SU}(d)}$ defined as

$$P_g := \frac{1}{d} U_g \varphi U_g^\dagger,$$

where U_g is a unitary representation of the group $\mathbb{SU}(d)$, and φ is a pure state. In Ref. [28] the canonical dual set $\{\tilde{P}_g\}_g$ has been explicitly calculated, and it holds that

$$\|\tilde{P}_g\|_1 = 2d - 1, \quad \forall \; g,$$

where d is the dimension of the Hilbert space on which the POVM $\{P_g\}_g$ is measured, in our case $\mathcal{H}^{E'}$. Since we saw that its dimension can be upper bounded as $\dim \mathcal{H}^{E'} \leqslant \dim \mathcal{H}^Q \times \dim \mathcal{H}^{Q'}$, we obtain the following:

$$\|\rho^{R'E'} - \rho^{R'} \otimes \rho^{E'}\|_1^2 \leqslant 2(2dd' - 1)^2 [S(\rho^Q) - E_f(\rho^{R'Q'})], \tag{11}$$

where $d := \dim \mathcal{H}^Q$ and $d' := \dim \mathcal{H}^{Q'}$. Anyway, the only assumption we need about the ancilla POVM $\{P_i^{E'}\}_i$ is that it is informationally complete. We could hence use the one, among informationally complete POVMs, whose dual set minimizes K. How to choose such an "optimal" informationally complete measurement is left as a wide open question.

At the end we can state the following:

Theorem 1. Given an input state ρ^Q, defined on the Hilbert space $\mathcal{H}^Q$, and a channel $\mathcal{E}$ mapping states on $\mathcal{H}^Q$ to states on $\mathcal{H}^{Q'}$, let us define $\varepsilon_f := S(\rho^Q) - E_f(\rho^{R'Q'})$. Then, there exists a channel $\mathcal{R}$, from states on $\mathcal{H}^{Q'}$ to states on $\mathcal{H}^Q$, such that

$$F(\rho^Q, \mathcal{R} \circ \mathcal{E}) \geqslant 1 - \sqrt{2(2dd' - 1)^2 \varepsilon_f}, \tag{12}$$

where $d := \dim \mathcal{H}^Q$ and $d' := \dim \mathcal{H}^{Q'}$.

Proof. With Eq. (11) at hand, the proof is straightforward. It makes use of the well-known relation existing between fidelity and trace distance, that is

$$\mathcal{F}(\rho, \sigma) \geqslant 1 - \frac{\|\rho - \sigma\|_1}{2},$$

and of the main result of Ref. [8], thanks to which the existence of a channel $\mathcal{R}$ such that

$$F(\rho^Q, \mathcal{R} \circ \mathcal{E}) \geqslant \mathcal{F}^2(\rho^{R'E'}, \rho^{R'} \otimes \rho^{E'})$$

is guaranteed. ■

VI. OTHER ENTANGLEMENT MEASURES

Up to now, we considered the entanglement of formation E_f as the entanglement measure quantifying quantum correlations. Such a choice is motivated by the fact that it is known [30] that E_f is an upper bound to the coherent information itself as well as to many other genuine entanglement measures $E_\bullet$ (among these, for example, one finds the *distillable entanglement* [31], the *relative entropy of entanglement*

[32], and the *squashed entanglement* [33], just to cite three of them). The following corollary directly stems from Theorem 1.

Corollary 1. If $E_\bullet \leqslant E_f$, the following inequality holds

$$F(\rho^Q, \mathcal{R} \circ \mathcal{E}) \geqslant 1 - \sqrt{2(2dd' - 1)^2 \varepsilon_\bullet}, \tag{13}$$

where $\varepsilon_\bullet := S(\rho^Q) - E_\bullet(\rho^{R'Q'})$.

Proof. Trivial. ■

Then, thanks to the above mentioned "extremality property" enjoyed by the entanglement of formation among entanglement measures, Corollary 1 can be applied to many different situations, making the conclusions we drew from Theorem 1 quite general.

On the other hand, the so-called hashing inequality [34]

$$I_c(\rho^Q, \mathcal{E}) \leqslant E_{\mathrm{d}}^{R' \to Q'}(\rho^{R'Q'}) \quad [\leqslant E_{\mathrm{d}}(\rho^{R'Q'})], \tag{14}$$

where $E_d(\sigma^{AB})$ is the distillable entanglement and $E_d^{A\to B}(\sigma^{AB})$ is the *one-way* distillable entanglement (i.e., we restrict the classical communication to go from A to B only), implies the converse direction, namely, if $E_d^{A\to B} \leqslant E_\bullet$, then the analogous of Eq. (4),

$$S(\rho^Q) - E_\bullet(\rho^{R'Q'}) \leqslant h(1 - F(\rho^Q, \mathcal{R} \circ \mathcal{E})), \tag{15}$$

holds true. It is worth stressing here that while the condition $E_d^{A\to B} \leqslant E_\bullet$ is very general, the condition $E_\bullet \leqslant E_f$ is satisfied by many among known entanglement measures but not by all of them (a notable exception is, for example, the *logarithmic negativity* [35]). Nevertheless, it is known that whatever generic entanglement measure satisfying a certain number of conditions can be proved to lie between $E_d^{A\to B}$ and E_f [36]. Hence inequivalent entanglement measures, provided they behave "sufficiently well," lead to equivalent conditions for *approximate* quantum error correction, generalizing what was already noted in Ref. [9] in the case of *exact* correction.

By further specializing the entanglement measure, we can say more. If the entanglement measure is chosen to be "not too large," it is possible to refine the bound (13) as follows. More explicitly, the following result, that we state as a second theorem independent from Theorem 1, can be proved.

Theorem 2. Let $\mathcal{E}$ be a channel acting on states on the input Hilbert space $\mathcal{H}^Q$. Let $E_\bullet(\sigma^{AB})$ be an entanglement measure such that

$$E_\bullet(\sigma^{AB}) \leqslant \frac{I^{A:B}(\sigma^{AB})}{2}, \quad \forall \ \sigma^{AB} \tag{16}$$

holds, and define $\varepsilon_\bullet := S(\rho^Q) - E_\bullet(\rho^{R'Q'})$. Then, there exists a channel $\mathcal{R}$ such that

$$F(\rho^Q, \mathcal{R} \circ \mathcal{E}) \geqslant 1 - 2\sqrt{\varepsilon_\bullet}\,. \tag{17}$$

Proof. The proof goes as follows:

$$\begin{aligned}\frac{1}{2}\|\rho^{R'E'} - \rho^{R'} \otimes \rho^{E'}\|_1^2 &\leqslant D(\rho^{R'E'} \| \rho^{R'} \otimes \rho^{E'}) \\ &= 2S(\rho^Q) - I^{R':Q'}(\rho^{R'Q'}) \\ &\leqslant 2S(\rho^Q) - 2E_\bullet(\rho^{R'Q'}),\end{aligned}$$

where we used again Pinsker inequality and Eq. (2). At this point, by the same passages as in the proof of Theorem 1, we obtain the statement. ■

Relation (17) is clearly much tighter than the analogous relation (13), in that here we succeeded in eliminating the dependence on the dimensions of the input and output Hilbert space. Notice that condition (16) is proved to hold for the distillable entanglement and for the squashed entanglement [33]. On the other hand, such a derivation cannot be applied to the entanglement of formation, which can be smaller or larger than the quantum mutual entropy [10].

VII. RELATION BETWEEN ENTANGLEMENT OF FORMATION AND DISTILLABLE ENTANGLEMENT

It is interesting to directly compare the three relations [Eqs. (3), (13), and (17)] for approximate quantum error correction that we considered throughout the paper:

$$\begin{aligned}F(\rho^Q, \mathcal{R} \circ \mathcal{E}) &\geqslant 1 - \sqrt{2[S(\rho^Q) - I_c(\rho^Q, \mathcal{E})]}, \\ F(\rho^Q, \mathcal{R} \circ \mathcal{E}) &\geqslant 1 - 2\sqrt{S(\rho^Q) - E_\bullet(\rho^{R'Q'})}, \\ F(\rho^Q, \mathcal{R} \circ \mathcal{E}) &\geqslant 1 - \sqrt{2(2dd' - 1)^2[S(\rho^Q) - E_\bullet(\rho^{R'Q'})]},\end{aligned} \tag{18}$$

where $d := \dim \mathcal{H}^Q$ and $d' := \dim \mathcal{H}^{Q'}$. The first is proved in Ref. [8], the second holds if $E_\bullet(\sigma^{AB}) \leqslant I^{A:B}(\sigma^{AB})/2$, while the third holds if $E_\bullet(\sigma^{AB}) \leqslant E_f(\sigma^{AB})$. The numerical factor in front of the "loss figure" becomes larger as we move from coherent-information loss toward entanglement-of-formation loss. This feature is reminiscent of the fact that, in general, the gap $E_f > E_d$ between entanglement of formation and distillable entanglement can be generically large [10].

Concerning this point, it is interesting to notice that our approach can be somehow useful to understand to which extent such a gap can be authentically arbitrary. In fact, entanglement of formation and distillable entanglement coincide on pure states, and both of them are known to be asymptotically continuous in the mixed neighborhood of every pure state [36]. It is then reasonable that, sufficiently close to pure states, entanglement of formation and distillable entanglement become equivalent entanglement measures (in the sense that they can be reciprocally bounded), and the gap between them cannot be completely arbitrary. In fact we can say something more in the form of the following:

Corollary 2. For an arbitrary bipartite mixed state σ^{AB}, with $S(\sigma^A) \leqslant S(\sigma^B)$, let us define the coherent information

$$I_c^{A\to B}(\sigma^{AB}) := S(\sigma^B) - S(\sigma^{AB})$$

and the entanglement of formation deficit

$$\varepsilon_f(\sigma^{AB}) := S(\sigma^A) - E_f(\sigma^{AB}).$$

Then, the following inequality holds

$$S(\sigma^A) - I_c^{A\to B}(\sigma^{AB}) \leqslant h(\sqrt{2(2d_A d_B - 1)^2 \varepsilon_f(\sigma^{AB})}), \tag{19}$$

where $h(x)$ is a function as in Eq. (4), and $d_{A(B)} := \dim \mathcal{H}^{A(B)}$.

Proof. First of all, let us notice that whatever bipartite mixed state σ^{AB} can be written as $(\mathrm{id}^A\otimes\mathcal{E}^B)(\Psi^{\mathrm{AB}})$, for some channel $\mathcal{E}^B$ and some pure Ψ^{AB} such that $\mathrm{Tr}_B[\Psi^{\mathrm{AB}}]=\sigma^A$. This simple observation is in order to make sure that all equations, previously obtained for bipartite states $\rho^{R'Q'}=(\mathrm{id}^R\otimes\mathcal{E}^Q)\times(\Psi^{\mathrm{RQ}})$, can be in particular interpreted as equations valid for all bipartite mixed states σ^{AB} as well, simply paying attention to the directionality intrinsic in the definition of coherent information. Then, putting together Eqs. (4) and (12), we obtain the statement (19). ∎

The large numerical factor multiplying ε_f in Eq. (19) makes possible the above-mentioned generic gap exhibited by high-dimensional systems, that is, entanglement of formation can be close to the maximum value, while distillable entanglement is almost null. Equation (19) then says "how large," for fixed finite dimensions d_A and d_B, the gap can actually be; in fact we can affirm that if the entanglement of formation is "sufficiently close" to its maximum value, then also the coherent information and, thanks to the hashing inequality (14), the one-way distillable entanglement have to be "correspondingly large." Notice moreover that there may be room for a further improvement of Eq. (19), since we obtained it as coming from a probably oversimplified estimation. To tighten the evaluation of the constant K in Eq. (10) could then be useful in understanding the relationships between entanglement of formation and distillable entanglement as well, besides being an interesting mathematical problem by itself.

Before concluding, we would like to stress one more remark. It is clear from Eq. (18) how we are actually dealing with three different topologies on the set of quantum channels induced by different measures of bipartite entanglement [37]. Also this connection definitely deserves further investigation.

VIII. CONCLUSIONS

In summary, we generalized the information-theoretical analysis of approximate quantum error correction based on coherent information given in Ref. [8] by showing that approximate quantum error correction is possible if and only if the loss of entanglement along the quantum channel is small. We considered explicitly different entanglement measures, in particular the entanglement of formation and the distillable entanglement, showing how equivalent conclusions come from inequivalent entanglement measures. We moreover showed that the approach used here can be applied also to understand the interconnections existing between entanglement of formation and distillable entanglement, even though they are known to behave quite independently, in particular in high-dimensional quantum systems.

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$$F(\rho^Q,\mathcal{R}\circ\mathcal{E})\geqslant 1-2\sqrt{[S(\rho^Q)-I_c(\rho^Q,\mathcal{E})]},$$

which is indeed a little looser than Eq. (3). It is however clear, already from the arguments used there, that Eq. (3) actually holds true.

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