# Graph states as ground states of many-body spin-1/2 Hamiltonians

M. Van den Nest,[1] K. Luttmer,[1] W. Dür,[1,2] and H. J. Briegel[1,2]

[1]*Institut für Quantenoptik und Quanteninformation der Österreichischen Akademie der Wissenschaften, Innsbruck, Austria*
[2]*Institut für Theoretische Physik, Universität Innsbruck, Technikerstraße 25, A-6020 Innsbruck, Austria*
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We consider the problem of whether graph states can be ground states of local interaction Hamiltonians. For Hamiltonians acting on $n$ qubits that involve at most two-body interactions, we show that no $n$-qubit graph state can be the exact, nondegenerate ground state. We determine for any graph state the minimal $d$ such that it is the nondegenerate ground state of a $d$-body interaction Hamiltonian, while we show for $d'$-body Hamiltonians $H$ with $d' < d$ that the resulting ground state can only be close to the graph state at the cost of $H$ having a small energy gap relative to the total energy. When allowing for ancilla particles, we show how to utilize a gadget construction introduced in the context of the $k$-local Hamiltonian problem, to obtain $n$-qubit graph states as nondegenerate (quasi)ground states of a two-body Hamiltonian acting on $n' > n$ spins.

## I. INTRODUCTION

Graph states form a large family of multiparticle quantum states that are associated with mathematical graphs. More precisely, to any $n$-vertex graph $G$ an $n$-qubit graph state $|G\rangle$ is associated. Graph states play an important role in several applications in quantum information theory and quantum computation [1]. Most prominently, the graph states that correspond to a two-dimensional square lattice, also known as the 2D cluster states [2], are known to be universal resources for measurement based quantum computation [3,4]. Other graph states serve as algorithmic specific resources for measurement based quantum computation, are codewords of error correcting codes known as Calderbank-Shor-Steane codes [6,7], or are used in multiparty communication schemes [8–10]. Graph states are specific instances of stabilizer states, which allow for an efficient description and manipulation of the states by making use of the stabilizer formalism [5]. This also makes them attractive as test-bed states to investigate the complex structure of multipartite entangled quantum systems [1]. We refer to Ref. [1] for an extensive review about graph states and their applications.

While it is known how to efficiently prepare $n$-qubit graph states in highly controlled quantum systems using only $O(n^2)$ phase gates, the question of whether certain graph states can occur naturally as nondegenerate ground states of certain, physically reasonable, local interaction Hamiltonians remains unanswered so far. This question is of direct practical relevance, as if this would be the case, graph states could simply be prepared by cooling a system governed by such a local interaction Hamiltonian, without further need to access or manipulate individual qubits in a controlled way. Moreover, measurement based quantum computation could then be realized by simply cooling and measuring.

The first results in this context have recently been obtained by Nielsen in Ref. [11], where classes of graph states which *cannot* possibly occur as ground states of two-body Hamiltonians have been constructed. Moreover, numerical evidence was reported which raised the conjecture that in fact no $n$-qubit graph state can occur as the nondegenerate ground state of an $n$-qubit two-body spin-1/2 Hamiltonian—

and this conjecture is believed to hold by several researchers in the field. Nonetheless, a definitive answer to the question of whether there exists graph states which are nondegenerate ground states of two-body Hamiltonian, is to date missing. In this article we provide a proof of this conjecture for arbitrary $n$-qubit graph states and two-body interaction Hamiltonians acting on $n$ qubits.

In fact, we provide some more general results, where we consider $n$-qubit graph states $|G\rangle$ that are ground states of $d$-body Hamiltonians $H$ acting on $n$ qubit systems, where $d$ is possibly larger than two. We find the following.

(i) For every graph state $|G\rangle$, we determine the minimal $d$ such that $|G\rangle$ is a, possibly degenerate, ground state of a $d$-body Hamiltonian $H$, and show that $d$ is equal to the minimal weight of the stabilizer group of $|G\rangle$.

(ii) For every graph state $|G\rangle$, we determine the minimal $d$ such that $|G\rangle$ is the *nondegenerate* ground state of a $d$-body Hamiltonian $H$, where we show that $d$ is again related to the stabilizer via a quantity which we denote by $\eta(|G\rangle)$. In addition, we find that $\eta(|G\rangle)$ cannot be smaller than 3 for all graph states, i.e., *no graph state can be the nondegenerate ground state of a two-body Hamiltonian*.

(iii) For $d' < \eta(|G\rangle)$, we show that the ground state of any $d'$-body Hamiltonian $H$ can only be $\epsilon$-close to $|G\rangle$ at the cost of $H$ having an energy gap (relative to the total energy) that is proportional to $\epsilon$.

If we allow for ancilla particles, i.e., consider systems of $n' > n$ spins where we are interested only in the state of a subset of $n$ qubits, we find the following.

(iv) Any graph state $|G\rangle$ of $n$ qubits can be the nondegenerate (quasi)ground state of a two-body Hamiltonian that acts on $n' > n$ qubits.

The latter result follows from a so-called *gadget construction* introduced in Refs. [19,20] in the context of the $k$-local Hamiltonian problem. Here, the ancilla particles act as mediating particles to generate an effective many-body Hamiltonian on the $n$ system particles, using only two-body interactions. Although this leads in principle to a way to obtain graph states as nondegenerate ground states of two-body interaction Hamiltonians, a high degree of control is required in the interaction Hamiltonians as the parameters describing

the interaction need to be precisely adjusted. We remark that this result is closely related to a recent finding of Bartlett and Rudolph [12], who show how to obtain an *encoded* graph state corresponding to a universal resource for measurement based quantum computation (a 2D cluster state or a state corresponding to a honeycomb lattice [13]) using a construction based on projected entangled pairs.

The paper is organized as follows. In Sec. II, we provide definitions and settle notation. In Sec. III, we consider degenerate ground states and provide a simple argument to determine the minimal $d$ such that $|G\rangle$ can be the exact ground state of a $d$-body Hamiltonian. In Sec. IV we treat the nondegenerate case, and compute, for every graph state, the minimal $d$ such that this graph state is the nondegenerate ground state of a $d$-body Hamiltonian; this quantity is denoted by $\eta(|G\rangle)$. In Sec. V we consider approximate ground states, i.e., approximations of $|G\rangle$ using $d'$-body Hamiltonians where $d'$ is strictly smaller than $\eta(|G\rangle)$. In Sec. VI we consider the case of ancilla particles, and show how to obtain any $n$-qubit graph state as a nondegenerate (quasi)ground state of a two-body Hamiltonian on $n' > n$ qubits. We demonstrate the gadget construction for the one-dimensional cluster state and the honeycomb lattice. Finally, we conclude in Sec. VII.

## II. DEFINITIONS AND NOTATIONS

The Pauli spin matrices are denoted by $\sigma_x$, $\sigma_y$, $\sigma_z$, and $\sigma_0$ denotes the $2 \times 2$ identity operator. A Pauli operator on $n$ qubits is an operator of the form $\sigma = \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}$, where $i_1, \ldots, i_n \in \{0, x, y, z\}$. The *weight* wt($\sigma$) of $\sigma$ is the number of qubit systems on which this operator acts nontrivially. A $d$-body spin-1/2 Hamiltonian on $n$ qubits is a Hermitian operator of the form $H = \Sigma_\sigma h_\sigma \sigma$, where the sum runs over all $n$-qubit Pauli operators $\sigma$ with wt($\sigma$) $\leq d$, and where the $h_\sigma$ are real coefficients.

Let $G = (V, E)$ be a graph with vertex set $V = \{1, \ldots, n\}$ and edge set $E$. The graph state $|G\rangle$ is defined to be the simultaneous fixed point of the $n$ operators

$$\sigma_x^{(a)} \prod_{b \in N(a)} \sigma_z^{(b)}, \qquad (1)$$

for every $a \in V$, where $N(a) \subseteq V$ denotes the set of vertices $b$ connected to $a$ by an edge. The superscripts denote on which system a Pauli operator acts. The stabilizer $\mathcal{S}$ of $|G\rangle$ is the group of all operators of the form $g = \pm \sigma$, where $\sigma$ is a Pauli operator on $n$ qubits, satisfying $g|G\rangle = |G\rangle$. Recall that $|\mathcal{S}| = 2^n$ and that $\mathcal{S}$ is an Abelian group, i.e., $[g, g'] = 0$ for every $g, g' \in \mathcal{S}$. We will frequently use the expansion

$$|G\rangle\langle G| = \frac{1}{2^n} \sum_{g \in \mathcal{S}} g. \qquad (2)$$

We refer to Ref. [1] for extensive material regarding graph states and the stabilizer formalism.

To avoid technical issues arising in trivial cases, we will only consider fully entangled graph states (corresponding to connected graphs) on $n \geq 3$ qubits.

## III. DEGENERATE GROUND STATES

In this section we investigate under which conditions a graph state is the ground state of a $d$-body Hamiltonian. At this point we will not yet require that this ground state is nondegenerate—this case will be considered below—which is a considerable simplification, and we will see that elementary arguments suffice to gain total insight in this matter. In Secs. III A and III B these insights are obtained, and examples are given in Sec. III C.

### A. General results

In order to exclude trivial cases, we will only consider Hamiltonians which are both nonzero and not a multiple of the identity; such Hamiltonians will be called *nontrivial*. We will need the following definition.

*Definition 1.* Letting $|G\rangle$ be an $n$-qubit graph state with stabilizer $\mathcal{S}$, define

$$\delta(|G\rangle) := \min_{g \in \mathcal{S} \setminus \{I\}} \text{wt}(g), \qquad (3)$$

i.e., $\delta(|G\rangle)$ is defined to be the minimal weight of $\mathcal{S}$.

One immediately finds that the nontrivial $\delta(|G\rangle)$-body Hamiltonian

$$H := - \sum_{g \in \mathcal{S}, \text{wt}(g) = \delta(|G\rangle)} g \qquad (4)$$

has the state $|G\rangle$ as a ground state. Furthermore, let $d < \delta(|G\rangle)$ and suppose that $H'$ is a nontrivial $d$-body Hamiltonian having $|G\rangle$ as a ground state. We prove that this leads to a contradiction. Defining the nontrivial Hamiltonian

$$H'' := H' - 2^{-n} \text{Tr}(H')I, \qquad (5)$$

it follows that $\text{Tr}(H'') = 0$ and that $|G\rangle$ is also a ground state of $H''$. Letting $E_0$ be the ground state energy of $H''$, it follows from Eq. (2) that

$$E_0 = \langle G|H''|G\rangle = \frac{1}{2^n} \sum_{g \in \mathcal{S}} \text{Tr}(gH''). \qquad (6)$$

As $d < \delta(|G\rangle)$ and $H''$ is a $d$-body Hamiltonian, one has $\text{Tr}(gH'') = 0$ for every $I \neq g \in \mathcal{S}$. Furthermore, one has

$$\text{Tr}(H'' \cdot I) = \text{Tr}(H'') = 0 \qquad (7)$$

by construction of $H''$, and therefore $\text{Tr}(gH'') = 0$ for *every* $g \in \mathcal{S}$. Together with Eq. (6), this shows that $E_0 = 0$. As $H''$ has zero trace and $E_0 = 0$ is the smallest eigenvalue, this implies that $H'' = 0$, yielding a contradiction, since $H'$ was assumed to be nontrivial. This shows that $|G\rangle$ cannot be the ground state of $H'$.

We have proven the following result.

*Theorem 1.* Let $|G\rangle$ be a graph state on $n$ qubits and let $\delta(|G\rangle)$ be defined as above. Then (i) there exists a nontrivial $\delta(|G\rangle)$-body Hamiltonian on $n$ qubits having $|G\rangle$ as a—possibly degenerate—ground state; (ii) any nontrivial $n$-qubit Hamiltonian having $|G\rangle$ as a ground state must involve at least $\delta(|G\rangle)$-body interactions.

Thus $\delta(|G\rangle)$ is the optimal $d$ such that the graph state $|G\rangle$ is the ground state of a nontrivial $d$-body Hamiltonian. Note

that some graph states can occur as degenerate ground states of two-body Hamiltonians, namely whenever the stabilizer of $|G\rangle$ contains elements of weight 2. This, e.g., occurs when the graph has one or more vertices of degree 1 [see Eq. (1)]. We refer to Sec. III C for examples.

In the next section we focus on some techniques to compute $\delta(|G\rangle)$.

## B. Computing $\delta(|G\rangle)$

We will show that $\delta(|G\rangle)$ can be computed directly from the graph $G$ in two distinct ways: One method being algebraic and the other graphical.

In the graphical approach, one considers a graph transformation rule called *local complementation*, defined as follows. Let $a \in V$ be a vertex of $G$ and let $N(a) \subseteq V$ denote the neighborhood of this vertex. The local complement $G * a$ of the graph $G$ at the vertex $a$ is defined to be the graph obtained by complementing (i.e., replacing edges with non-edges and vice versa) the subgraph of $G$ induced on the subset $N(a)$ of vertices, and leaving the rest of the graph unchanged. Local complementations of graphs correspond to local (Clifford) operations on the corresponding graph states [15]. Moreover, it was proven in Ref. [14] that $\delta(|G\rangle) - 1$ is equal to the minimal vertex degree of any graph which can be obtained from $G$ by applying local complementations. This immediately yields a graphical method to calculate $\delta(|G\rangle)$, as one simply has to draw all graphs which can be obtained from $G$ by local complementations and determine the smallest vertex degree in this class of graphs. We refer to Refs. [1,15] for more details on local complementation of graphs and local operations on graph states.

In the algebraic approach, one considers the adjacency matrix $\Gamma$ of $G$ and defines, for every subset $A \subseteq V$ of the vertex set of $G$, the $|A| \times (n - |A|)$ matrix

$$\Gamma\langle A\rangle := (\Gamma_{ab})_{a \in A, b \in V\backslash A}. \tag{8}$$

We can then formulate the following result.

*Theorem 2.* Let $G = (V, E)$ be a graph with adjacency matrix $\Gamma$. Then $\delta(|G\rangle)$ is equal to the smallest possible cardinality $|A|$ of a subset $A \subseteq V$ such that $|A| > \mathrm{rank}_2 \Gamma\langle A\rangle$ [where "$\mathrm{rank}_2 X$" denotes the rank of a matrix $X$ over the finite field GF(2)].

*Proof.* The proof uses standard graph state techniques; see Ref. [1]. First one shows that $2^{|A|-\mathrm{rank}_2\Gamma\langle A\rangle}$ is equal to the number of elements $g = \pm\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}$ in the stabilizer $\mathcal{S}$ of $|G\rangle$ satisfying $i_\alpha = 0$ for every $\alpha \in V\backslash A$. Therefore, if $|A| = \mathrm{rank}_2 \Gamma\langle A\rangle$ for every subset $A$ with cardinality $|A| = d$, then $\mathcal{S}$ does not contain elements of weight $d$ or less. The result then immediately follows. ∎

Note that the above result yields a polynomial time algorithm to test, for a given constant $k$ (independent of $n$), whether $\delta(|G\rangle)$ is smaller than $k$. In order to do so, one needs to calculate the rank over GF(2) of all matrices $\Gamma\langle A\rangle$ with $|A| \leq k$; since the number of such matrices is $\sum_{i=0}^{k}\binom{n}{i}$ (which is polynomial in $n$) and the rank of a matrix can be calculated in polynomial time in the dimensions of the matrix, one obtains a polynomial algorithm to verify whether $\delta(|G\rangle)$ is

smaller than $k$. However, a direct computation of $\delta(|G\rangle)$ is likely to be a hard problem.

### C. Examples

Here we give some examples of the theoretical results obtained regarding graph states as nondegenerate ground states.

Following theorem 2, one finds that a graph state is the—degenerate—ground state of a two-body Hamiltonian if and only if its stabilizer contains elements of weight 2. For example, the GHZ state $(|0\rangle^n + |1\rangle^n)/\sqrt{2}$ (which is locally equivalent to the graph state defined by the fully connected graph) is the degenerate ground state of the Ising Hamiltonian with zero magnetic field:

$$H = -\sum_{i=1}^{n-1} \sigma_z^{(i)}\sigma_z^{(i+1)}. \tag{9}$$

One can easily verify that this ground state is twofold degenerate.

The linear cluster $|L_n\rangle$ state with open boundary conditions, where $L_n$ is the linear chain on $n$ vertices, also is the degenerate ground state of a two-body Hamiltonian. This immediately follows by considering the stabilizer generators associated to the two boundary vertices of the graph $L_n$, which both have degree 1, therefore $\delta(|L_n\rangle) = 2$ by using the graphical approach to determine $\delta(|G\rangle)$, as explained in Sec. III A. The state $|L_n\rangle$ is the ground state of the Hamiltonian

$$-\sigma_x^{(1)}\sigma_z^{(2)} - \sigma_z^{(n-1)}\sigma_x^{(1)}. \tag{10}$$

However, the degeneracy of the ground state energy is very large, namely $2^{n-2}$. (Moreover, an argument similar to the proof of theorem 3 shows that any two-body Hamiltonian having $|L_n\rangle$ as a ground state must exhibit at least this degeneracy.)

Consider the 1D cluster state $|C_n\rangle$ with periodic boundary conditions, which is a graph state where the underlying graph is a cycle graph $C_n$. The adjacency matrix of $C_n$ is given by

$$\Gamma = \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}, \tag{11}$$

where we give the example for $n = 6$. The periodic boundary conditions are chosen such as to eliminate boundary effects. Using theorem 2, we easily find that $\delta(|C_n\rangle) = 3$. Indeed, $\Gamma\langle\{a\}\rangle$ has rank 1 for every $a \in V$, and $\Gamma\langle\{a,b\}\rangle$ has rank 2 for every two-element subset $\{a,b\}$ of $V$. Moreover, the rank of

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$$\Gamma\langle\{1,2,3\}\rangle = \begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{bmatrix} \qquad (12)$$

is equal to 2, and therefore $\delta(G)=|\{1,2,3\}|=3$.

The 2D cluster state $|C_{k\times k}\rangle$ has $\delta(|C_{k\times k}\rangle)=3$ (open boundary conditions) or $\delta(|C_{k\times k}\rangle)=5$ (periodic boundary conditions), which can be verified by applying theorem 2.

## IV. NONDEGENERATE GROUND STATES

While in the previous section we have found that graph states can be degenerate ground states of two-body Hamiltonians, next we show that they can never be nondegenerate ground states of such Hamiltonians. As in the previous section, first we present the general results in Sec. IV A, after which we give examples in Sec. IV B.

### A. General results

In order to investigate nondegenerate ground states and their relation to graph states, we will need the following definition.

*Definition 2.* Let $|G\rangle$ be a graph state on $n$ qubits with stabilizer $\mathcal{S}$. Let $\mathcal{S}_d$ be the subgroup of $\mathcal{S}$ generated by all elements of weight at most $d$ [17]. Then $\eta(|G\rangle)$ is defined to be the minimal $d$ such that $\mathcal{S}_d=\mathcal{S}$.

We can now state the second main result of this article.

*Theorem 3.* Let $|G\rangle$ be a graph state on $n\geq 3$ qubits, and let $\eta(|G\rangle)$ be defined as above. Then (i) there exists an $\eta(|G\rangle)$-body Hamiltonian on $n$ qubits having $|G\rangle$ as a nondegenerate ground state; (ii) any $n$-qubit Hamiltonian having $|G\rangle$ as a nondegenerate ground state must involve at least $\eta(|G\rangle)$-body interactions; (iii) $\eta(|G\rangle)\geq 3$, i.e., no $n$-qubit graph state is the nondegenerate ground state of a two-body Hamiltonian on $n$ qubits.

*Proof.* First we show that $|G\rangle$ is the nondegenerate ground state of an $\eta(|G\rangle)$-body Hamiltonian. To see this, note that by definition of $\eta(|G\rangle)$ there exists a set of generators $\{g_1,\ldots,g_n\}$ of $\mathcal{S}$ such that $\mathrm{wt}(g_i)\leq\eta(|G\rangle)$ for every $i=1,\ldots,n$. Therefore, the Hamiltonian $H:=-\sum_{i=1}^n g_i$ involves at most $\eta(|G\rangle)$-body interactions. Moreover, $H$ has the state $|G\rangle$ as a nondegenerate ground state. To see this, note that the operators $g_i$ mutually commute, and that they have eigenvalues $\pm 1$. Therefore, the smallest eigenvalue of $H$ is equal to $-n$. As $H|G\rangle=-n|G\rangle$ trivially, this shows that $|G\rangle$ is a ground state of $H$. Furthermore, this ground state is nondegenerate, as any state $|\psi\rangle$ satisfying $H|\psi\rangle=-n|\psi\rangle$ must also satisfy $g_i|\psi\rangle=|\psi\rangle$ for every $i=1,\ldots,n$, and therefore $|\psi\rangle$ must be equal to $|G\rangle$ up to a global phase. This shows that $H$ has $|G\rangle$ as the nondegenerate ground state.

Conversely, we prove that any Hamiltonian having $|G\rangle$ as a nondegenerate ground state must involve at least $\eta(|G\rangle)$-body terms. To see this, suppose that $H'$ is a $d$-body Hamiltonian, with $d<\eta(|G\rangle)$, having $|G\rangle$ as a ground state. Let $\{g_1,\ldots,g_s\}$ be a set of independent [16] generators of $\mathcal{S}_d$, where $s=\log_2|\mathcal{S}_d|$. As $d<\eta(|G\rangle)$ it follows that $\mathcal{S}_d$ cannot be equal to $\mathcal{S}$. Hence there exists a nonempty independent set of

elements $\{g_{s+1},\ldots,g_n\}\subseteq\mathcal{S}$, where $\mathrm{wt}(g_i)\geq d+1$ for every $i=s+1,\ldots,n$, such that $\{g_1,\ldots,g_n\}$ is an (independent) generating set of $\mathcal{S}$. We then define $|G;\gamma\rangle$ to be the stabilizer state with stabilizer $\mathcal{S}_\gamma$ generated by the set

$$\{g_1,\ldots,g_s,(-1)^{\gamma_{s+1}}g_{s+1},\ldots,(-1)^{\gamma_n}g_n\}, \qquad (13)$$

for every $\gamma:=(\gamma_{s+1},\ldots,\gamma_n)\in\{0,1\}^{n-s}$.

We now claim that

$$\langle G|\tau|G\rangle = \langle G;\gamma|\tau|G;\gamma\rangle \qquad (14)$$

for every Pauli operator $\tau$ of weight at most $d$. This property can be shown by considering Eq. (2) and a similar expansion for $|G;\gamma\rangle$, and making a distinction between the following cases: (a) $\alpha\tau\in\mathcal{S}$ for some $\alpha=\pm 1$; (b) both $\tau$ and $-\tau$ do not belong to $\mathcal{S}$.

First, if (a) $\alpha\tau\in\mathcal{S}$ for some $\alpha=\pm 1$ then by construction $\alpha\tau\in\mathcal{S}_\gamma$, and therefore

$$\langle G|\tau|G\rangle = \langle G;\gamma|\tau|G;\gamma\rangle = \alpha. \qquad (15)$$

Second, if (b) both $\tau$ and $-\tau$ do not belong to $\mathcal{S}$, then none of these two operators can belong to the stabilizer $\mathcal{S}_\gamma$: Suppose that $\alpha\tau\in\mathcal{S}_\gamma$ for some $\alpha=\pm 1$; then $\alpha\tau$ can be written in a unique way as a product

$$\alpha\tau = \prod_{i=1}^{s} g_i^{a_i} \prod_{j=s+1}^{n} [(-1)^{\gamma_j}g_j]^{a_j} \qquad (16)$$

for some $(a_1,\ldots,a_n)\in\{0,1\}^n$. But then clearly either $\tau$ or $-\tau$ is equal to $g_1^{a_1}\cdots g_n^{a_n}\in\mathcal{S}$, which yields a contradiction. We can now conclude, as both $\tau$ and $-\tau$ belong to neither $\mathcal{S}$ nor $\mathcal{S}_\gamma$, that

$$\langle G|\tau|G\rangle = \langle G;\gamma|\tau|G;\gamma\rangle = 0. \qquad (17)$$

This proves property (14). Using this identity, we find that $\langle G|H'|G\rangle=\langle G;\gamma|H'|G;\gamma\rangle$ for every $\gamma$. As $\langle G|H'|G\rangle$ is equal to the ground state energy $E_0$ of $H'$, this shows that every state $|G;\gamma\rangle$ is an eigenstate of $H'$ with eigenvalue $E_0$. Hence the ground state of $H'$ is degenerate.

Finally, it we prove that $\mathcal{S}_2$ cannot be equal to $\mathcal{S}$ for any (fully entangled) graph state on $n\geq 3$ qubits, which implies that $\eta(|G\rangle)\geq 3$.

Note that a fully entangled graph state cannot have stabilizer elements of weight 1. Suppose that $g_1,\ldots,g_n$ are $n$ Pauli operators of weight 2 which generate $\mathcal{S}$. We will show that this leads to a contradiction. For two arbitrary such operators $g_i$ and $g_j$, we distinguish between three possible cases: (i) $g_i$ and $g_j$ act nontrivially on disjoint pairs of qubits, (ii) $g_i$ and $g_j$ act nontrivially on the same pair of qubits, and (iii) $g_i$ acts nontrivially on qubits $a$ and $b$, and $g_j$ acts on qubits $a$ and $c$, for some $a,b,c\in\{1,\ldots,n\}$. If (ii) is the case, one finds that the operators $g_i$, $g_j$, and $g_ig_j$, which act on the same pair of qubits, belong to the stabilizer. Due to the commutativity of these operators, one finds that up to a local unitary operation, one has $g_i=X\otimes X\otimes I_{n-2}$, $g_j=Z\otimes Z\otimes I_{n-2}$, and $g_ig_j=-Y\otimes Y\otimes I_{n-2}$, where $I_k$ is the identity on $k$ qubits. As

$$\{I_2, X \otimes X, Z \otimes Z, -Y \otimes Y\} \qquad (18)$$

is the complete stabilizer of a two-qubit (Bell) state $|\psi_B\rangle$, it then follows that $|G\rangle$ can be written as a tensor product $|G\rangle = |\psi_B\rangle \otimes |\tilde{\psi}\rangle$ for some $|\tilde{\psi}\rangle$. This yields a contradiction, as we have assumed that $|G\rangle$ is fully entangled. Thus only cases (i) and (iii) occur. It is then easy to show that any set of independent Pauli operators which satisfies these condition, can contain at most $n-1$ elements. However, $n$ elements are required to obtain a full generating set of a stabilizer. This shows that no fully entangled graph state can be generated by weight two elements. This proves the result. ∎

A few remarks are in order. First, the above result proves that graph states cannot be nondegenerate ground states of two-body Hamiltonians. This settles the question raised by Nielsen in Ref. [11].

Second, it follows from the proof of theorem 3 that, given any $d < \eta(|G\rangle)$ and any Hamiltonian $H'$ having $|G\rangle$ as a (necessarily degenerate) ground state, then the degeneracy is at least equal to

$$2^{n-s} = 2^n |\mathcal{S}_d|^{-1}, \qquad (19)$$

since we have proven that the $2^{n-s}$ orthogonal states $\{|G; \gamma\rangle\}_\gamma$ are all ground states of $H'$. Also, note that the $d$-body Hamiltonian

$$H'' := - \sum_{g \in \mathcal{S}, \mathrm{wt}(g) \leq d} g \qquad (20)$$

has $|G\rangle$ as a degenerate ground state, where the degeneracy is exactly equal to Eq. (19).

Finally, we note that computing $\eta(|G\rangle)$ for an arbitrary graph state is likely to be hard. A brute force approach would be the following: Enumerate all generating sets $S := \{g_1, \ldots, g_n\}$ of the stabilizer $\mathcal{S}$ of $|G\rangle$, and determine $\mathrm{wt}(S)$, which is defined to be the maximal weight of an element in $S$. Then the minimal value of $\mathrm{wt}(S)$, when $S$ ranges over of all generating sets, is then equal to $\eta(|G\rangle)$. Clearly this approach is nonpolynomial, as the stabilizer of a graph state on $n$ qubits has $O(2^{n^2})$ generating sets. Nevertheless, in the next section we will encounter some interesting examples of graph states where $\eta(|G\rangle)$ can be computed quickly.

### B. Examples

In this section we consider some examples of the calculation of $\eta(|G\rangle)$. Note that one always has $\eta(|G\rangle) \geq \delta(|G\rangle)$.

Consider the linear cluster state with periodic boundary conditions. The stabilizer of the state $|C_n\rangle$ is generated by the elements

$$\sigma_z^{(a-1)} \sigma_x^{(a)} \sigma_z^{(a+1)} \qquad (21)$$

$(a = 1, \ldots, n)$. This implies that $|C_n\rangle$ is the nondegenerate ground state of the three-body Hamiltonian

$$-\sum_{a=1}^{n} \sigma_z^{(a-1)} \sigma_x^{(a)} \sigma_z^{(a+1)}. \qquad (22)$$

Theorem 3 (iii) then shows that $H$ is optimal in the sense that no two-body Hamiltonian exists having $|C_n\rangle$ as a ground state. Thus we have proven that $\eta(|C_n\rangle) = 3$.

For the linear cluster state with open boundary conditions $|L_n\rangle$ one also finds that $\eta(|L_n\rangle) = 3$, as $|L_n\rangle$ is the nondegenerate ground state of a three-body Hamiltonian analogous to Eq. (22).

As for the 2D cluster states, one finds that $\eta(|C_{k \times k}\rangle) = 5$, showing that at least five-body interactions are needed to have these states as nondegenerate ground states. We note that Nielsen already proved in Ref. [11] that the 2D cluster states on $n$ qubits cannot be ground states of two-body Hamiltonians on $n$ qubits.

### V. APPROXIMATE GROUND STATES

Having determined that at least $\eta(|G\rangle)$-body interactions are needed to obtain a graph state $|G\rangle$ as an *exact* nondegenerate ground state, next we investigate whether it is possible to obtain nondegenerate ground states *close to* graph states when only $d$-body Hamiltonians are considered with $d < \eta(|G\rangle)$. We will show that this is only possible if the Hamiltonians have small energy gaps between the ground state and the first excited level. In order to obtain this result, we prove a technical result which relates the fidelity between a graph state and the ground state of a Hamiltonian, and the spectrum of this Hamiltonian.

*Theorem 4.* Let $d$ be a positive integer. Let $|G\rangle$ be an $n$-qubit graph state with stabilizer $\mathcal{S}$, let $\mathcal{S}_d \subseteq \mathcal{S}$ be defined as above, and denote $r := 2^n |\mathcal{S}_d|^{-1}$. Let $H$ be a $d$-body, $n$-qubit Hamiltonian with ground state $|\psi\rangle$, and let $\mathbf{E} = (E_0, E_1, \ldots, E_{2^n-1})$ be the energies of $H$ in ascending order. Then

$$\frac{1}{\sqrt{2}\|\mathbf{E}\|}\left(\frac{E_0 + \cdots + E_{r-1}}{r} - E_0\right) \leq (1 - |\langle G|\psi\rangle|^2)^{1/2}.$$

*Proof.* Consider the Hermitian operator

$$\rho_d := \frac{1}{2^n} \sum_{g \in \mathcal{S}_d} g. \qquad (23)$$

Note that this operator satisfies

$$(\rho_d)^2 = \frac{1}{2^{2n}} \sum_{g \in \mathcal{S}_d} g \sum_{h \in \mathcal{S}_d} h = \frac{1}{2^{2n}} \sum_{g \in \mathcal{S}_d} \sum_{h \in \mathcal{S}_d} h = \frac{|\mathcal{S}_d|}{2^n} \rho_d = r^{-1} \rho_d.$$

$$(24)$$

The second equality holds since $\mathcal{S}_d$ is a group. It follows that $(r\rho_d)^2 = r\rho_d$, showing that $r\rho_d$ is a projection operator. Thus all nonzero eigenvalues of this operator are equal to 1, implying that the trace of $r\rho_d$ is equal to the rank of this matrix. As the trace of $\rho_d$ is equal to 1, it follows that the rank of $\rho_d$ (and $r\rho_d$) is equal to $r$.

We now use Ky Fan's maximum principle, which states the following: Letting $H$ be any Hermitian operator, the

minimum value of $\text{Tr}(PH)$, when the minimization is taken over all projection operators $P$ of rank $r$, is equal to the sum of the $r$ smallest eigenvalues $E_0, E_1, \ldots, E_{r-1}$ of $H$. As $r\rho_d$ is a rank $r$ projector, this shows that

$$E_0 + E_1 + \cdots + E_{r-1} \leq r\,\text{Tr}(\rho H). \qquad (25)$$

Note that $\text{Tr}(\tau\rho_d) = \langle G|\tau|G\rangle$ for every Pauli operator $\tau$ of weight at most $d$, and we therefore have $\text{Tr}(\rho_d H) = \langle G|H|G\rangle$, showing that

$$\frac{1}{r}(E_0 + E_1 + \cdots + E_{r-1}) \leq \langle G|H|G\rangle. \qquad (26)$$

We now determine an upper bound to $\langle G|H|G\rangle$ in terms of the fidelity $F := |\langle G|\psi\rangle|$. By the Cauchy-Schwarz inequality the trace $\text{Tr}\{H(|G\rangle\langle G| - |\psi\rangle\langle\psi|)\}$ cannot be greater than the product

$$\{\text{Tr}(H^2)\}^{1/2}\{\text{Tr}(|G\rangle\langle G| - |\psi\rangle\langle\psi|)^2\}^{1/2}. \qquad (27)$$

The first factor of Eq. (27) is equal to $\|\mathbf{E}\|$ and the second factor is equal to the square root of $2(1-F^2)$. We now find that

$$\begin{aligned}
\langle G|H|G\rangle &= \text{Tr}\{H(|G\rangle\langle G| - |\psi\rangle\langle\psi|)\} + \langle\psi|H|\psi\rangle \\
&\leq \sqrt{2}\|\mathbf{E}\|(1-F^2)^{1/2} + \langle\psi|H|\psi\rangle \\
&= \sqrt{2}\|\mathbf{E}\|(1-F^2)^{1/2} + E_0.
\end{aligned} \qquad (28)$$

Combining this identity with Eq. (26) proves the result. ∎

An important implication of this result is the following. Letting $\Delta E = E_1 - E_0$ be the energy gap of $H$ between the ground state energy and the first excited level, one finds that $E_i \geq E_0 + \Delta E$ for every $i = 1, \ldots, 2^n - 1$, and therefore

$$\frac{(r-1)\Delta E}{r} \leq \frac{E_0 + \cdots + E_{r-1}}{r} - E_0. \qquad (29)$$

This shows that

$$\frac{r-1}{\sqrt{2}\,r}\frac{\Delta E}{\|\mathbf{E}\|} \leq (1 - |\langle G|\psi\rangle|^2)^{1/2}. \qquad (30)$$

As $r \geq 2$ for any $d < \eta(|G\rangle)$, this proves that any ground state of a $d$-body Hamiltonian $H$ with $d < \eta(|G\rangle)$ can only be $\epsilon$-close to the graph state $|G\rangle$ at the cost of $H$ having an energy gap which is $\epsilon$-small relative to the total energy in the system.

In a first approximation, this result indicates it might be difficult to robustly create states close to the graph state $|G\rangle$ by cooling a system governed by a $d$-body Hamiltonian into its nondegenerate ground state, as minor thermal fluctuations may easily bring the system into an excited state (which is orthogonal to the ground state), since the energy gap is necessarily small. However, two important remarks regarding the precise interpretation of this result are in order.

First, Eq. (29) is stated in terms of the quantity $\Delta E_{\text{rel}} := \Delta E / \|\mathbf{E}\|$, i.e., the energy gap relative to the total energy, rather than in terms of the absolute energy gap. This implies that, in physical systems where $\Delta E$ is held constant and where $\|\mathbf{E}\|$ is very large, one finds that $\Delta E_{\text{rel}}$ is arbitrarily small. This situation might, e.g., occur if there exist a large

number of energy levels in the system, each a constant distance $\Delta E$ apart. Hence, in such cases the bound (29) does not seem to be very useful. The fact that $\Delta E_{\text{rel}}$ appears in Eq. (29), and not the absolute gap $\Delta E$, is due to the fact that the bound is totally general, in that it holds for *all* Hamiltonians; in particular, it does not exclude situations where, e.g., the ground state level is maximally degenerate.

Second, we also note that the fidelity might not be the best suited distance measure in certain applications. If one is for instance interested in creating 2D cluster states in order to build a one-way quantum computer, it is known that if the 2D cluster states are subject to local noise which is below a certain threshold, the resulting states still enable universal quantum computation—this follows from the fault tolerance of the one-way model, where computation is performed on encoded states [23]. However, when the fidelity is used as a distance measure, the original 2D cluster states and the noisy ones can be very far apart (their fidelity can even be exponentially small). Thus, for such applications, one should be cautious in using the bound (29).

Finally, we note that (an analogue of) theorem 4 can also be derived from theorem 1 in Ref. [18], where a general bound on the fidelity between an arbitrary state $|\phi\rangle$ and the ground state $|\psi\rangle$ of a Hamiltonian $H$ is obtained in terms of the spectrum of $H$.

In the next section we consider Hamiltonians on $n' > n$ qubits, thus allowing for ancilla particles, which are to be constructed in such a way that the desired $n$-qubit graph states occur as states on a subset of the initial system of $n'$ qubits.

## VI. ANCILLA PARTICLES AND GADGET CONSTRUCTIONS

In this section we demonstrate how one can construct a two-body Hamiltonian with approximately the same ground state as a given many-body Hamiltonian. The basic idea is from Refs. [19,20] where they use perturbation theory to show that by adding ancilla qubits one can construct a two-body Hamiltonian whose ground state is arbitrarily close to the ground state of any many-body Hamiltonian. In this construction, the ancilla qubits serve to mediate interactions. More precisely, a strong interaction term within the auxiliary system favors a certain subspace, leading effectively to a projection onto this subspace. Pairwise interactions between the auxiliary system and remaining particles, together with the effective projection onto this subspace, lead to effective many-body interactions. Notice that by adding ancilla qubits we can avoid the problems discussed in the previous sections.

We first illustrate the construction with the help of two examples. First, we consider in Sec. VI A a linear cluster state, which is the exact ground state of a three-body Hamiltonian. We show how to find a two-body Hamiltonian with approximately the same ground state using a gadget construction with three ancilla qubits per three-body interaction term. Second, in Sec. VI B we consider a graph state corresponding to a honeycomb (or hexagonal) lattice. A different type of gadget construction is required in this case, which

first reduces the required four-body interaction terms to three-body interactions. These three-body interactions are then subsequently reduced to two-body interactions using the same construction as in the first example. We remark that the honeycomb lattice graph state is a universal resource for measurement-based quantum computation [13]. Therefore, the two-body Hamiltonian we construct in this section will have a universal resource for quantum computation as a ground state. We conclude with Sec. VI C by discussing how any graph state can be the approximate ground state of a two-body Hamiltonian.

### A. Linear cluster state

In order to create a two-body Hamiltonian with the same ground state as a given three-body Hamiltonian $H$, we create a new perturbed Hamiltonian $\widetilde{H}=K+V$ where $K$ is a Hamiltonian with large spectral gap and degenerate ground space associated with eigenvalue 0. $V$ is a Hamiltonian with norm much smaller than the gap of $K$ that attempts to recreate the spectrum of $H$ in the gap of $K$ by raising the degeneracy.

Our starting Hamiltonian will be Eq. (22) which has the linear cluster state on $n$ qubits as a ground state. We will only consider Hamiltonians with periodic boundary conditions, i.e., the cluster state corresponds to a ring graph. We construct a two-body Hamiltonian with a ground state that is close to the ground state of Eq. (22). In order the apply the perturbation theory argument given in [19] we require our Hamiltonian to be in a certain form. We simply rewrite Eq. (22) as

$$ H = \sum_{i=1}^{n} \frac{-n^9}{6} (H_1^{(i)} - 6B_z^{(i-1)}B_x^{(i)}B_z^{(i+1)}), \qquad (31) $$

where

$$ B_j^{(i)} = \left( \frac{2}{n^3}I + \frac{1}{n^3}\sigma_j^{(i)} \right) \qquad (32) $$

and $H_1^{(i)}$ is a Hamiltonian that contains only two-body, one-body, and identity terms, namely

$$ H_1^{(i)} = \frac{48}{n^9}I + \frac{24}{n^9}(\sigma_x^{(i)} + \sigma_z^{(i+1)} + \sigma_z^{(i-1)}) $$
$$ + \frac{12}{n^9}(\sigma_x^{(i)}\sigma_z^{(i+1)} + \sigma_x^{(i)}\sigma_z^{(i-1)} + \sigma_z^{(i-1)}\sigma_z^{(i+1)}). \qquad (33) $$

Let $i_1$, $i_2$, and, $i_3$ be the ancilla qubits for qubit $i$. For a sufficiently small $\delta$ we can show that the two-body perturbed Hamiltonian $\widetilde{H}=-(n^9/6)(K+V)$ has approximately the same ground state as Eq. (22), where

$$ K = -\frac{\delta^{-3}}{4}\sum_{i=1}^{n}(\sigma_z^{(i_1)}\sigma_z^{(i_2)} + \sigma_z^{(i_1)}\sigma_z^{(i_3)}) - \frac{\delta^{-3}}{4}\sum_{i=1}^{n}(\sigma_z^{(i_2)}\sigma_z^{(i_3)} - 3I) $$

$$ (34) $$

and
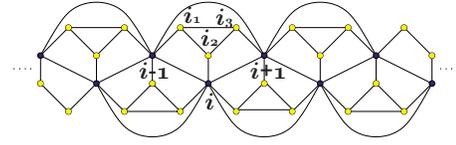


FIG. 1. (Color online) Interaction diagram of a two-body Hamiltonian with the linear cluster state as the ground state. The original qubits are shown in dark gray and the light gray qubits are the added ancilla qubits.

$$ V = \sum_{i=1}^{n} H_1^{(i)} + \delta^{-1}\sum_{i=1}^{n}[(B_x^{(i-1)})^2 + (B_z^{(i)})^2 + (B_z^{(i+1)})^2] $$
$$ - \delta^{-2}\sum_{i=1}^{n}(B_z^{(i-1)}\sigma_x^{(i_1)} + B_x^{(i)}\sigma_x^{(i_2)} + B_z^{(i+1)}\sigma_x^{(i_3)}). \qquad (35) $$

Notice that the Hamiltonian $K$ has eigenvalues 0 and $\delta^{-3}$. We use $V$ to manipulate this gap in order to approximate Eq. (22).

By substituting in the appropriate values for the $B_j^{(i)}$ in $V$ we get our final two-body Hamiltonian, $\widetilde{H}=-(n^9/6)(K+V)$, where $K$ is as above and

$$ V = \sum_{i=1}^{n} H_1^{(i)} + \frac{\delta^{-1}}{n^6}\sum_{i=1}^{n}(15I + 4\sigma_z^{(i-1)} + 4\sigma_x^{(i)} + 4\sigma_z^{(i+1)}) $$

$$ - \frac{\delta^{-2}}{n^3}\sum_{i=1}^{n}(2\sigma_x^{(i_1)} + 2\sigma_x^{(i_2)} + 2\sigma_x^{(i_3)}) $$

$$ - \frac{\delta^{-2}}{n^3}\sum_{i=1}^{n}(\sigma_x^{(i-1)}\sigma_x^{(i_1)} + \sigma_z^{(i)}\sigma_x^{(i_2)} + \sigma_z^{(i+1)}\sigma_x^{(i_3)}). \qquad (36) $$

Figure 1 shows the interaction diagram of the linear cluster state which is the approximate ground state of a two-body Hamiltonian. The original qubits are shown in blue and the red qubits are the added ancilla qubits. This diagram is significantly less complex than the interaction diagram of a two-body Hamiltonian constructed from the four-body honeycomb lattice, which we consider below.

Although this section has shown us a method for creating a two-body Hamiltonian with approximately the same ground state as a three-body Hamiltonian, it has also demonstrated a major disadvantage with this method. In order for the perturbation theory to apply we need to choose a small $\delta$. The smaller $\delta$ is the greater the difference between the coefficients of $K$ and the parts of $V$ becomes. As this gap becomes bigger it becomes more difficult to create such a Hamiltonian in the laboratory as it requires a high degree of precision over several orders of magnitude. Added to this problem is the fact that the coefficients depend on the number of qubits in the original lattice; see Eq. (36) for an example. As the number of qubits increases the difference between the coefficients grows even larger.

### B. Honeycomb lattice graph state

Next, we use the gadgets from above to show that the honeycomb lattice graph state can occur as the approximate
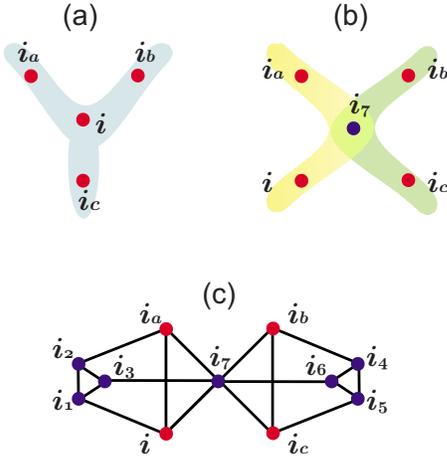
FIG. 2. (Color online) Illustration of the reduction of (a) a single four-qubit interaction term $\sigma_x^{(i)}\sigma_z^{(i_a)}\sigma_z^{(i_b)}\sigma_z^{(i_c)}$ to (b) two three-body terms $\sigma_x^{(i)}\sigma_z^{(i_a)}\sigma_x^{(i_7)}$ and $\sigma_z^{(i_b)}\sigma_z^{(i_c)}\sigma_x^{(i_7)}$ using one ancilla qubit $i_7$ and a strong single body term, $|1\rangle\langle 1|^{(i_7)}$. (c) Further reduction to two-qubit interactions of each of the three-body terms with help of three ancilla particles $(i_1,i_2,i_3)$ and $(i_4,i_5,i_6)$, respectively.

ground state of a two-body Hamiltonian. We show this by using a two step process. We start with a four-body Hamiltonian with the honeycomb lattice graph state as a ground state. Then, using results from [20] we create a three-body Hamiltonian with approximately the same ground state. We then use the procedure described in the previous section to reduce each of the three-body interactions to two-body interactions, hence constructing a two-body Hamiltonian with approximately the same ground state as the initial four-body Hamiltonian.

Our starting Hamiltonian is given by

$$H = - \sum_i \sigma_x^{(i)}\sigma_z^{(i_a)}\sigma_z^{(i_b)}\sigma_z^{(i_c)}, \qquad (37)$$

where $i_a$, $i_b$, and $i_c$ are the neighboring qubits to qubit $i$ in the hexagonal lattice graph and each term is a member of the generator for the stabilizer of the graph state. It is clear that the ground state of this four-body Hamiltonian is the honeycomb lattice state. We note that the hexagonal lattice graph state is a universal resource for measurement based quantum computation.

Let us first illustrate the construction for a single four-body term in the Hamiltonian (see also Fig. 2). In the first step, the four-body interaction term $\sigma_x^{(i)}\sigma_z^{(i_a)}\sigma_z^{(i_b)}\sigma_z^{(i_c)}$ is reduced to three-body interactions by adding an ancilla particle, $i_7$. The ancilla particle is coupled via three-body interactions to two distinct pairs of particles, $(i,i_a)$ and $(i_b,i_c)$. That is, the four-body interaction is essentially replaced by two three-body interactions $\sigma_x^{(i)}\sigma_z^{(i_a)}\sigma_x^{(i_7)}$ and $\sigma_z^{(i_b)}\sigma_z^{(i_c)}\sigma_x^{(i_7)}$, together with a strong single body term $|1\rangle\langle 1|^{(i_7)}$. In the second step, each of the three-body terms is reduced to two-body interactions following the procedure described in the previous section. That is, three auxiliary particles which strongly interact pairwise in a triangular configuration are coupled via pairwise interactions to the three system particles, i.e.,

$(i_1,i_2,i_3)$ interact pairwise and are coupled to $(i,i_a,i_7)$, and $(i_4,i_5,i_6)$ interact pairwise and are coupled to $(i_b,i_c,i_7)$. This finally reduces the involved interactions to two-body terms, at the cost of adding in total seven ancilla qubits. Notice that this is just a sketch of the construction, and a more careful analysis reveals the appearance of additional single-body terms (see below). The construction can now be repeated for each four-body term appearing in the original Hamiltonian, leading to a total of $7N$ ancilla qubits for an initial lattice of $N$ qubits. The qubits only interact pairwise among each other, and the ground state of the resulting Hamiltonian approximates the honeycomb lattice graph state.

After applying the two steps above the resulting Hamiltonian is

$$\tilde{H} = -a\sum_i (A_i + bB_i + cC_i + dD_i + eI), \qquad (38)$$

where $A_i$, $B_i$, $C_i$, and $D_i$ are defined below and $a$, $b$, $c$, $d$, and $e$ are constants depending on $n$ and the spectral gap of the Hamiltonian $H$.

In the seven added ancilla qubits there are two sets of three qubits which interact. The interaction diagram for each of the two sets forms a triangle. The first part of our Hamiltonian, $A_i$, is the part that describes the interactions between these six ancilla qubits. Let $i_1$, $i_2$, $i_3$, $i_4$, $i_5$, and $i_6$ be six ancilla qubits for qubit $i$. Then,

$$A_i = \sigma_z^{(i_1)}\sigma_z^{(i_2)} + \sigma_z^{(i_1)}\sigma_z^{(i_3)} + \sigma_z^{(i_2)}\sigma_z^{(i_3)} + \sigma_z^{(i_4)}\sigma_z^{(i_5)} + \sigma_z^{(i_4)}\sigma_z^{(i_6)}$$
$$+ \sigma_z^{(i_5)}\sigma_z^{(i_6)}. \qquad (39)$$

The second part of our two-body Hamiltonian functions is much the same way as the first part. This part of the Hamiltonian is a result of the first step when we created a three-body Hamiltonian. We partitioned each original four part stabilizer term into two parts, each with two operators. We then coupled each pair with a common ancilla essentially forming triangles which share a common vertex, the seventh ancilla qubit. We have

$$B_i = \sigma_x^{(i)}\sigma_z^{(i_a)} + \sigma_x^{(i)}\sigma_x^{(i_7)} + \sigma_z^{(i_a)}\sigma_x^{(i_7)} + \sigma_z^{(i_b)}\sigma_z^{(i_c)} + \sigma_z^{(i_b)}\sigma_x^{(i_7)}$$
$$+ \sigma_z^{(i_c)}\sigma_x^{(i_7)}. \qquad (40)$$

The correlations described by $C_i$ are those correlations which connect the triangles created in $A_i$ and $B_i$. Each vertex of a triangle from $A_i$ is coupled to a vertex from a triangle in $B_i$. $C_i$ can be written as

$$C_i = \sigma_x^{(i)}\sigma_x^{(i_1)} + \sigma_z^{(i_a)}\sigma_x^{(i_2)} + \sigma_x^{(i_3)}\sigma_x^{(i_7)} + \sigma_z^{(i_b)}\sigma_x^{(i_4)} + \sigma_z^{(i_c)}\sigma_x^{(i_5)}$$
$$+ \sigma_x^{(i_6)}\sigma_x^{(i_7)}. \qquad (41)$$

The final part of our two-body Hamiltonian describes the remaining operators acting on single qubits,

$$D_i = \sigma_x^{(i)} + \sigma_z^{(i_a)} + \sigma_z^{(i_b)} + \sigma_z^{(i_c)} + 2\sigma_z^{(i_7)} + d_1\sum_{j=1}^{6}\sigma_x^{(i_j)} + d_2|1\rangle\langle 1|^{(i_7)}. \qquad (42)$$

Again, the constants $d_1$ and $d_2$ depend on the number of qubits and the spectral gap of the Hamiltonian $H$.

### C. Generic graph states

Given any graph state $|G\rangle$ it is always possible to create a two-body Hamiltonian that has $|G\rangle$ (along with some ancilla qubits) as the approximate ground state. If $|G\rangle$ is a graph state on $n$ qubits then we know from Sec. IV that there exists a set $\{g_1, g_2, \ldots, g_n\}$ of generators for the stabilizer of $|G\rangle$ such that each generator has weight at most $\eta(|G\rangle)$. We can construct the $\eta(|G\rangle)$-body Hamiltonian

$$H = -\sum_{i=1}^{n} g_i, \qquad (43)$$

which has $|G\rangle$ as the ground state. We then use the steps described above to reduce this Hamiltonian to one which has only two-body interactions. For every $k$-body interaction in the Hamiltonian (43) we require an additional $O(k)$ ancilla qubits for our two-body Hamiltonian. In the worst case we would need to add $O(\eta(|G\rangle)n)$ ancilla qubits.

In this section we have shown that by adding ancilla qubits we can overcome the problems we addressed in the previous sections. We have shown that one can construct a two-body Hamiltonian whose ground state is close to a universal resource for measurement based quantum computation. Unfortunately, the Hamiltonians we have constructed are only of theoretical interest. Due to the high degree of control and precision that is required to create these Hamiltonians they seem to be of little value for practical applications. Fluctuations in the interaction strength as well as thermal noise are expected to significantly influence the resulting ground state. What might help to overcome these difficulties is the creation of *encoded* graph states as, e.g., suggested in [12]. The encoding provides additional robustness against noise for the resulting states, as error correction techniques can be applied.

Finally, we note that a bound similar to Eq. (29) can also be obtained for the gadget construction. Note that in this situation, the total energy $\|\mathbf{E}\|$ is typically large [24], such that the relative gap $\Delta E / \|\mathbf{E}\|$ is small.

## VII. CONCLUSION

In this paper we have settled the issue of whether graph states can occur as ground states of two-body Hamiltonians. More generally, we have shown that the quantity $\eta(|G\rangle)$, defined as the minimal $d$ such that $\mathcal{S}_d = \mathcal{S}$, is of central interest in the present context. It determines the minimum number of interactions in the sense that $\eta(|G\rangle)$-body interactions are required to obtain $|G\rangle$ as the exact, nondegenerate ground

state. In addition, we found that $\eta(|G\rangle) \gtrsim 3$ for all graph states, which implies that any $n$-qubit graph state $|G\rangle$ cannot be the exact nondegenerate ground state of a two-body Hamiltonian acing on $n$ qubits. We have also related the accuracy $\epsilon$ of approximating the graph state $|G\rangle$ using a Hamiltonian with $d'$-body interactions and $d' < \eta(|G\rangle)$ to the energy gap of the Hamiltonian relative to the total energy, which turns out to be proportional to $\epsilon$. When allowing the usage of ancilla particles that act as mediating particles to generate an effective many-body Hamiltonian on a subsystem, we have shown that the gadget construction introduced in Refs. [19,20] can be used to obtain the $n$-qubit graph state $|G\rangle$ as an nondegenerate (quasi)ground state of a two-body Hamiltonian acting on $n' > n$ qubits. However, an incredible high accuracy in the control of the parameters of the interaction Hamiltonian is required. We also remark that our results do not directly apply to the generation of graph states in an *encoded* form. On the one hand, an (exponential) small fidelity of the physical state might still be acceptable to obtain high fidelity with respect to the encoded (logical) graph states when using redundant encodings corresponding to quantum error correcting codes. On the other hand, as demonstrated in Ref. [12], there exist (approximate) ground states of two-body Hamiltonians which are arbitrary close to encoded graph states with respect to a certain encoding. The energy gap of the corresponding Hamiltonian is constant, independent of the system size, and the encoded graph states constitute a universal resource for measurement based quantum computation using only single qubit measurements. The usage of encoded graph states for measurement based quantum computation is the subject of ongoing research [21,22].

We finally remark that the quantity $\eta(|G\rangle)$ serves as a natural complexity measure of graph states, as it assesses how difficult it is to exactly prepare a state by cooling a system into its ground state. The present results show that graph states typically exhibit a large complexity in this sense, whereas they have small computational complexity, since all graph states can be prepared with a polysized quantum circuit.

[1] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Van den Nest, and H.-J. Briegel, in *Quantum Computers, Algorithms and Chaos*, International School of Physics "Enrico Fermi," Vol. 162, edited by G. Casati, D. Shepelyansky, P. Zoller, and G. Benenti (IOS Press, Amsterdam, 2006). See also e-print arXiv:quant-ph/0602096.

[2] H. J. Briegel and R. Raussendorf, Phys. Rev. Lett. **86**, 910

(2001).

[3] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001); Quantum Inf. Comput. **2**, 443 (2002).

[4] R. Raussendorf, D. E. Browne, and H. J. Briegel, Phys. Rev. A **68**, 022312 (2003).

[5] D. Gottesman, Ph.D. thesis, Caltech, 1997.

[6] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098

(1996).

[7] A. M. Steane, Phys. Rev. Lett. **77**, 793 (1996).

[8] M. Hillery, V. Buzek, and A. Berthiaume, Phys. Rev. A **59**, 1829 (1999).

[9] Kai Chen and Hoi-Kwong Lo, e-print arXiv:quant-ph/0404133.

[10] W. Dür, J. Calsamiglia, and H.-J. Briegel, Phys. Rev. A **71**, 042336 (2005).

[11] M. A. Nielsen, e-print arXiv:quant-ph/0504097.

[12] S. D. Bartlett and T. Rudolph, Phys. Rev. A **74**, 040302(R) (2006).

[13] M. Van den Nest, A. Miyake, W. Dür, and H. J. Briegel, Phys. Rev. Lett. **97**, 150504 (2006).

[14] M. Mhalla and S. Perdrix, e-print arXiv:quant-ph/0412071.

[15] M. Van den Nest, J. Dehaene, and B. De Moor, Phys. Rev. A **69**, 022316 (2004).

[16] A set of stabilizer elements is called *independent* if no non-trivial product of operators in this set yields the identity.

[17] In other words, $\mathcal{S}_d$ is the set of all elements $g$ in $\mathcal{S}$ which can be written as a product $g = g_1 \ldots g_k$ for some $g_1, \ldots, g_k \in \mathcal{S}$ and for some $k$, such that $\text{wt}(g_i) \leq d$ for all $i = 1, \ldots, k$.

[18] H. L. Haselgrove, M. A. Nielsen, and T. J. Osborne, Phys. Rev. A **69**, 032303 (2004).

[19] Julia Kempe, Alexi Kitaev, and Oded Regev, SIAM J. Comput. **35**, 1070 (2006).

[20] Roberto Oliveira and Barbara M. Terhal, e-print arXiv:quant-ph/0504050.

[21] D. Gross and J. Eisert, Phys. Rev. Lett. **98**, 220503 (2007).

[22] W. Dür, M. Van den Nest, A. Miyake, and H.-J. Briegel, New J. Phys. **9**, 204 (2007).

[23] R. Raussendorf, Ph.D. thesis, LMU Munich, 2003.

[24] This is, e.g., reflected in Eq. (34), where the Hamiltonian $K$ has a large prefactor $d^{-3}$ (since $\delta$ is small) and therefore has a large norm.