Sudden death of entanglement at finite temperature

Asma Al-Qasimi and Daniel F. V. James
Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7

(Received 31 July 2007; revised manuscript received 11 January 2008; published 31 January 2008)

We consider the decay of quantum entanglement quantified by the concurrence of a pair of two-level systems, each of which is interacting with a reservoir at finite temperature T. For a broad class of initially entangled states, we demonstrate that the system always becomes disentangled in a finite time, i.e., "entanglement sudden death" occurs. This class includes *all* states which previously had been found to have long-lived entanglement in zero-temperature reservoirs. Our general result is illustrated by an example.

DOI: 10.1103/PhysRevA.77.012117 PACS number(s): 03.65.Yz, 03.65.Ud, 42.50.Lc

I. INTRODUCTION

In the past few years there has been considerable interest in the properties of entangled quantum systems. Spurred on by the emergence of compelling applications in quantum information processing, useful methods by which the entanglement of quantum systems can be established and characterized have emerged. Perhaps the most impactful to date has been the simple procedure derived by Wootters [1] for quantifying entanglement for an arbitrary mixed state of a pair of two-level systems. This has provided a very useful tool for measurement of experimental quantum states [2] and is today commonly used in assessing the capabilities of emerging quantum technologies. Building on Wootters' work, recently Yu and Eberly [3] investigated the time evolution of entanglement (quantified using the concurrence) of a bipartite qubit system undergoing various modes of decoherence. Remarkably, they found that, even when there is no interaction (either directly or through a correlated environment), there are certain states whose entanglement decays exponentially with time, while for other closely related states, the entanglement vanishes completely in a finite time. This "entanglement sudden death" (ESD) is an intriguing discovery. Nor is this effect limited to the case of two-qubit systems: prior to Yu and Eberly's work, Diósi [4] demonstrated, using Werner's criteria for separability [5], that ESD occurs in two-state quantum systems. Further investigations of different systems have been made by various groups [7–16]. Extending Yu and Eberly's model by considering correlated reservoirs and interactions [6,8,11,13,15,16], it was shown that entanglement may be created by spontaneous emission (something which has been known for some time [21] in a different context). The ESD has also been predicted for more complicated systems using other entanglement measures [18–20], and an attempt to give a geometric interpretation for the phenomenon of ESD has also been made [22]. Very recently, experimental studies have also been carried out to demonstrate ESD, using carefully engineered interactions between systems and environments: Sudden death has been observed both in photons [17], the method being proposed in [23], and in atomic ensembles [24].

The entropy of systems undergoing irreversible dynamics increases; further, as was established some time ago, there is a limit on the amount of entanglement that can be present in a mixed system [25]: the more mixed a state is, the less entangled it can be, and when the entropy reaches a certain level, entanglement will necessarily disappear. However, these heuristic arguments do not tell us the time taken for entanglement to disappear, which cannot be answered without careful study of the dynamics.

One might think that, from the quantum technological point of view, states that exhibit exponential decay of entanglement, and therefore retain some vestige of this all-important correlation for long periods, are of significance. Although the vanishingly small entanglement present in the exponential tail will be of little practical importance, nevertheless it is of interest to identify precisely in what circumstances ESD will occur.

In this paper, we consider qubits in finite-temperature reservoirs: instead of the energy of the qubits being lost via spontaneous decay to the environment, now additionally the reservoirs can cause excitation of the qubits. For a broad class of mixed quantum states, which includes all of the states studied by Yu and Eberly and others in connection with this problem, we demonstrate that *all states undergo sudden death of entanglement at finite temperature.*

II. TWO-QUBIT MODEL SYSTEM

As in [3], we study a system of two qubits initially entangled and interacting with uncorrelated reservoirs. However, unlike [3], in which the system is studied at temperature $T=0$, we include the effects of heat in our system (Fig. 1). Here the dynamics of the density matrix $\hat{\rho}$ describing the two qubits is given by

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{1}{i\hbar}[\hat{H},\hat{\rho}] + L_1[\hat{\rho}] + L_2[\hat{\rho}], \tag{1}$$

where $[H,\rho]$ is the unitary part of the evolution (which we shall ignore as it has no effect on our study of decoherence). The Liouvillian of the ith qubit is given by

$$L_i[\hat{\rho}] = \frac{(\bar{n}+1)\Gamma}{2}\{[\hat{\sigma}_-^i,\hat{\rho}\hat{\sigma}_+^i] + [\hat{\sigma}_-^i\hat{\rho},\hat{\sigma}_+^i]\} + \frac{\bar{n}\Gamma}{2}\{[\hat{\sigma}_+^i,\hat{\rho}\hat{\sigma}_-^i] + [\hat{\sigma}_+^i\hat{\rho},\hat{\sigma}_-^i]\}, \tag{2}$$

where Γ is the spontaneous decay rate of the qubits, $\hat{\sigma}_+^i = (|1\rangle\langle 0|)_i$, and $\hat{\sigma}_-^i = (|0\rangle\langle 1|)_i$, where the index $i \in \{1,2\}$ de-

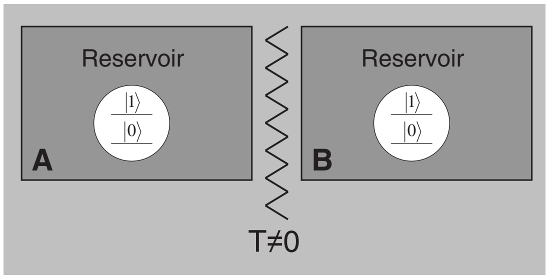

FIG. 1. Disentanglement by spontaneous emission of a two-qubit system in a heat bath. The reservoir is modeled by different harmonic oscillator modes. Each qubit, here depicted by a two-level atom, interacts with its reservoir. The only interaction between the qubits that ever exists is the one that leads to their entanglement at time $t=0$. Following this, however, the only interaction that remains is that with the corresponding reservoir. This leads to decoherence, which causes the qubits to disentangle. Here we include the effect of heat by studying the system at $T\neq 0$.

notes the qubits. The first term on the right-hand side of (2) corresponds to the depopulation of the atoms due to stimulated and spontaneous emissions, while the second term describes the reexcitations caused by the finite temperature; and $\bar{n}$ is the mean occupation number of the reservoir (assumed to be the same for both qubits).

We assume that our system is initially an "X state" described by the following density matrix:

$$\hat{\rho}(t)=\begin{pmatrix} a(t) & 0 & 0 & w(t) \\ 0 & b(t) & z(t) & 0 \\ 0 & z^*(t) & c(t) & 0 \\ w^*(t) & 0 & 0 & d(t) \end{pmatrix}. \tag{3}$$

Such states are general enough to include states such as the Werner states, the maximally entangled mixed states (MEMSs) [25], the Bell states, and what we will refer to as the $\hat{\rho}_{YE}$ states, studied in [3] and which will be described later.

Substituting (3) into (1), the master equation of our system, we obtain the following first-order coupled differential equations:

$$\dot{a}(t)=\Gamma[-2(\bar{n}+1)a(t)+b(t)\bar{n}+c(t)\bar{n}],$$

$$\dot{b}(t)=\Gamma[(\bar{n}+1)a(t)-(2\bar{n}+1)b(t)+\bar{n}d(t)],$$

$$\dot{c}(t)=\Gamma[(\bar{n}+1)a(t)-(2\bar{n}+1)c(t)+\bar{n}d(t)],$$

$$\dot{d}(t)=\Gamma[(\bar{n}+1)b(t)+(\bar{n}+1)c(t)-2\bar{n}d(t)],$$

$$\dot{z}(t)=\Gamma[-(2\bar{n}+1)z(t)],$$

$$\dot{w}(t)=\Gamma[-(2\bar{n}+1)w(t)]. \tag{4}$$

These may be solved to yield the following expressions:

$$a(t)=\frac{1}{(2\bar{n}+1)^2}\{\bar{n}^2+[2(a_0-d_0)\bar{n}^2+(a_0-d_0+1)\bar{n}]X + [(2a_0+2d_0-1)\bar{n}^2+(3a_0+d_0-1)\bar{n}+a_0]X^2\},$$

$$b(t)=\frac{1}{(2\bar{n}+1)^2}\{\bar{n}(\bar{n}+1)-[2(a_0+2c_0+d_0-1)\bar{n}^2 + (a_0+4c_0+3d_0-2)\bar{n}+(c_0+d_0-1)]X - [(2a_0+2d_0-1)\bar{n}^2+(3a_0+d_0-1)\bar{n}+a_0]X^2\},$$

$$c(t)=\frac{1}{(2\bar{n}+1)^2}\{\bar{n}(\bar{n}+1)+[2(a_0+2c_0+d_0-1)\bar{n}^2 + (3a_0+4c_0+d_0-2)\bar{n}]X-[(2a_0+2d_0-1)\bar{n}^2 + (3a_0+d_0-1)\bar{n}+a_0]X^2\},$$

$$d(t)=\frac{1}{(2\bar{n}+1)^2}\{(\bar{n}+1)^2-(\bar{n}+1)[2\bar{n}(a_0-d_0) + (a_0-d_0+1)]X+[(2a_0+2d_0-1)\bar{n}^2 + (3a_0+d_0-1)\bar{n}+a_0]X^2\},$$

$$w(t)=w_0X,$$

$$z(t)=z_0X, \tag{5}$$

where $X=e^{-\Gamma(2\bar{n}+1)t}$, $a_0=a(0)$, etc.

III. SOLUTION FOR THE SUDDEN DEATH TIME

Using Wootters' formula [1], the concurrence for a state of the form given by (3) is

$$C=2\max\{0,|z(t)|-\sqrt{a(t)d(t)},|w(t)|-\sqrt{b(t)c(t)}\}. \tag{6}$$

This implies that the disentanglement time will be the largest positive solutions of the following equations:

$$|z(t)|-\sqrt{a(t)d(t)}=0,\quad |w(t)|-\sqrt{b(t)c(t)}=0. \tag{7}$$

We cannot solve Eqs. (7) in closed form. Further, since they are not polynomial functions of X, neither can we make any straightforward deductions about the nature of their roots. However, multiplying both equations in (7) by the positive quantities $|z(t)|+\sqrt{a(t)d(t)}$ and $|w(t)|+\sqrt{b(t)c(t)}$, respectively, yields

$$|z(t)|^2-a(t)d(t)=0,\quad |w(t)|^2-b(t)c(t)=0. \tag{8}$$

Substituting from Eqs. (8) gives two quartic equations in X, which we will use in the proof of our main result.

The quantity X is the time-dependent parameter that we use to monitor the evolution of entanglement in the system. Notice that at $t=0$, $X=1$, and that at $t=\infty$, $X=0$. Physically meaningful values for X are, therefore, between 0 and 1. Asymptotic decay of entanglement implies a solution at $X=0$. However, if the entanglement of the system decays in a finite time (ESD), the solution of (8) must lie in the range $0<X<1$.

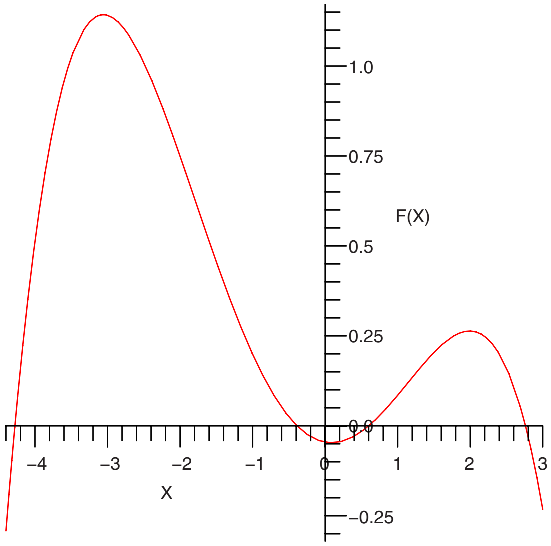

FIG. 2. (Color online) Plot of $F(x)$ vs X. This is the plot of the first quartic equation in (8) for $\bar{n}=0.8$, $a=0.1$, $d=0.05$, and $z=0.3$.

Both equations in (8) are polynomial equations in X and continuous. At $X=0$, these equations take the following value:

$$\frac{-\bar{n}^2(\bar{n}^2+2\bar{n}+1)}{(2\bar{n}+1)^4}. \tag{9}$$

Notice that since $\bar{n}$ is a positive quantity, (9) is negative. On the other hand, if we evaluate (8) for $X=1$, which corresponds to $t=0$, we obtain $|z_0|^2-a_0d_0$ and $|w_0|^2-b_0c_0$ for the first and second equations, respectively. If we assume that our systems are initially entangled, at least one of these quantities has to be positive, which is the case here. The fact that the quartic equations are continuous and have a negative value at $X=0$ and a positive one at $X=1$ implies that they have at least one root in our interval of interest $0<X<1$ (see Fig. 2 for an example illustrating this point).

For finite $\bar{n}$, and therefore for finite T, the constant term (9) is always finite and nonzero for $\bar{n}>0$. Hence, there is no $X=0$ solution—i.e., no asymptotic decay. Thus Eq. (8) has at least one solution in the range $0<X<1$, implying that the entanglement falls to zero in a *finite time.*

IV. EXAMPLE

As an example, let us consider a special case when $w(t)=0$. In this case, the only quartic equation that has to be satisfied for $C=0$ is the first one in (8). This equation can be easily solved with appropriate reparametrization. The four solutions for X are given by

$$X=(r+s)\pm\sqrt{(r+s)^2+q^2},$$

$$X=(r-s)\pm\sqrt{(r-s)^2+q^2}, \tag{10}$$

where

$$q=\sqrt{\frac{\bar{n}(\bar{n}+1)}{\{(2\bar{n}+1)[a_0(\bar{n}+1)+d_0\bar{n}]-\bar{n}(\bar{n}+1)\}}},$$

$$r=\frac{1+(a_0-d_0)(2\bar{n}+1)}{4\{(2\bar{n}+1)[a_0(\bar{n}+1)+d_0\bar{n}]-\bar{n}(\bar{n}+1)\}},$$

$$s=\frac{(2\bar{n}+1)^2\sqrt{(1-a_0-d_0)^2+4(z^2-a_0d_0)}}{4\{(2\bar{n}+1)[a_0(\bar{n}+1)+d_0\bar{n}]-\bar{n}(\bar{n}+1)\}}.$$

Further, following (3), let us consider states of the form

$$\hat{\rho}_{YE}=\frac{1}{3}\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1-\alpha\end{pmatrix}. \tag{11}$$

In [3], Yu and Eberly have shown that for $0\leq\alpha\leq\frac{1}{3}$, the entanglement of this state is long lived at zero temperature. Here, we will illustrate that as soon as $\bar{n}$ becomes finite, the range vanishes and there is no long-lived entanglement for any value of α.

For $\hat{\rho}_{YE}$, the physical solutions—i.e., those between 0 and 1—are only for $X=(r-s)\pm\sqrt{(r-s)^2+q^2}$ in (10). Here, we have the following expressions for q, r, and s:

$$q=\sqrt{\frac{3\bar{n}(\bar{n}+1)}{2\alpha\bar{n}-\bar{n}^2+\alpha-2\bar{n}}},$$

$$r=\frac{1}{2}\left\{\frac{1+2\alpha\bar{n}+\alpha-\bar{n}}{2\alpha\bar{n}-\bar{n}^2+\alpha-2\bar{n}}\right\},$$

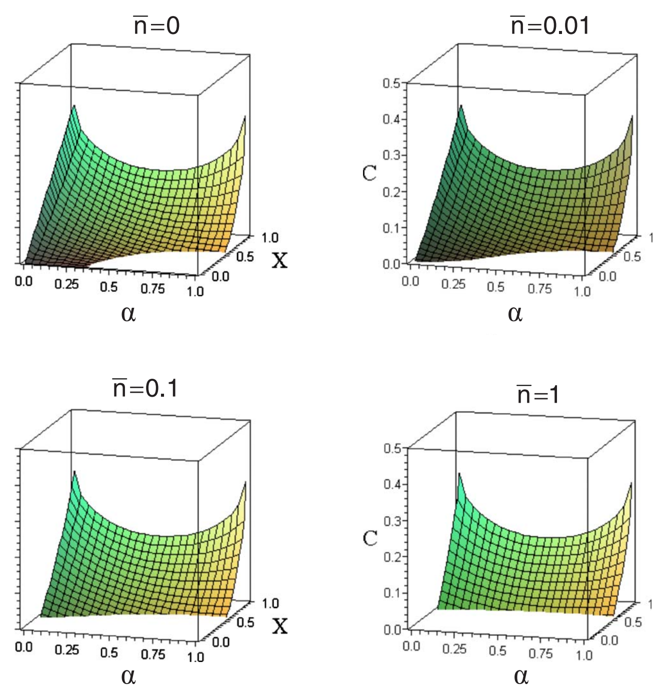

FIG. 3. (Color online) Plot of C (concurrence) vs X $(=e^{-\Gamma(2\bar{n}+1)t})$ vs α. $C=0$ corresponds to no entanglement. $X=1$ corresponds to $t=0$, while $X=1$ corresponds to $t=\infty$. Notice that as soon as $\bar{n}$ becomes finite, for all values of α, C becomes zero at $X<0$, i.e., entanglement decays in a finite time. As $\bar{n}$ becomes larger, all states disentangle at approximately $X=0.5$.

$$s = \frac{1}{2}\left\{ \frac{(2\bar{n}+1)^2\sqrt{2+\alpha^2-\alpha}}{2\alpha\bar{n}-\bar{n}^2+\alpha-2\bar{n}} \right\}. \tag{12}$$

The first solution (with the plus sign) is valid for $\alpha > \bar{n}(\bar{n}+2)/2\bar{n}+1$, while the second solution is valid for $\alpha < \bar{n}(\bar{n}+2)/2\bar{n}+1$. Examples are plotted in Fig. 3.

As a specific physical example (described in Fig. 2), which is experimentally accessible with current technology, consider trapped ion qubits interacting with a thermal reservoir of phonons [26]. With a temperature of 60 μK, we find that, for a 1-MHz trap, $\bar{n}=0.8$. Then the disentanglement time is $T_{dis} = -\ln X_0/(2\bar{n}+1)\Gamma \approx 0.2/\Gamma$, Γ being the controllable coupling parameter between the bath and the ions; with, for example, $\Gamma = 10^3\ \mathrm{s}^{-1}$, we find a disentanglement time of 200 μs.

V. CONCLUSION

In conclusion, in this paper we presented a proof that in two-qubit systems interacting with uncorrelated reservoirs and described by X states, ESD always occurs at any finite temperature. Although X states are quite general states, they are not the most general ones. Thus, the next question to ask is, do *all* states exhibit ESD? In other words, for a state described by a density matrix, without any zero elements, is entanglement still lost in a finite time? This question is not straightforward. One has to find the equation for concurrence in that case, determine its order, study the properties of its coefficients, and from there maybe be able to comment on the nature of the roots and, therefore, be able to predict what will happen in the actual physical system. Alternatively, one could adopt the approach of considering the general dynamics from the perspective of the finite domain of separability surrounding the thermal equilibrium state [27]. In this case, one must find a means for establishing that the evolution to the entanglement-separable boundary occurs in finite time: in the mathematical language of this paper, this is equivalent to proving that the concurrence is a single-valued function of the argument X at $X=1$. One important conclusion can nevertheless be drawn from our results: in any finite-temperature reservoir, all states that have been shown to be long lived in a zero-temperature bath will undergo sudden death. Thus to draw the distinction between sudden-death and exponentially long-lived entanglement seems to be redundant: in all realistic circumstances, all entanglement disappears in a finite time.

ACKNOWLEDGMENTS

We would like to thank Joseph Eberly, René Stock, and Bill Munro for valuable discussions. This work was supported by NSERC.

[1] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[2] D. F. V. James, P. G. Kwiat, W. J. Munro, and A. G. White, Phys. Rev. A **64**, 052312 (2001).
[3] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).
[4] L. Diósi, Lect. Notes Phys. **622**, 157 (2003).
[5] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[6] L. Diósi, Acta Phys. Hung. B **20**, 29 (2004).
[7] T. Yu and J. H. Eberly, Opt. Commun. **264**, 393 (2006).
[8] Muhammed Yönaç *et al.*, J. Phys. B **39**, S621 (2006).
[9] C. Anastopoulos, S. Shresta, and B. L. Hu, e-print arXiv:quant-ph/0610007.
[10] Mahmoud Abdel-Aty, e-print arXiv:quant-ph/0610186.
[11] H. T. Cui *et al.*, Phys. Lett. A **365**, 44 (2007).
[12] T. Yu and J. H. Eberly, Quantum Inf. Commun. **7**, 459 (2007).
[13] M. Yönaç *et al.*, J. Phys. B **40**, S45 (2007).
[14] F. Lastra, G. Romero, C. E. Lopez, M. Franca Santos, and J. C. Retamal, Phys. Rev. A **75**, 062324 (2007).
[15] G.-F. Zhang *et al.*, Opt. Commun. **275**, 274 (2007).
[16] A. Vaglica and G. Vetri, Phys. Rev. A **75**, 062120 (2007).
[17] M. P. Almeida *et al.*, Science **316**, 579 (2007).
[18] Mahmoud Abdel-Aty and H. Moya-Cessa, e-print arXiv:quant-ph/0703077.
[19] I. Sainz and G. Björk, Phys. Rev. A **76**, 042313 (2007).
[20] A. R. R. Carvalho *et al.*, Eur. Phys. J. D **41**, 425 (2007).
[21] M. B. Plenio, S. F. Huelga, A. Beige, and P. L. Knight, Phys. Rev. A **59**, 2468 (1999).
[22] M. O. Terra Cunha, New J. Phys. **9**, 237 (2007).
[23] M. F. Santos, P. Milman, L. Davidovich, and N. Zagury, Phys. Rev. A **73**, 040305(R) (2006).
[24] J. Laurat, K. S. Choi, H. Deng, C. W. Chou, and H. J. Kimble, Phys. Rev. Lett. **99**, 180504 (2007).
[25] W. J. Munro, D. F. V. James, A. G. White, and P. G. Kwiat, Phys. Rev. A **64**, 030302(R) (2001).
[26] S. Schneider and G. J. Milburn, Phys. Rev. A **65**, 042107 (2002).
[27] Yu. Ting and J. H. Eberly, e-print arXiv:0707.3215.