Foldy-Wouthyusen transformation and semiclassical limit for relativistic particles in strong external fields

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A general method of the Foldy-Wouthyusen (FW) transformation for relativistic particles of arbitrary spin in strong external fields has been developed. The use of the found transformation operator is not restricted by any definite commutation relations between even and odd operators. The final FW Hamiltonian can be expanded into a power series in the Planck constant which characterizes the order of magnitude of quantum corrections. Exact expressions for low-order terms in the Planck constant can be derived. Finding these expressions allows one to perform a simple transition to the semiclassical approximation which defines a classical limit of the relativistic quantum mechanics. As an example, interactions of spin-1/2 and scalar particles with a strong electromagnetic field have been considered. Quantum and semiclassical equations of motion of particles and their spins have been deduced. Full agreement between quantum and classical theories has been established.

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I. INTRODUCTION

The Foldy-Wouthuysen (FW) representation [1] occupies a special place in quantum theory. Properties of this representation are unique. The Hamiltonian and all operators are block diagonal (diagonal in two spinors). Relations between the operators in the FW representation are similar to those between the respective classical quantities. For relativistic particles in external fields, operators have the same form as in nonrelativistic quantum theory. For example, the position operator is $\boldsymbol{r}$ and the momentum one is $\boldsymbol{p}=-i\hbar\nabla$. These properties considerably simplify the transition to the semiclassical description. As a result, the FW representation provides the best possibility of obtaining a meaningful classical limit of the relativistic quantum mechanics. The basic advantages of the FW representation are described in Refs. [1–3].

Interactions of relativistic particles with strong external fields can be considered on three levels: (i) classical physics, (ii) relativistic quantum mechanics, and (iii) quantum field theory. The investigation of such interactions on every level is necessary. The use of the FW representation allows one to describe strong-field effects on level (ii) and to find a unambiguous connection between classical physics and relativistic quantum mechanics. To solve the problem, one should carry out an appropriate FW transformation (transformation to the FW representation). The deduced Hamiltonian should be exact up to first-order terms in the Planck constant $\hbar$. This precision is necessary for the establishment of an exact connection between the classical physics and the relativistic quantum mechanics. Known methods of exact FW transformation either can be used only for some definite classes of initial Hamiltonians in the Dirac representation [3,4] or need cumbersome derivations [5].

In the present work, a general method of the FW transformation for relativistic particles in strong external fields is proposed. This method gives exact expressions for low-order terms in $\hbar$. The proposed method is based on the developments performed in Ref. [3] and can be utilized for particles of arbitrary spin. Any definite commutation relations between even and odd operators in the initial Hamiltonian are not needed. An expansion of the FW Hamiltonian into a power series in the Planck constant is used. Since just this constant defines the order of magnitude of quantum corrections, the transition to the semiclassical approximation becomes trivial. As an example, interaction of scalar and spin-1/2 particles with a strong electromagnetic field is considered. We use the designations $[\ldots,\ldots]$ and $\{\ldots,\ldots\}$ for commutators and anticommutators, respectively.

II. FOLDY-WOUTHYUSEN TRANSFORMATION FOR PARTICLES IN EXTERNAL FIELDS

In this section, we review previously developed methods of the FW transformation for particles in external fields. The relativistic quantum mechanics is based on the Klein-Gordon equation for scalar particles, the Dirac equation for spin-1/2 particles, and corresponding relativistic wave equations for particles with higher spins (see, e.g., Ref. [6]). The quantum field theory is not based on these fundamental equations of the relativistic quantum mechanics (see Ref. [7]). Relativistic wave equations for particles with any spin can be presented in the Hamilton form. In this case, the Hamilton operator acts on the bispinor wave function $\Psi=\begin{pmatrix}\phi\\ \chi\end{pmatrix}$:

$$i\hbar\frac{\partial\Psi}{\partial t}=\mathcal{H}\Psi. \qquad (1)$$

A particular case of Eq. (1) is the Dirac equation.

We can introduce the unit matrix I and the Pauli matrices those components act on the spinors

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \rho_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \rho_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},$$

$$\rho_3\equiv\beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

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The Hamiltonian can be split into operators commuting and noncommuting with the operator β:

$$\mathcal{H} = \beta\mathcal{M} + \mathcal{E} + \mathcal{O}, \quad \beta\mathcal{M} = \mathcal{M}\beta, \quad \beta\mathcal{E} = \mathcal{E}\beta, \quad \beta\mathcal{O} = -\mathcal{O}\beta, \tag{2}$$

where the operators $\mathcal{M}$ and $\mathcal{E}$ are even and the operator $\mathcal{O}$ is odd. We suppose that the operator $\mathcal{E}$ is multiplied by the unit matrix I which is everywhere omitted.

Explicit form of the Hamilton operators for particles with arbitrary half-integer spin has been obtained in Ref. [8]. Similar equations have been derived for spin-0 [9] and spin-1 [10,11] particles. To study semiclassical limits of these equations, one should perform appropriate FW transformations.

The wave function of a spin-1/2 particle can be transformed to a new representation with the unitary operator U:

$$\Psi' = U\Psi.$$

The Hamilton operator in the new representation takes the form [1,3,12]

$$\mathcal{H}' = U\mathcal{H}U^{-1} - i\hbar U\frac{\partial U^{-1}}{\partial t} \tag{3}$$

or

$$\mathcal{H}' = U\left(\mathcal{H} - i\hbar\frac{\partial}{\partial t}\right)U^{-1} + i\hbar\frac{\partial}{\partial t}.$$

The FW transformation has been justified in the best way. In the classical work by Foldy and Wouthuysen [1], the exact transformation for free relativistic particles and the approximate transformation for nonrelativistic particles in electromagnetic fields have been carried out. There exist several other nonrelativistic transformation methods which give the same results (see Ref. [3], and references therein). A few methods can be applied for relativistic particles in external fields. However, the transformation methods explained in Refs. [13,14] require cumbersome calculations.

The block-diagonalization of two-body Hamiltonians for a system of two spin-1/2 particles and a system of spin-0 and spin-1/2 particles can be performed by the methods found by Chraplyvy [15] and Tanaka *et al.* [16], respectively. Some methods allow one to reach the FW representation without the use of unitary transformations. The so-called elimination method [17] makes it possible to exclude the lower spinor from relativistic wave equations. Variants of this method useful for relativistic particles have been elaborated in Refs. [18,19]. Another method which essentially differs from the FW and elimination methods has been presented in Ref. [20]. This method defines a diagonalization procedure based on a formal expansion in powers of the Planck constant $\hbar$ and can be used for a large class of Hamiltonians directly inducing Berry phase corrections [20]. An important feature of this method is a possibility to take into account strong-field effects.

Any method different from the FW one should also be justified. The validity of the elimination method is proved only by the coincidence of results obtained by this method and the FW one [21]. Impressive agreement between results presented in Ref. [20] and corresponding results obtained by the FW method is reached for first-order terms in $\hbar$. These terms define momentum and spin dynamics that can be well described in the framework of classical physics. To prove the validity of the method, one should show such an agreement for terms derived from second-order commutators (e.g., for the Darwin term [1]). Therefore, we desist from a definitive estimate of the method developed in Ref. [20].

We suppose the consistence with the genuine FW transformation to be necessary for any diagonalization method. For example, the Eriksen-Korlsrud method [22] does not transform the wave function to the FW representation even for free particles [23]. The use of this method in Refs. [24,25] instead of the FW one could cause a misunderstanding of the nature of spin-gravity coupling (see the discussion in Refs. [23,26]).

In the general case, the exact FW transformation has been found by Eriksen [5]. The validity of the Eriksen transformation has also been argued by de Vries and Jonker [21]. The Eriksen transformation operator has the form [5]

$$U = \frac{1}{2}(1 + \beta\lambda)\left[1 + \frac{1}{4}(\beta\lambda + \lambda\beta - 2)\right]^{-1/2}, \quad \lambda = \frac{\mathcal{H}}{(\mathcal{H}^2)^{1/2}}, \tag{4}$$

where $\mathcal{H}$ is the Hamiltonian in the Dirac representation. This operator brings the Dirac wave function and the Dirac Hamiltonian to the FW representation in one step. However, it is difficult to use the Eriksen method for obtaining an explicit form of the relativistic FW Hamiltonian because the general final formula is very cumbersome and contains roots of Dirac matrix operators. Therefore, the Eriksen method was not used for relativistic particles in external fields.

To perform the FW transformation in the strong external fields, we develop the much simpler method elaborated in Ref. [3] for relativistic spin-1/2 particles. In this work, the initial Dirac Hamiltonian is given by

$$\mathcal{H} = \beta m + \mathcal{E} + \mathcal{O}, \tag{5}$$

where m is the particle mass. In Eqs. (5)–(11), the system of units $\hbar = c = 1$ is used.

When $[\mathcal{E},\mathcal{O}]=0$, the FW transformation is exact [3]. This transformation is fulfilled with the operator

$$U = \frac{\epsilon + m + \beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon + m)}}, \quad \epsilon = \sqrt{m^2 + \mathcal{O}^2} \tag{6}$$

and the transformed Hamiltonian takes the form

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E}. \tag{7}$$

The same transformation is valid for Hamiltonian (2) when not only does the operator $\mathcal{E}$ commutates with $\mathcal{O}$ but also the operator $\mathcal{M}$:

$$[\mathcal{M},\mathcal{O}] = 0. \tag{8}$$

In this case, Eq. (7) remains valid but the operator ϵ takes the form

$$\epsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2}.$$

In the general case, the FW Hamiltonian has been obtained as a power series in external field potentials and their derivatives [3]. As a result of the first stage of transformation performed with operator (6), the following Hamiltonian can be found:

$$\mathcal{H}' = \beta\epsilon + \mathcal{E}' + \mathcal{O}', \quad \beta\mathcal{E}' = \mathcal{E}'\beta, \quad \beta\mathcal{O}' = -\mathcal{O}'\beta. \tag{9}$$

The odd operator $\mathcal{O}'$ is now comparatively small:

$$\epsilon = \sqrt{m^2 + \mathcal{O}^2},$$

$$\mathcal{E}' = i\frac{\partial}{\partial t} + \frac{\epsilon + m}{\sqrt{2\epsilon(\epsilon + m)}}\left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\frac{\epsilon + m}{\sqrt{2\epsilon(\epsilon + m)}} - \frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon + m)}}\left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon + m)}},$$

$$\mathcal{O}' = \frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon + m)}}\left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\frac{\epsilon + m}{\sqrt{2\epsilon(\epsilon + m)}} - \frac{\epsilon + m}{\sqrt{2\epsilon(\epsilon + m)}}\left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon + m)}}. \tag{10}$$

The second stage of transformation leads to the approximate equation for the FW Hamiltonian

$$\mathcal{H}_{\mathrm{FW}} = \beta\epsilon + \mathcal{E}' + \frac{1}{4}\beta\left\{\mathcal{O}'^2, \frac{1}{\epsilon}\right\}. \tag{11}$$

To reach a better precision, additional transformations can be used [3].

This method has been applied for deriving the Hamiltonian and the quantum mechanical equations of momentum and spin motion for Dirac particles interacting with electroweak [3] and gravitational [23,27] fields. The semiclassical limit of these equations has been obtained [3,23,27]. To determine the exact classical limit of the relativistic quantum mechanics of arbitrary-spin particles in strong external fields, we need to generalize the method.

General properties of the Hamiltonian depend on the particle spin. The Hamiltonian is Hermitian ($\mathcal{H}=\mathcal{H}^\dagger$) for spin-1/2 particles and pseudo-Hermitian for spin-0 and spin-1 ones (more precisely, β-pseudo-Hermitian, see Ref. [28], and references therein). In the latter case, it possesses the property ($\beta^{-1}=\beta$)

$$\mathcal{H}^\dagger = \beta\mathcal{H}\beta$$

which is equivalent to

$$\mathcal{H}^\ddagger \equiv \beta\mathcal{H}^\dagger\beta = \mathcal{H}.$$

The normalization of wave functions is given by

$$\int \Psi^\dagger\Psi dV = \int (\phi\phi^* + \chi\chi^*)dV = 1$$

for spin-1/2 particles and

$$\int \Psi^\ddagger\Psi dV \equiv \int \Psi^\dagger\beta\Psi dV = \int (\phi\phi^* - \chi\chi^*)dV = 1$$

for spin-0 and spin-1 particles. We suppose $\mathcal{M}=\mathcal{M}^\dagger$, $\mathcal{E}=\mathcal{E}^\dagger$, $\mathcal{O}=\mathcal{O}^\dagger$ when $\mathcal{H}=\mathcal{H}^\dagger$ and $\mathcal{M}=\mathcal{M}^\ddagger$, $\mathcal{E}=\mathcal{E}^\ddagger$, $\mathcal{O}=\mathcal{O}^\ddagger$ when $\mathcal{H}=\mathcal{H}^\ddagger$. These conditions can be satisfied in any case.

Since the FW Hamiltonian is block diagonal and a lower spinor describes negative-energy states, this spinor should be equal to zero. The FW transformation should be performed with the unitary operator $U^\dagger=U^{-1}$ for spin-1/2 particles and with the pseudounitary operator $U^\ddagger\equiv\beta U^\dagger\beta=U^{-1}$ for spin-0 and spin-1 particles.

III. FOLDY-WOUTHYUSEN TRANSFORMATION IN STRONG EXTERNAL FIELDS

We propose the method of the FW transformation for relativistic particles in strong external fields which can be used for particles of arbitrary spin. The FW Hamiltonian can be expanded into a power series in the Planck constant which defines the order of magnitude of quantum corrections. The obtained expressions for low-order terms in $\hbar$ are exact. The proposed FW transformation makes the transition to the semiclassical approximation to be trivial. The power expansion can be available only if

$$pl \gg \hbar, \tag{12}$$

where p is the momentum of the particle and l is the characteristic size of the nonuniformity region of the external field. This relation is equivalent to

$$\lambda \ll l, \tag{13}$$

where λ is the de Broglie wavelength. Equations (12) and (13) result from the fact that the Planck constant appears in the final Hamiltonian due to commutators between the operators $\mathcal{M}$, $\mathcal{E}$, and $\mathcal{O}$.

The expansion of the FW Hamiltonian into the power series in the Planck constant is formally similar to the previously obtained expansion [3] into a power series in the external field potentials and their derivatives. However, the equations derived in Ref. [3] do not define the semiclassical limit of the Dirac equation for particles in strong external fields, while these equations exhaustively describe the weak-field expansion. The proposed method can also be used in the weak-field expansion even when relations (12) and (13) are not valid.

When the power series in the Planck constant is deduced, zero power terms define the quantum analog of the classical Hamiltonian. On this level, classical and quantum expressions should be very similar because the classical theory gives the right limit of the quantum theory. Terms proportional to powers of $\hbar$ may describe quantum corrections. As a rule, interactions described by these terms also exist in the classical theory. However, classical expressions may differ from the corresponding quantum ones because the quantum corrections to the classical theory may appear.

We generalize the method developed in Ref. [3] in order to take into account a possible noncommutativity of the operators $\mathcal{M}$ and $\mathcal{O}$. The natural generalization of transformation operator (6) used in Ref. [3] is

$$U=\frac{\beta\epsilon+\beta\mathcal{M}-\mathcal{O}}{\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}}\beta,\quad U^{-1}=\beta\frac{\beta\epsilon+\beta\mathcal{M}-\mathcal{O}}{\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}},$$

$$\epsilon=\sqrt{\mathcal{M}^2+\mathcal{O}^2}. \tag{14}$$

where $U^\dagger=U^{-1}$ when $\mathcal{H}=\mathcal{H}^\dagger$ and $U^\ddagger=U^{-1}$ when $\mathcal{H}=\mathcal{H}^\ddagger$. This form of the transformation operator allows to perform the FW transformation in the general case. The special case $\mathcal{M}=mc^2$ has been considered in Ref. [3] and commutation relation (8) has been used in Refs. [23,29].

We consider the general case when external fields are nonstationary. The exact formula for the transformed Hamiltonian has the form

$$\begin{aligned}\mathcal{H}'=\beta\epsilon+\mathcal{E}+\frac{1}{2T}([T,[T,(\beta\epsilon+\mathcal{F})]]+\beta[\mathcal{O},[\mathcal{O},\mathcal{M}]]\\-[\mathcal{O},[\mathcal{O},\mathcal{F}]]-[(\epsilon+\mathcal{M}),[(\epsilon+\mathcal{M}),\mathcal{F}]]\\-[(\epsilon+\mathcal{M}),[\mathcal{M},\mathcal{O}]]-\beta\{\mathcal{O},[(\epsilon+\mathcal{M}),\mathcal{F}]\}\\+\beta\{(\epsilon+\mathcal{M}),[\mathcal{O},\mathcal{F}]\})\frac{1}{T},\end{aligned} \tag{15}$$

where $\mathcal{F}=\mathcal{E}-i\hbar\frac{\partial}{\partial t}$ and $T=\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}$.

Hamiltonian (15) still contains odd terms proportional to the first and higher powers of the Planck constant. This Hamiltonian can be presented in the form

$$\mathcal{H}'=\beta\epsilon+\mathcal{E}'+\mathcal{O}',\quad \beta\mathcal{E}'=\mathcal{E}'\beta,\quad \beta\mathcal{O}'=-\mathcal{O}'\beta, \tag{16}$$

where $\epsilon=\sqrt{\mathcal{M}^2+\mathcal{O}^2}$. The even and odd parts of Hamiltonian (16) are defined by the well-known relations

$$\mathcal{E}'=\frac{1}{2}(\mathcal{H}'+\beta\mathcal{H}'\beta)-\beta\epsilon,\quad \mathcal{O}'=\frac{1}{2}(\mathcal{H}'-\beta\mathcal{H}'\beta).$$

Additional transformations performed according to Ref. [3] bring $\mathcal{H}'$ to the block-diagonal form. The approximate formula for the final FW Hamiltonian is

$$\mathcal{H}_{\rm FW}=\beta\epsilon+\mathcal{E}'+\frac{1}{4}\beta\left\{\mathcal{O}'^2,\frac{1}{\epsilon}\right\}. \tag{17}$$

This formula is similar to the corresponding one obtained in Ref. [3]. The additional transformations allow one to obtain more precise expression for the FW Hamiltonian.

Equations (15)–(17) solve the problem of the FW transformation for relativistic particles of arbitrary spin in strong external fields. Equation (15) can be significantly simplified in some special cases. When $[\mathcal{M},\mathcal{O}]=0$ and the external fields are stationary, it is reduced to

$$\begin{aligned}\mathcal{H}'=\beta\epsilon+\mathcal{E}+\frac{1}{2T}([T,[T,\mathcal{E}]]-[\mathcal{O},[\mathcal{O},\mathcal{E}]]\\-[(\epsilon+\mathcal{M}),[(\epsilon+\mathcal{M}),\mathcal{E}]]\\-\beta\{\mathcal{O},[(\epsilon+\mathcal{M}),\mathcal{E}]\}+\beta\{(\epsilon+\mathcal{M}),[\mathcal{O},\mathcal{E}]\})\frac{1}{T}.\end{aligned} \tag{18}$$

In this case, $[\epsilon,\mathcal{M}]=[\epsilon,\mathcal{O}]=0$ and the operator $T=\sqrt{2\epsilon(\epsilon+\mathcal{M})}$ is even.

IV. SPIN-1/2 AND SCALAR PARTICLES IN STRONG ELECTROMAGNETIC FIELD

As an example, the FW transformation for spin-1/2 and scalar particles interacting with a strong electromagnetic field can be considered. The initial Dirac-Pauli Hamiltonian for a particle possessing an anomalous magnetic moment (AMM) has the form [30]

$$\mathcal{H}_{\rm DP}=c\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+\beta mc^2+e\Phi+\mu'(-\boldsymbol{\Pi}\cdot\boldsymbol{H}+i\boldsymbol{\gamma}\cdot\boldsymbol{E}),$$

$$\boldsymbol{\pi}=\boldsymbol{p}-\frac{e}{c}\boldsymbol{A},\quad \mu'=\frac{g-2}{2}\frac{e\hbar}{2mc}, \tag{19}$$

where μ' is the AMM and Φ, $\boldsymbol{A}$ and $\boldsymbol{E}$, $\boldsymbol{H}$ are the potentials and strengths of the electromagnetic field.

Here and below the following designations for the matrices are used:

$$\boldsymbol{\gamma}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta\equiv\gamma^0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

$$\boldsymbol{\alpha}=\beta\boldsymbol{\gamma}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix},$$

$$\boldsymbol{\Pi}=\beta\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix},$$

where $0,1,-1$ mean the corresponding 2×2 matrices and $\boldsymbol{\sigma}$ is the Pauli matrix. Terms describing the electric dipole moment (EDM) d have been added in Ref. [31]. The resulting Hamiltonian is given by

$$\begin{aligned}\mathcal{H}=c\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+\beta mc^2+e\Phi+\mu'(-\boldsymbol{\Pi}\cdot\boldsymbol{H}+i\boldsymbol{\gamma}\cdot\boldsymbol{E})\\-d(\boldsymbol{\Pi}\cdot\boldsymbol{E}+i\boldsymbol{\gamma}\cdot\boldsymbol{H}),\quad d=\frac{\eta}{2}\frac{e\hbar}{2mc},\end{aligned} \tag{20}$$

Where the η factor for the EDM is an analog of the g factor for the magnetic moment. It is important that μ' and d are proportional to $\hbar$.

In the case considered

$$\mathcal{M}=mc^2,\quad \mathcal{E}=e\Phi-\mu'\boldsymbol{\Pi}\cdot\boldsymbol{H}-d\boldsymbol{\Pi}\cdot\boldsymbol{E},$$

$$\mathcal{O}=c\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+i\mu'\boldsymbol{\gamma}\cdot\boldsymbol{E}-id\boldsymbol{\gamma}\cdot\boldsymbol{H}.$$

Since only terms of zero and first powers in the Planck constant define the semiclassical equations of motion of particles and their spins, we retain only such terms in the FW Hamiltonian. The terms of order of $\hbar$ are proportional either to field gradients or to products of field strengths (H^2, E^2, and EH). We do not calculate the terms proportional to products of field strengths because they are usually small in comparison with the terms proportional to field gradients.

The calculated Hamiltonian is given by

$$\mathcal{H}_{\rm FW}=\beta\epsilon'+e\Phi-\mu'\boldsymbol{\Pi}\cdot\boldsymbol{H}-\frac{\mu_0}{2}\left\{\frac{mc^2}{\epsilon'},\boldsymbol{\Pi}\cdot\boldsymbol{H}\right\}+\frac{\mu'c}{4}\left\{\frac{1}{\epsilon'},[\boldsymbol{\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{E})-\boldsymbol{\Sigma}\cdot(\boldsymbol{E}\times\boldsymbol{\pi})]\right\}+\frac{\mu_0mc^3}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}[\boldsymbol{\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{E})$$
$$-\boldsymbol{\Sigma}\cdot(\boldsymbol{E}\times\boldsymbol{\pi})]\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}+\frac{\mu'c^2}{2\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Pi}\cdot\boldsymbol{\pi}),(\boldsymbol{H}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{H})\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}-d\boldsymbol{\Pi}\cdot\boldsymbol{E}$$
$$+\frac{dc^2}{2\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Pi}\cdot\boldsymbol{\pi}),(\boldsymbol{E}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{E})\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}-\frac{dc}{4}\left\{\frac{1}{\epsilon'},[\boldsymbol{\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{H})-\boldsymbol{\Sigma}\cdot(\boldsymbol{H}\times\boldsymbol{\pi})]\right\},\qquad(21)$$

where

$$\epsilon'=\sqrt{m^2c^4+c^2\boldsymbol{\pi}^2}\qquad(22)$$

and $\mu_0=\frac{e\hbar}{2mc}$ is the Dirac magnetic moment. The quantum evolution of the kinetic momentum operator $\boldsymbol{\pi}$ is defined by the operator equation of particle motion

$$\frac{d\boldsymbol{\pi}}{dt}=\frac{i}{\hbar}[\mathcal{H}_{\rm FW},\boldsymbol{\pi}]-\frac{e}{c}\frac{\partial\boldsymbol{A}}{\partial t}.\qquad(23)$$

The equation of spin motion describes the evolution of the polarization operator $\boldsymbol{\Pi}$:

$$\frac{d\boldsymbol{\Pi}}{dt}=\frac{i}{\hbar}[\mathcal{H}_{\rm FW},\boldsymbol{\Pi}].\qquad(24)$$

Because the operator $\boldsymbol{\pi}$ does not contain the Dirac spin matrices, the commutator of this operator with the Hamiltonian is proportional to $\hbar$. The equation of spin-1/2 particle motion in the strong electromagnetic field to within first-order terms in the Planck constant has the form

$$\frac{d\boldsymbol{\pi}}{dt}=e\boldsymbol{E}+\beta\frac{ec}{4}\left\{\frac{1}{\epsilon'},([\boldsymbol{\pi}\times\boldsymbol{H}]-[\boldsymbol{H}\times\boldsymbol{\pi}])\right\}+\mu'\nabla(\boldsymbol{\Pi}\cdot\boldsymbol{H})+\frac{\mu_0}{2}\left\{\frac{mc^2}{\epsilon'},\nabla(\boldsymbol{\Pi}\cdot\boldsymbol{H})\right\}-\frac{\mu'c}{4}\left(\frac{1}{\epsilon'},[\nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}])\right.$$
$$\left.-\nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])]\right)-\frac{\mu_0mc^3}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}[\nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}])-\nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])]\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}$$
$$-\frac{\mu'c^2}{2\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Pi}\cdot\boldsymbol{\pi}),[\nabla(\boldsymbol{H}\cdot\boldsymbol{\pi})+\nabla(\boldsymbol{\pi}\cdot\boldsymbol{H})]\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}.\qquad(25)$$

This equation can be divided into two parts. The first part does not contain the Planck constant and describes the quantum equivalent of the Lorentz force. The second part is of order of $\hbar$. This part defines the relativistic expression for the Stern-Gerlach force. Since terms proportional to d are small, they are omitted.

The equation of spin motion is given by

$$\frac{d\boldsymbol{\Pi}}{dt}=\frac{2\mu'}{\hbar}\boldsymbol{\Sigma}\times\boldsymbol{H}+\frac{\mu_0}{\hbar}\left\{\frac{mc^2}{\epsilon'},\boldsymbol{\Sigma}\times\boldsymbol{H}\right\}-\frac{\mu'c}{2\hbar}\left\{\frac{1}{\epsilon'},[\boldsymbol{\Pi}\times(\boldsymbol{\pi}\times\boldsymbol{E})-\boldsymbol{\Pi}\times(\boldsymbol{E}\times\boldsymbol{\pi})]\right\}-\frac{\mu_0mc^3}{\hbar\sqrt{\epsilon'(\epsilon'+mc^2)}}[\boldsymbol{\Pi}\times(\boldsymbol{\pi}\times\boldsymbol{E})$$
$$-\boldsymbol{\Pi}\times(\boldsymbol{E}\times\boldsymbol{\pi})]\frac{1}{\sqrt{\epsilon'(\epsilon'+mc^2)}}-\frac{\mu'c^2}{\hbar\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Sigma}\times\boldsymbol{\pi}),(\boldsymbol{H}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{H})\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}+\frac{2d}{\hbar}\boldsymbol{\Sigma}\times\boldsymbol{E}$$
$$-\frac{dc^2}{\hbar\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Sigma}\times\boldsymbol{\pi}),(\boldsymbol{E}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{E})\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}+\frac{dc}{2\hbar}\left\{\frac{1}{\epsilon'},[\boldsymbol{\Pi}\times(\boldsymbol{\pi}\times\boldsymbol{H})-\boldsymbol{\Pi}\times(\boldsymbol{H}\times\boldsymbol{\pi})]\right\}.\qquad(26)$$

Equations (21), (25), and (26) agree with the corresponding equations derived in Refs [3,31]. However, unlike the latter equations, Eqs. (21), (25), and (26) describe strong-field effects.

We can also consider the interaction of spinless particles with the strong electromagnetic field. The initial Klein-Gordon equation describing this interaction has been transformed to the Hamilton form in Ref. [9].

In this case, the Hamiltonian acts on the two-component wave function which is the analog of the spinor. The explicit form of this Hamiltonian is [9]

$$\mathcal{H}=\rho_3mc^2+(\rho_3+i\rho_2)\frac{\boldsymbol{\pi}^2}{2m}+e\Phi.\qquad(27)$$

Therefore,

$$\mathcal{M}=mc^2+\frac{\boldsymbol{\pi}^2}{2m},\quad \mathcal{E}=e\Phi,\quad \mathcal{O}=i\rho_2\frac{\boldsymbol{\pi}^2}{2m},\quad [\mathcal{M},\mathcal{O}]=0. \tag{28}$$

For spinless particles,

$$\epsilon=\sqrt{m^2c^4+c^2\boldsymbol{\pi}^2},\quad T=\sqrt{\frac{\epsilon}{mc^2}}(\epsilon+mc^2). \tag{29}$$

The Hamiltonian transformed to the FW representation is given by

$$\mathcal{H}_{FW}=\beta\epsilon+\mathcal{E}=\beta\sqrt{m^2c^4+c^2\boldsymbol{\pi}^2}+e\Phi. \tag{30}$$

There are not any terms of order of $\hbar$ in this Hamiltonian, while it contains terms of second and higher orders in the Planck constant. We do not calculate the latter terms because their contribution into equations of particle motion is usually negligible.

The operator equation of particle motion takes the form

$$\frac{d\boldsymbol{\pi}}{dt}=e\boldsymbol{E}+\beta\frac{ec}{4}\left\{\frac{1}{\epsilon},([\boldsymbol{\pi}\times\boldsymbol{H}]-[\boldsymbol{H}\times\boldsymbol{\pi}])\right\}. \tag{31}$$

The right-hand side of this equation coincides with the spin-independent part of the corresponding equation for spin-1/2 particles.

Equation (30) for the FW Hamiltonian agrees with Eq. (12) in Ref. [32]. In this reference, the weak-field approximation has been used and the operator equation of particle motion in the strong electromagnetic field has not been obtained.

V. SEMICLASSICAL LIMIT OF RELATIVISTIC QUANTUM MECHANICS FOR PARTICLES IN STRONG EXTERNAL FIELDS

To obtain the semiclassical limit of the relativistic quantum mechanics, one needs to average the operators in the quantum mechanical equations. When the FW representation is used and relations (12) and (13) are valid, the semiclassical transition consists in trivial replacing operators by corresponding classical quantities. In this representation, the problem of extracting even parts of the operators does not appear. Therefore, the derivation of equations for particles of arbitrary spin in strong external fields made in the precedent section solves the problem of obtaining the semiclassical limit of the relativistic quantum mechanics. If the momentum and position operators are chosen to be the dynamical variables, relations (12) and (13) are equivalent to the condition

$$|\langle p_i\rangle|\cdot|\langle x_i\rangle|\gg|\langle[p_i,x_i]\rangle|=\hbar,\quad i=1,2,3. \tag{32}$$

The angular brackets which designate averaging in time will be hereinafter omitted. Obtained semiclassical equations may differ from corresponding classical ones.

As a result of replacing operators by corresponding classical quantities, the semiclassical equations of motion of particles and their spins take the form

$$\begin{aligned}\frac{d\boldsymbol{\pi}}{dt}&=e\boldsymbol{E}+\frac{ec}{\epsilon'}(\boldsymbol{\pi}\times\boldsymbol{H})+\mu'\nabla(\boldsymbol{P}\cdot\boldsymbol{H})+\frac{\mu_0}{mc^2\epsilon'}\nabla(\boldsymbol{P}\cdot\boldsymbol{H})\\&-\frac{\mu'c}{\epsilon'}\nabla(\boldsymbol{P}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}])-\frac{\mu_0mc^3}{\epsilon'(\epsilon'+mc^2)}\nabla(\boldsymbol{P}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}])\\&-\frac{\mu'c^2}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\cdot\boldsymbol{\pi})\nabla(\boldsymbol{H}\cdot\boldsymbol{\pi}),\quad \boldsymbol{P}=\frac{\boldsymbol{S}}{S},\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{d\boldsymbol{P}}{dt}&=2\mu'\boldsymbol{P}\times\boldsymbol{H}+\frac{2\mu_0mc^2}{\epsilon'}(\boldsymbol{P}\times\boldsymbol{H})-\frac{2\mu'c}{\epsilon'}(\boldsymbol{P}\times[\boldsymbol{\pi}\times\boldsymbol{E}])\\&-\frac{2\mu_0mc^3}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\times[\boldsymbol{\pi}\times\boldsymbol{E}])-\frac{2\mu'c^2}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\times\boldsymbol{\pi})\\&\times(\boldsymbol{\pi}\cdot\boldsymbol{H})+2d\boldsymbol{P}\times\boldsymbol{E}-\frac{2dc^2}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\times\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{E})\\&+\frac{2dc}{\epsilon'}(\boldsymbol{P}\times[\boldsymbol{\pi}\times\boldsymbol{H}]).\end{aligned} \tag{34}$$

In Eqs. (33) and (34), ϵ' is defined by Eq. (22), $\boldsymbol{P}$ is the polarization vector, and $\boldsymbol{S}$ is the spin vector (i.e., the average spin).

For scalar particles

$$\frac{d\boldsymbol{\pi}}{dt}=e\boldsymbol{E}+\frac{ec}{\sqrt{m^2c^4+c^2\boldsymbol{\pi}^2}}(\boldsymbol{\pi}\times\boldsymbol{H}). \tag{35}$$

The first two terms on the right-hand sides of Eqs. (33) and (35) are the same as in the classical expression for the Lorentz force. This is a manifestation of the correspondence principle. The part of Eq. (34) dependent on the magnetic moment coincides with the well-known Thomas-Bargmann-Michel-Telegdi (TBMT) equation. It is natural because the TBMT equation has been derived without the assumption that the external fields are weak. The relativistic formula for the Stern-Gerlach force can be obtained from the Lagrangian consistent with the TBMT equation (see Ref. [33]). The semiclassical and classical formulae describing this force also coincide. High-order corrections to the quantum equations of motion of particles and their spins should bring a difference between quantum and classical approaches.

VI. DISCUSSION AND SUMMARY

The method of the FW transformation for relativistic particles of arbitrary spin in strong external fields described in the present work is based on previous developments [3]. However, the use of transformation operator (14) is not restricted by any definite commutation relations [see Eq. (8)] between even and odd operators. The proposed method utilizes the expansion of the FW Hamiltonian into a power series in the Planck constant which defines the order of magnitude of quantum corrections. In the FW Hamiltonian, exact expressions for low-order terms in $\hbar$ can be obtained. If the de Broglie wavelength is much less than the characteristic size of the nonuniformity region of the external field [see Eqs. (12) and (13)], the transition to the semiclassical approximation becomes trivial. In this case, it consists in replacing operators by corresponding classical quantities. The simplest semiclassical transition is one of main preferences of the FW representation.

If Eqs. (12) and (13) are not valid, the proposed method can be used in the weak-field expansion. This expansion previously used in Ref. [3] presents the FW Hamiltonian as a power series in the external field potentials and their derivatives. In this case, the operator equations characterizing dynamics of the particle momentum and spin can also be derived. Solutions of these equations define the quantum evolution of main operators. Semiclassical evolution of classical quantities corresponding to these operators can be obtained by averaging the operators in the solutions. An example of such an evolution is the time dependence of average energy and momentum in a two-level system.

When the FW Hamiltonian can be expanded into a power series in the Planck constant, we obtain the semiclassical limit of the relativistic quantum mechanics. Since the correspondence principle must be satisfied, classical and semiclassical Hamiltonians and equations of motion must agree. As an example, we consider the interaction of scalar and spin-1/2 particles with the strong electromagnetic field. We carried out the FW transformation and derived the quantum equations of particle motion. We have also deduced the quantum equations of spin motion for spin-1/2 particles. Averaging operators in the quantum equations consists in substitution of classical quantities for these operators and allows one to obtain the semiclassical equations which are in full agreement with the corresponding classical equations. The proved agreement confirms the validity of both the correspondence principle and the aforesaid method. All calculations have been carried out for relativistic particles in strong external fields.

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