Stabilization of vortex solitons in nonlocal nonlinear media

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We study the evolution of vortex solitons in optical media with a nonlocal nonlinear response. We employ a modulation theory for the vortex parameters based on an averaged Lagrangian, and analyze the azimuthal evolution of both the vortex width and diffractive radiation. We describe *analytically* the physical mechanism for vortex stabilization due to the long-range nonlocal nonlinear response, the effect observed earlier in numerical simulations only.

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I. INTRODUCTION

Optical vortices are usually introduced as phase singularities in diffracting optical beams [1]. In self-focusing nonlinear media, such optical vortices are transformed into spatially localized self-trapped beams carrying a nonzero angular momentum, also known as *vortex solitons* [2]. In all conservative media with a local nonlinear response, vortex solitons are known to experience a symmetry-breaking azimuthal instability, and they decay into several fundamental solitons [3], as illustrated in Figs. 1(a)–1(d). In some dissipative local media it has been found that vortices can be stabilised, as discussed in the overview in Ref. [3]. However, in recent numerical studies it was revealed that spatially localized vortex solitons can be stabilized even in self-focusing nonlinear media when the vortex symmetry-breaking instability is eliminated by a nonlocal, nonlinear response [4–6].

In this paper, we study the stability of vortex solitons in nonlocal, nonlinear media analytically, taking into account the specific form of the nonlocal equations governing, for example, the propagation of light through a nematic liquid crystal [7]. We employ a suitable trial function and develop the modulation theory to find an averaged Lagrangian for the evolution of the vortex parameters. The trial function used in this averaged Lagrangian includes a term representing the low-wave-number diffractive radiation which accumulates under the vortex as it evolves. To study the vortex stability, we assume the width of the trial function to depend on the angular variable, so that the vortex can break up. The conjugate variable to the width, which is the height of the shelf of low wavenumber diffractive radiation under the vortex, then also becomes a function of the angular variable. We then derive modulation equations from the averaged Lagrangian in the usual way, and they form a system of equations for the amplitude and width of the vortex. This system has a fixed point representing a stationary vortex. We find that in the local limit (i.e., when $\nu \to 0$, ν being the nonlocality parameter), this fixed point is unstable to azimuthal perturbations of low wave number $l=1,2,3$, with the most unstable wave number being $l=2$, and stable otherwise, in perfect agreement with numerical simulations [4]. On the other hand, as the nonlocality parameter ν increases, it is found that the instability growth rate decreases and disappears at $\nu=\nu_{\mathrm{cr}}$ [4]. In addition, we predict that the instability decreases as the amplitude of the vortex increases. We believe that the analysis presented below provides a clear physical mechanism for the vortex stabilization in nonlocal, nonlinear media observed in earlier numerical work only.

The paper is organized as follows. In Sec. II we formulate the vortex stability problem using a single equation for the electric field with a nonlocal response function. Using a specific ansatz, in Sec. III we calculate the averaged Lagrangian based on this single equation. In Sec. IV we derive and study the modulation equations in both the local and nonlocal regimes, and determine the stability regions for the vortex soliton.

II. BASIC EQUATIONS

As a specific case of nonlocal media, we consider the propagation of an electromagnetic wave in a nematic liquid crystal in the presence of an applied constant, static electric field. The dimensionless equations for the field envelope $E(x,y;z)$ and the director deviation ϕ from its pretilted position, due to the static electric field, can be written in the form [7]

$$i\frac{\partial E}{\partial z} + \frac{1}{2}\nabla^2 E + \phi E = 0,$$

$$\nu\nabla^2\phi - \phi = -|E|^2. \tag{1}$$

Here z is the direction of light propagation, ∇^2 is the transverse (to z) Laplacian, and ν is the nonlocality parameter being the ratio of the elastic energy of the nematic liquid crystal to the energy of the applied static electric field [8].

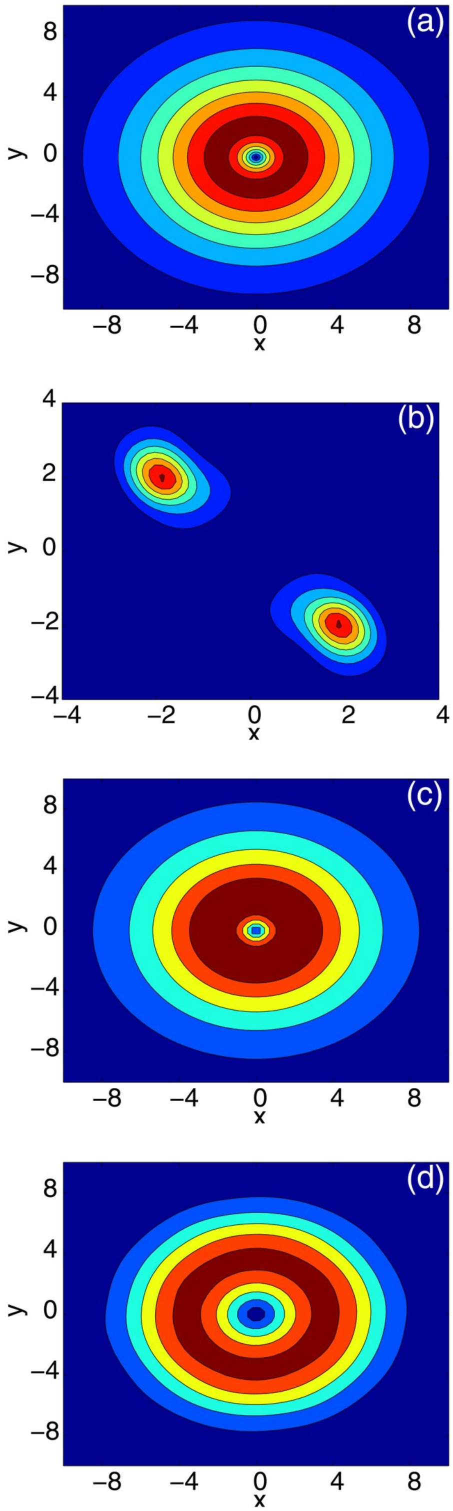

FIG. 1. (Color online) Numerical solution of nematicon equations (1) for initial condition (8) with $g=0$. (a) $a=2$, $w=2$, $\nu=0.5$ at $z=0$; (b) $a=2$, $w=2$, $\nu=0.5$ at $z=10$; (c) $a=4.077$, $w=2$, $\nu=150$ at $z=0$; (d) $a=4.077$, $w=2$, $\nu=150$ at $z=100$.

The same model also describes light propagation in partially ionized plasmas [4].

The system (1) can be reduced to a single equation after using the solution of the second equation for the director expressed through a Green's function,

$$\phi(x,y;z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0(\psi)|E(x',y',z)|^2\, dx'\, dy', \tag{2}$$

where

$$\psi = \left[\left(\frac{x-x'}{\sqrt{\nu}}\right)^2 + \left(\frac{y-y'}{\sqrt{\nu}}\right)^2\right]^{1/2} \tag{3}$$

and K_0 is the modified Bessel function of zero order.

This nonlinear equation has the Lagrangian,

$$\mathcal{L} = \int_0^{2\pi}\int_0^{\infty} L\, dr\, d\theta, \tag{4}$$

where

$$L = ir(E^*E_z - EE_z^*) - r|E_r|^2 - \frac{1}{r}|E_\theta|^2 + r|E|^2\int_0^{2\pi}\int_0^{\infty} G(r,\theta,r',\theta')|E(r',\theta')|^2\, dr'\, d\theta', \tag{5}$$

with a star denoting the complex conjugate. The Green's function G is obtained from K_0 using the addition property of the modified Bessel functions in polar coordinates, resulting in

$$G(r,\theta,r',\theta') = \sum_{l=-\infty}^{\infty} G_l(r,r')e^{il(\theta-\theta')}, \tag{6}$$

with

$$G_l(r,r') = \begin{cases} I_l(r/\sqrt{\nu})K_l(r'/\sqrt{\nu}), & 0 \leq r \leq r', \\ K_l(r/\sqrt{\nu})I_l(r'/\sqrt{\nu}), & r' \leq r < \infty. \end{cases} \tag{7}$$

Here I_l and K_l are modified Bessel functions of order l.

III. VARIATIONAL APPROACH

To study the dynamics of the vortex and its stability analytically, we shall employ the variational approach and use a trial function for the vortex soliton of the form

$$E(r,\theta,z) = are^{-r/w}e^{i\sigma+i\theta} + ig(r,\theta,z)e^{i\sigma+i\theta}. \tag{8}$$

The first term in Eq. (8) is the vortex soliton, and the second term is introduced for the low frequency diffractive radiation shed by the vortex as it evolves [4]. As this radiation has zero group velocity, it sits under the evolving soliton, in analogy with other cases studied earlier [8–10].

A new feature of the vortex problem is that the parameters a, w, g, and σ should depend not only on the space variable z, but also on the azimuthal angle θ. This angular dependence for the trial function is motivated by the modal expansion

$$w(z,\theta) = w_0(z) + w_1(z)\cos\,\theta + w_2(z)\cos(2\theta) + \cdots. \tag{9}$$

This expansion, when w_2 becomes negative, produces the pinching of the vortex which is caused by the azimuthal instability. Since the conjugate variable to w is g, g must also be a function of θ. By similar reasoning, the amplitude a and phase σ are also functions of θ, accounting for the angular momentum which can be added or taken away from the vortex by its interaction with diffractive radiation.

Now substituting the trial function (8) into the Lagrangian (5) and, after integrating with respect to r, we obtain an averaged Lagrangian for the modulation parameters a, w, g, and σ. This averaged Lagrangian consists of two parts,

$$L = L_0 + L_\nu, \tag{10}$$

where L_0 is independent of the nonlinearity, and has average $\mathcal{L}_0$, while L_ν, with average $\mathcal{L}_\nu$, depends on the specific form of the nonlocal nonlinearity. These components of the averaged Lagrangian are given by

$$\begin{aligned}\mathcal{L}_0 = &-\left(\frac{3}{4}a^2w^4 + 2\Lambda_1 g^2\right)\frac{\partial \sigma}{\partial z} - 4aw^3\frac{\partial g}{\partial z} + 4gw^3\frac{\partial a}{\partial z}\\ &+ 12agw^2\frac{\partial w}{\partial z} - \frac{3}{8}w^2a^2 - \frac{1}{4}w^2\left(\frac{\partial a}{\partial \theta}\right)^2 - \frac{1}{2}aw\frac{\partial a}{\partial \theta}\frac{\partial w}{\partial \theta}\\ &- \frac{3}{8}a^2\left(\frac{\partial w}{\partial \theta}\right)^2 + 2wg\frac{\partial a}{\partial \theta} + 2ag\frac{\partial w}{\partial \theta} - 2aw\frac{\partial g}{\partial \theta}\\ &- \Lambda_2\left[\left(\frac{\partial g}{\partial \theta}\right)^2 + g^2\right] - \frac{1}{2}a^2w^2\frac{\partial \sigma}{\partial \theta} - 2aw\frac{\partial \sigma}{\partial \theta}\frac{\partial g}{\partial \theta}\\ &+ 2wg\frac{\partial \sigma}{\partial \theta}\frac{\partial a}{\partial \theta} + 2ag\frac{\partial \sigma}{\partial \theta}\frac{\partial w}{\partial \theta}\\ &- \Lambda_2 g^2\left[\left(\frac{\partial \sigma}{\partial \theta}\right)^2 + 2\frac{\partial \sigma}{\partial \theta}\right] - \frac{1}{4}a^2w^2\left(\frac{\partial \sigma}{\partial \theta}\right)^2\end{aligned} \tag{11}$$

and

$$\mathcal{L}_\nu = \int_0^{2\pi}\int_0^\infty |E|^2 \int_0^{2\pi}\int_0^\infty G(r,\theta,r',\theta')|E'|^2 r'\,dr'\,d\theta'\,r\,dr\,d\theta,$$

where $E=E(r,\theta,z)$ and $E'=E(r',\theta',z)$. To calculate $\mathcal{L}_\nu$, below we shall use the trial function (8), and evaluate the nonlocal integral term for small and large ν.

The terms Λ_1 and Λ_2 in Eq. (11) describe the effect of the shelf of radiation [9]. This radiation is assumed to be radially symmetric in space, centered about the vortex peak at $r=w$, as follows from numerical solutions. Hence g is only nonzero in the region $r_{\min} < r < r_{\max}$, where $r_{\min,\max} = w \mp R/2$. Then

$$\Lambda_1 = wR \quad \text{and} \quad \Lambda_2 = \ln\left(\frac{r_{\max}}{r_{\min}}\right), \tag{12}$$

where R will be determined later in the analysis.

In the local limit $\nu \to 0$, the nonlocal, nonlinear term in Eq. (2) reduces to $|E|^2$, so that the averaged Lagrangian term $\mathcal{L}_\nu$ becomes

$$\mathcal{L}_\nu = \frac{5!}{2^{12}}a^4w^6 + \frac{3!}{2^3}a^2w^4g^2. \tag{13}$$

To calculate $\mathcal{L}_\nu$ in the nonlocal limit, we note from the Lagrangian (5) that there is a contribution involving $|E|^4$ and a contribution involving $|E|^2$ and g^2. To evaluate the first contribution we note that since $|E|^2$ is independent of θ, the Green's function G reduces to the term with $l=0$. However, the integral

$$\Theta(r) = \int_0^\infty G_0(r,r')a^2r'^2e^{-2r'/w}r'\,dr' \tag{14}$$

cannot be evaluated in closed form. Numerical results [4] show that in the local limit $\nu \to 0$ the profile for the director ϕ for a vortex is concentrated around the electric field and in the nonlocal limit $\nu \to \infty$ the major portion of the director profile is a flat shelf from $r=0$ to the region where the vortex in the electric field is concentrated. To capture this behavior as $\nu \to \infty$ we approximate $\Theta(r)$ by the solution of the one-dimensional equation

$$\nu\frac{d^2\Theta}{dr^2} - \Theta = -a^2r^2e^{-2r/w}, \tag{15}$$

with the boundary conditions $d\Theta/dr=0$ at $r=0$ and $\Theta \to 0$ for $r \to \infty$. With Θ defined in this way, all the integrals involved in the calculation of the averaged Lagrangian for $\nu \to \infty$ are computable. For the azimuthally independent term $\bar{\mathcal{L}}_\nu$ of the averaged Lagrangian term $\mathcal{L}_\nu$ we then obtain (for $\nu \to \infty$)

$$\bar{\mathcal{L}}_\nu = \frac{a^4w^6}{32\sqrt{\nu}}. \tag{16}$$

To calculate the azimuthally dependent cross terms in $\mathcal{L}_\nu$, we use Eqs. (5) and (6) and calculate integrals with respect to r' and r, on setting

$$\begin{aligned}&\int_0^\infty \sum_{l=-\infty}^{\infty} e^{il(\theta-\theta')}G_l(r,r')g^2(r',\theta')r'\,dr'\\ &= \sum_{l=-\infty}^{\infty} e^{il(\theta-\theta')}Q_l(r,\theta',z).\end{aligned} \tag{17}$$

As above, we take the shelf term g of the form

$$g(r,\theta,z) = \begin{cases} g(\theta,z), & \text{for } r_{\min} \le r \le r_{\max},\\ 0, & \text{otherwise}.\end{cases} \tag{18}$$

The integral involving G_l in Eq. (17) cannot be evaluated in closed form. However, for large ν, G_l is just the Green's function of the Laplace operator, so that Q_l is the solution of

$$\frac{d^2Q_l}{dr^2} + \frac{1}{r}\frac{dQ_l}{dr} - \frac{l^2}{r^2}Q_l = \frac{g_l^2(r,z)}{\nu}, \tag{19}$$

and is regular at the origin and decays as $r \to \infty$. The forcing g_l^2 is the lth Fourier coefficient of $g^2(r,\theta',z)$. The other integral can be evaluated using the explicit analytical solution of the differential equation (19). The final result is

$$\mathcal{L}_\nu = \frac{2}{\nu}a^2w^4\int_0^{2\pi}\int_0^{2\pi}\sum_{l=-\infty}^{\infty}A_le^{il(\theta-\theta')}g^2(r,\theta',z)\,d\theta'\,d\theta, \tag{20}$$

where for large ν the coefficients A_l are given by

$$A_l = \frac{405e^{-2}}{256}\frac{1}{|l|}w^2. \tag{21}$$

This result is valid for $l \neq 0$. The $l=0$ case can be dealt with using the same Green's function solution (14), but it has no influence on the azimuthal instability.

IV. MODULATION EQUATIONS

Modulation equations for the vortex parameters can be obtained by averaging the Lagrangian (10) and then taking variations with respect to the parameters a, w, σ, and g. However, the full equations are not needed, as in the present work we are only interested in studying the linear stability of the vortex. Hence, we use only the modulation equations linearized around the stationary vortex profile.

The fixed point of the modulation equations corresponds to a stationary vortex; it can be found by setting $g=0$, $a'=w'=0$ and σ' a constant and then taking variations. In the local limit ($\nu \to 0$), we find the amplitude-width relation between the steady amplitude a_0 and width w_0

$$a_0^2 w_0^4 = 2^6/5, \tag{22}$$

and in the nonlocal limit ($\nu \to \infty$) the amplitude-width relation becomes

$$a_0^2 w_0^4 = 48\nu^{1/2}. \tag{23}$$

It can be seen that, for a fixed steady amplitude, the vortex width increases as ν increases, in agreement with numerical solutions [4].

As stated above, we study the vortex stability in the two limits $\nu \to 0$ and $\nu \to \infty$, and linearize the modulation equations about the fixed points given by Eqs. (22) and (23). In principle, perturbations around all the variables must be considered to fully explore stability.

First, we shall consider the local limit $\nu \to 0$. In this limit, the averaged Lagrangian is given by Eqs. (11) and (13). The general stability problem consists of linearizing about the fixed point (22), which results in an eigenvalue problem for a full (4×4) matrix, which is difficult to solve analytically. However, the azimuthally unstable modes can be determined from the special class of perturbations around the fixed point with $a=a_0+a_1$, $w=w_0+w_1$, $g=g_1$, and $\sigma=\sigma_0(z)$, with $|a_1| \ll a_0$, $|w_1| \ll w_0$, and $|g_1| \ll 1$. Variations of the phase σ with θ play no role in the dominant azimuthal instability. In this case, since σ_0' is constant, the linearized equations can be reduced to three equations for a_1, w_1, and g_1.

The leading-order mass conservation equation, obtained from variations with respect to σ, gives a relation between a_1 and w_1 of the form

$$a_1 = -\frac{2a_0}{w_0}w_1. \tag{24}$$

Using this relation, together with the fact that $\sigma_0=\sigma_0(z)$, we obtain the equations for the conjugate variables w and g linearized about the fixed point as

$$8a_0 w_0^2 \frac{\partial w_1}{\partial z} = \left(\frac{2\Lambda_1}{w_0^2} + \Lambda_2\right) g_1 - 2\Lambda_2 \frac{\partial^2 g_1}{\partial \theta^2} - 3a_0^2 w_0^4 g_1,$$

$$16 a_0 w_0^3 \frac{\partial g_1}{\partial z} = 3a_0^2 w_0 \frac{\partial^2 w_1}{\partial \theta^2} + 4a_0 w_0 \frac{\partial g_1}{\partial \theta}. \tag{25}$$

We then seek solutions of this system in the form

$$w_1 = W(z)e^{il\theta} \quad \text{and } g_1 = G(z)e^{il\theta} \tag{26}$$

and determine stability from the resulting system of ordinary differential equations. These eigenvalues are real provided that

$$\Delta_l(l) = l^2\left[\frac{9}{4}a_0^2 w_0^4 - 1 - 3\Lambda_1 w_0^{-2} - 3\Lambda_2(l^2+1)\right] \tag{27}$$

is positive. The maximum growth rate is determined by the root of $\Delta'(l)=0$.

The only quantity left to determine is the width R of the radiation shelf. In previous studies [8–10] this value was determined by the requirement that the frequency of oscillation of the modulation equations linearized about the fixed point matched the oscillation frquency of the pulse σ', a result which was confirmed by Yang [11]. However, for the local vortex, when $l=0$, the eigenvalues of the linearized equations are 0, so that there is no oscillation. To obtain an estimate of the shelf width, we note from numerical solutions presented in Kath and Smyth [9] that the width of the radiation shelf is of the same order as the width of the pulse. Hence an estimate of the width of the shelf is $R=w_0$. Using this value gives the most unstable azimuthal mode as $l=1.8$, which since l must be an integer becomes $l=2$, is agreement with the numerical results [4]. In this regard we note from Eq. (22) that $a_0^2 w_0^4$ is constant, so that the instability does not depend on the vortex parameters. Furthermore, the discriminant (27) becomes negative at $l=4$ (rounded to an integer), which means that the symmetry breaking unstable modes are $l=1,2,3$, again in agreement with the numerical results [4]. These instability results hinge on the choice of the shelf length R. The instability modes are, however, only weakly dependent on R. To have the most unstable mode as $l=1$ would require $R=1.23$, which is far too wide, being nearly twice the vortex width, which for the present case is $w_0=0.7312$ from numerical solutions [4]. On the other hand, to have the most unstable mode as $l=3,4,5$ would require $R=0.323, 0.192, 0.126$, respectively, which are far too small compared with the vortex width. The conclusion that the symmetry breaking unstable modes are $l=1,2,3$, with the most unstable mode at $l=2$, is then robust with respect to reasonable choices for the shelf width.

We now turn to the strongly nonlocal limit $\nu \to \infty$. We note from the fixed point relation (23) that for fixed a_0, $w_0 \to \infty$ as $\nu \to \infty$. It is then clear from the averaged Lagrangian (11) that in this limit the variational equation for σ decouples. Since the mass conservation equation does not depend on the nonlinear, nonlocal term, we can eliminate a_1 in the linearized equations in terms of w_1. The resulting linearized equations are the same as before, Eqs. (25), with the term involving g_1 now replaced by the nonlocal term, giving

$$8a_0w_0^2\frac{\partial w_1}{\partial z}=(2\Lambda_1 w_0^{-2}+\Lambda_2)g_1-2\Lambda_2\frac{\partial^2 g_1}{\partial\theta^2}-\frac{4a_0^2w_0^6}{\nu}\int_0^{2\pi}\sum_{l=-\infty}^{\infty}A_l e^{il(\theta-\theta')}g(\theta',z)d\theta',$$

$$16a_0w_0^3\frac{\partial g_1}{\partial z}=3a_0^2w_0\frac{\partial^2 w_1}{\partial\theta^2}+4a_0w_0\frac{\partial g_1}{\partial\theta}. \quad (28)$$

In a similar manner to the local limit $\nu\to 0$, the study of the stability of the vortex reduces to an eigenvalue problem with the eigenvalues given by the roots of a quadratic equation. The discriminant of this quadratic equation is

$$\Delta_{\nu l}(l)=\frac{405e^{-2}}{64|l|\nu}a_0^2w_0^6-1-3\Lambda_1w_0^{-2}-3\Lambda_2(l^2+1). \quad (29)$$

In contrast to the discriminant (27) in the local limit, it can be seen that as a consequence of the nonlocal fixed point relation (23) the numerator of the destabilizing term in $\Delta_{\nu l}$ scales, for fixed amplitude a_0, as $\nu^{3/4}$, while the denominator of this term scales as ν, resulting in stability as ν increases. It is this change of scaling from the local limit which is responsible for the stabilization of the vortex as ν increases.

The stability boundary given by the discriminant (29) can be compared with the numerical results of Yakimenko *et al.* [4]. Let us denote by A the amplitude of the steady vortex numerically determined in Ref. [4]. Then in the present scaling of Eqs. (1), we obtain

$$w_0=\frac{\sqrt{48}}{Ae\nu^{1/4}}. \quad (30)$$

Finally, using the large ν fixed point relation (23), the discriminant (29) for $l=2$ becomes

$$\Delta_{\nu l}(2)=\frac{7290e^{-4}}{A^2\nu}-1-3\Lambda_1w_0^{-2}-15\Lambda_2. \quad (31)$$

The results of Yakimenko *et al.* [4] used $A=0.3$. Then $\Delta_{\nu l}(2)\leqslant 0$ for $\nu\geqslant 100.5$, which is in very good agreement with the value $\nu\approx 90.7$ determined numerically from the linear eigenvalue problem above which the vortex becomes stable [12].

In Figs. 1(a)–1(d) examples of vortex evolution in weakly local [Figs. 1(a) and 1(b)] and strongly nonlocal [Figs. 1(c) and 1(d)] media are shown. Similar simulations were performed for a range of values of the nonlocality parameter ν with the initial condition (8). It was observed that as soon as the parameter ν exceeded the critical value ν_{cr}, the vortex became stable, as for the case illustrated in Figs. 1(c) and 1(d), with its profile reshaping as it evolved.

In a similar manner the stability of vortices of general charge m can be analyzed. For such a general charge vortex, the appropriate trial function is

$$E(r,\theta,z)=ar^me^{-r/w}e^{i\sigma+im\theta}+ig(r,\theta,z)e^{i\sigma+im\theta}. \quad (32)$$

The calculation of the modulation equations for this trial function proceeds exactly as for the $m=1$ case discussed above, so that only the final results will be given, without any details of the derivation. In the local limit $\nu\to 0$, it is found that the fixed point relation for a steady vortex is

$$a_0^2w_0^{2m+2}=\frac{(2m+1)!}{(4m+1)!}2^{6m+2} \quad (33)$$

and the vortex is unstable if the discriminant

$$\Delta_l(l)=l^2\left[\frac{(m+2)(2m-1)![(2m+1)!]^2}{(4m+1)!}2^{2m+3}-(m!)^2-\frac{(m+2)(2m-1)!}{2^{2m-1}}[2\Lambda_1w_0^{-2}+2\Lambda_2(l^2+m^2)]\right] \quad (34)$$

is positive.

The linear stability analysis of Yakimenko *et al.* [4] indicates that for multicharge vortices with $m\geqslant 2$ all azimuthal wave numbers l lead to unstable modes. This general result is not conclusive however as the stability eigenvalue problem was not solved for large azimuthal wave numbers l. The stability boundary (34) gives that, for any charge m, the azimuthal mode l will be stable if l is sufficiently large. However, the mode l at which the stability changes grows as $m!$ with increasing m. So while the results of the present modulation approximation are not in total agreement with numerical stability results for $m\geqslant 2$, the present analysis does show that the number of unstable azimuthal modes grows rapidly as m increases.

V. CONCLUSIONS

We have developed a simple modulation theory to analyze the dynamics and stability of vortex solitons in nonlocal, nonlinear media. We have shown analytically, for the first time to our knowledge, how the nonlocality of the medium's response and amplitude enhancement can stabilize vortices which are azimuthally unstable in almost all local nonlinear media. We have demonstrated that this analytical approach, which includes azimuthal dependence of both the vortex width and the diffractive radiation shelf, provides very good agreement with the numerical data of Yakimenko *et al.* [4].

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