Mediated homogenization

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Homogenization protocols model the quantum mechanical evolution of a system to a fixed state independently from its initial configuration by repeatedly coupling it with a collection of identical ancillas. Here we analyze these protocols within the formalism of "relaxing" channels providing an easy-to-check sufficient condition for homogenization. In this context we describe mediated homogenization schemes where a network of connected qudits relaxes to a fixed state by only partially interacting with a bath. We also study configurations which allow us to introduce entanglement among the elements of the network. Finally we analyze the effect of having competitive configurations with two different baths and we prove the convergence to dynamical equilibrium for Heisenberg chains.

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I. INTRODUCTION

Homogenization protocols have been extensively studied in recent years as a powerful model for the equilibration of a quantum mechanical system interacting with a large bath [1–6]. In these schemes one considers a collisionlike coupling of the system with a collection of ancillas that have been prepared in identical states. This corresponds to a Markovian approximation in a discrete dynamical evolution. Compared to typical quantum Markov equations the advantage of this model is that it allows one to concentrate on the effective unitaries and completely positive maps rather than the underlying Hamiltonians and Lindblad generators [7]. By "homogenization" one means that the system converges to a state that is the same as the ancilla states. This was demonstrated for a class of qudit systems in [1–3]. The bath-system entanglement was studied in [4], and a continuous-time model (quantum master equation) was derived from the discrete model in [5]. Furthermore, the emergence of irreversibility was investigated in [6].

An important aspect of quantum homogenization is that is a stable method of driving a system into some fixed state, independent of its initial conditions. In this context, we will also refer to the bath as a "controller" system and its state as a "controller state." Hence apart from its fundamental role of studying quantum convergence, homogenization has possible applications for quantum cloning [2], for the hiding of quantum information [2,8] and spin chain quantum communication [9].

The prototypical homogenization scenario [1,2] is described in Fig. 1. It is composed of two parts: a system A with an always on Hamiltonian H_A, and a large ensemble of identical controller systems $B_1, B_2, \ldots, B_n$. The latter are prepared in the same initial state ω_B and are assumed to have no independent free evolution. The system A is coupled in sequential order with each one of the Bs through a series of identical stepwise interactions described by the Hamiltonian H_I. In this setting the evolution of the system A is described by the successive application of the completely positive (CP) map

$$\mathcal{E}(\rho_A) \equiv \mathrm{Tr}_B[U(\rho_A \otimes \omega_B)U^\dagger], \tag{1}$$

with $U \equiv \exp[-i(H_A + H_I)t]$ and $t>0$ being the time interval associated with a single A-B coupling. After the interaction with n controllers the state of A becomes

$$\rho_A^{(n)} = \underbrace{\mathcal{E} \circ \mathcal{E} \circ \cdots \circ \mathcal{E}}_{n \text{ times}}(\rho_A) \equiv \mathcal{E}^n(\rho_A). \tag{2}$$

We are interested in the behavior of the sequence (2) for large n: in the case where the system A and the controllers B_k are identical, $H_A=0$ and H_I is a swap Hamiltonian, it was shown [1–3] that the state of A asymptotically converges to the state ω_B of the controllers, independently from the initial state ρ_A.

In the above, homogenization also gives rise to thermalization [10]—if the bath is initialized in Gibbs states, then the system converges to a Gibbs state. However, it is an open question if this still holds in a situation where system and bath particles have different dimensionality [6]. Moreover, in [1–6] the bath is modeled to interact with the *whole* system whereas an interaction with a subsystem (such as the *surface*) seems more plausible.

A first generalization toward this direction was observed by the authors of the present paper when studying the propagation of quantum information along spin chain communication channels [9]. In that case A represented a collection of N coupled qubits, while the Hamiltonians H_I implemented a sequence of strong instantaneous swaps among the last ele-

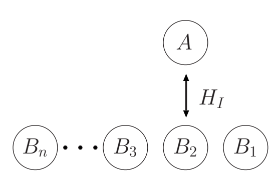

FIG. 1. Setup of a standard homogenization protocol: the controlled system A interacts with a collection of controller systems B which have been initialized into the same input state ω_B. Homogenization takes place when the final state of A is driven into the same state ω_A of the controllers in the limit of infinitely many couplings with the B's.

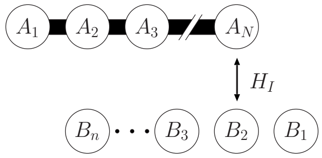

FIG. 2. Generalized homogenization protocol: in this case the controlled system A is a composite one (e.g., a network of coupled spins). Only a proper subset of the network interacts directly with the controllers B (as in the case of Fig. 1, the B_k are assumed to be prepared in the same initial states).

ment of the chain and a collection of controller qubits (the B's). By assuming the B's to be prepared into the spin-down state $|0\rangle_B$ we showed that in the limit of large n, any initial state of A will be coherently transferred into the B's while the chain will be mapped into the all spin-down configuration $|00\cdots 0\rangle$ (the only requirement being a nontrivial connection among the qubits of A). Concerning Ref. [9] it is worth stressing that in contrast to Refs. [1–6] A and the controllers are quite distinct objects (namely A is a network of coupled qubits while each of the $B_1\cdots B_n$ is just a single qubit). Moreover only a proper subset of A (specifically the Nth element of the network) interacts directly with B: the remaining qubits are only affected by the controllers through the free Hamiltonian H_A of the network (see Fig. 2). The possibility of preparing the ancilla state $|0\rangle_B$ into each one of the spins of the network is therefore a remarkable feature of the system which requires some further investigation. We call it *mediated homogenization process*.

In this paper we tackle this issue by analyzing spin networks which show similar properties. In particular in Sec. III we present a first example of a mediated homogenization process which allows one to transfer on the network element *any* input state ω_B of the controllers. Before doing so however we introduce a general convergence criterion for relaxing channels in Sec. II that will turn out to be extremely useful in our discussion. Our generalized setup gives rise to much richer quantum convergence effect—in Sec. IV we give numerical evidence for equilibrium states which are entangled. Finally in Sec. V we study the effects of having competitive baths at different temperature.

II. MIXING CRITERIA IN THE HOMOGENIZATION SETUP

Our starting point is the observation that what makes the CP map in Eq. (2) converge into a specific state independently from the initial input ρ_A is a well-known property called *relaxing* [10] (also referred to as "mixing" [11,12] or "absorbing" [13]). In this language the convergence point

$$\rho_A^* \equiv \lim_{n\to\infty} \rho_A^{(n)}, \tag{3}$$

is the only *fixed point* of $\mathcal{E}$, i.e., the only solution of the equation

$$\mathcal{E}(\rho_A) = \rho_A, \tag{4}$$

(we refer the reader to Refs. [11,12] for a detailed introduction to relaxing channels). Therefore an homogenization procedure is associated with a relaxing channel whose fixed point ω_A coincides with the state of the controller.

We now prove a very simple but important result. Suppose that given U we know that the map (1) is relaxing for a specific choice of the controlled state ω_B. Consider now the map $\tilde{\mathcal{E}}$ which is obtained from Eq. (1) by replacing ω_B with a state $\tilde{\omega}_B$ which is a (nontrivial) convex combination of ω_B, i.e., $\tilde{\omega}_B = p\omega_B + (1-p)\omega_B'$, with $p\in]0,1]$ and ω_B' being a generic density matrix. In this case the map $\tilde{\mathcal{E}}$ can be expressed as a convex combination of $\mathcal{E}$, i.e.,

$$\tilde{\mathcal{E}} = p\mathcal{E} + (1-p)\mathcal{E}', \tag{5}$$

with the map $\mathcal{E}'$ as in Eq. (1) with ω_B replaced by ω_B'. We can then use a theorem by Haag [14] which shows that a convex combination of CP maps containing at least one relaxing channel is relaxing, to conclude that $\tilde{\mathcal{E}}$ is relaxing—for completeness, we provide an alternative (and much simpler) proof of this important theorem. It is based on the fact that the relaxing map is equivalent to the asymptotic deformation property [11]. Hence for all $\rho \neq \rho'$ there is a k such that

$$\|\mathcal{E}^k(\rho) - \mathcal{E}^k(\rho')\|_1 < \|\rho - \rho'\|_1, \tag{6}$$

where $\|\Theta\|_1 \equiv \mathrm{Tr}[\sqrt{\Theta^\dagger\Theta}]$ indicates the trace norm of the operator Θ. We write $\tilde{\mathcal{E}}^k = p^k\mathcal{E}^k + (1-p^k)\mathcal{S}'$, where $\mathcal{S}'$ is the CP map that contains all other terms of the expansion of $\tilde{\mathcal{E}}^k$. By the nonexpansiveness [15] of $\mathcal{S}'$ and the triangle inequality we obtain

$$\|\tilde{\mathcal{E}}^k(\rho) - \tilde{\mathcal{E}}^k(\rho')\|_1 < \|\rho - \rho'\|_1, \tag{7}$$

whence $\tilde{\mathcal{E}}$ is an asymptotic deformation. In the context of Fig. 2 this implies that the system still converges when substituting the controller state ω_B with $\tilde{\omega}_B$. For instance, if there exists *any* state ω_B for which the system converges, then it will converge to the fully mixed state if the B_k are initialized in the fully mixed state.

A natural question is then to determine the fixed point of $\tilde{\mathcal{E}}$. Specifically one may ask how the final state of the system A depends upon the controller state $\tilde{\omega}_B$. For instance, if homogenization takes place for ω_B, does it hold also for $\tilde{\omega}_B$? Or, how does the entropy of the fixed point depend on the entropy $S(\tilde{\omega}_B)$ of the controllers?

Before passing to apply the Haag criterion to the mediated homogenization scheme, it is worth presenting yet another interesting generalization of this simple but important theorem. Consider in fact the situation in which the states of the controllers $B_1,\ldots,B_n$ have not been properly initialized. In particular we are interested in studying what happens if instead of being prepared in the "good" initial state ω_B, the ℓth controller is described by the following imperfect state:

$$\bar{\omega}_B^{(\ell)} \equiv p_\ell\omega_B + (1-p_\ell)\varrho_B^{(\ell)}, \tag{8}$$

where for $\ell=1,\ldots,n$, $p_\ell>0$ are probabilities and $\varrho_B^{(\ell)}$ are density matrices. According to the analysis of Sec. I this yields a sequence $\mathcal{E}_1,\ldots,\mathcal{E}_n$ of CP maps which have the property that each of them is a convex combination of a fixed relaxing map $\mathcal{E}$, i.e.,

$$\mathcal{E}_\ell = p_\ell \mathcal{E} + (1-p_\ell)\mathcal{S}_\ell, \qquad (9)$$

where $\mathcal{E}$ and $\mathcal{S}_\ell$ are the channels (1) associated with ω_B and $\varrho_B^{(\ell)}$, respectively. Clearly without putting any restriction on the values of p_ℓ nothing can be said about the convergence property of the protocol. Therefore, we consider the case in which the "error" $(1-p_\ell)$ is bounded, by imposing the constraint

$$p_\ell \geq p > 0. \qquad (10)$$

This hypothesis does not yet guarantee that A will be driven toward ω_A. However we can at least verify that the process is still able to "forget" about the initial state of the controlled system as in the relaxing case (this is a typical feature of any homogenization protocol). The evolution of the controlled system A is in fact now described by the following sequence of concatenated maps:

$$\mathcal{M}_n = \mathcal{E}_n \circ \mathcal{E}_{n-1} \circ \cdots \circ \mathcal{E}_1. \qquad (11)$$

For arbitrary input density matrices ρ_A', ρ_A'' of the controlled system define

$$f_n = \|\mathcal{M}_n(\rho_A') - \mathcal{M}_n(\rho_A'')\|_1. \qquad (12)$$

Now since f_n is non-negative and nonincreasing [15] it certainly admits a limit $\lim_{n\to\infty} f_n \equiv f_*$. To show that the protocol forces the controlled system to forget about its initial conditions we need only to verify that this quantity is null for all ρ_A' and ρ_A''. Assume then by contradiction that $f_*>0$ for some choice of these input states. Let ρ_A^* be the fixed point of the unperturbed map $\mathcal{E}$. Then there is a value of k such that $\|\mathcal{E}^k(\rho_A)-\rho_A^*\|_1 < f_*/4$ for *all* ρ_A [10]. Let then $\delta = \frac{p^k f_*}{3(1-p^k)} > 0$. There is a n such that $f_n - f_* < \delta$. We write

$$\mathcal{M}_{k+n} = \widetilde{\mathcal{M}} \circ \mathcal{M}_n, \qquad (13)$$

where the superoperator $\widetilde{\mathcal{M}} = \mathcal{E}_{k+n} \circ \cdots \circ \mathcal{E}_{1+n}$ can be decomposed as

$$\widetilde{\mathcal{M}} = P_k \mathcal{E}^k + (1-P_k)\Gamma \qquad (14)$$

with Γ being CP and $P_k = p_{k+n}\cdots p_{1+n} \geq p^k$ by assumption. Hence

$$f_{k+n} = \|\widetilde{\mathcal{M}}[\mathcal{M}_n(\rho_A)] - \widetilde{\mathcal{M}}[\mathcal{M}_n(\rho_A')]\|_1 \leq P_k \|\mathcal{E}^k[\mathcal{M}_n(\rho_A)] - \mathcal{E}^k[\mathcal{M}_n(\rho_A')]\|_1 + (1-P_k)\|\Gamma[\mathcal{M}_n(\rho_A)] - \Gamma[\mathcal{M}_n(\rho_A)]\|_1 < P_k f_*/2$$
$$+ (1-P_k)(\delta + f_*) \leq P_k f_*/2 + (1-P_k)\left(\frac{P_k f_*}{3(1-P_k)} + f_*\right) = f_* - P_k f_*/6 < f_*.$$

Since f_n is nonincreasing this is a contradiction, and $f_*=0$. We have shown that the whole state space is contracted to a single point. In general, this point is still evolving under the action of $\mathcal{E}_n$, but contains no information about the initial state. The map $\mathcal{M}_n$ is relaxing if and only if there exists an *asymptotic* fixed point ϱ_A^*, i.e., a state with $\lim_{n\to\infty} \mathcal{M}_n(\varrho_A^*) = \varrho_A^*$.

III. MEDIATED HOMOGENIZATION IN SPIN NETWORKS

An interesting example of mediated homogenization is obtained by assuming A to be a network of N coupled qudits $A_1, \ldots, A_N$ mutually interacting through a sum of a local terms of the form

$$H_A = \sum_{k,k'} J_{kk'} S_{A_k A_{k'}}, \qquad (15)$$

where $J_{kk'}$ are coupling constants and where $S_{A_k A_{k'}} = (S_{A_k A_{k'}})^\dagger$ are unitary operators which swap the kth qudits of A with the k'th [16]. Regarding the coupling with the controller we consider the case in which only A_N interacts with the B's (also represented by d-dimensional systems) through a swap Hamiltonian similar to (15), i.e.,

$$H_I = S_{BA_N}. \qquad (16)$$

Under these conditions we can show that, for all choice of the controller states ω_B and for almost all choices of the interaction time t the map (1) is relaxing with fixed point

$$\rho_A^* = (\omega_A)^{\otimes N}, \qquad (17)$$

given that the graph associated with the coupling $J_{kk'}$ satisfies certain constraints. This corresponds to the case in which, in the limit of large n, the controller state ω_B is "copied" in all the N controlled qudits. We call this process a *mediated homogenization* of A. It fulfills all four homogenization criteria mentioned in Ref. [6]: First, the coupling between system and bath is independent of the bath state. Second, the equilibrium state is not only a fixed point of the CP map (1) but also of the unitary evolution $U \equiv \exp[-i(H_A + H_I)t]$ alone. Third, the system converges to the fixed point for all initial states. Finally the change of the bath due to the evolution can be made arbitrarily small by choosing a short interaction time t. An immediate consequence of the above result is the fact that the von Neumann entropy of A converges to N times the von Neumann entropy of the controller state $S_B = -\mathrm{Tr}[\rho_B \log_2 \rho_B]$. This is a distinctive trait of the mediated homogenization processes and it is similar to what happens when we put a system of interest in thermal contact with a reservoir. It should be pointed out though that in our case the convergence state is in general far away from any thermal state $\exp(-\beta H_A)/Z$. The thermalization feature observed in [1] thus seems to be specific to the case where A is a single qudit.

To prove the above result we first focus on the case in which ω_B is a pure vector $|\phi\rangle_B$. Define then the joint observable

$$M_{AB} = M_B + M_A, \tag{18}$$

where $M_A = \sum_{k=1}^{N} M_{A_k}$ and

$$M_B \equiv -|\phi\rangle_B\langle\phi|,$$

$$M_{A_k} \equiv -|\phi\rangle_{A_k}\langle\phi|. \tag{19}$$

The operator M_{AB} commutes with the total Hamiltonian $H = H_A + H_I$ and hence with the operator $U = \exp(-iHt)$: we thus say that free evolution of the network preserves the "excitations" associated with the projectors $|\phi\rangle\langle\phi|$.

Moreover, the state $|\phi\rangle_B$ is the (nondegenerate) eigenvector associated with the minimum eigenvalue of M_B. Under these conditions we can invoke the Lemma 3 of Ref. [11] which states that the map (1) is relaxing with fixed point $|\phi\rangle^{\otimes N} = |\phi\rangle_{A_1} \otimes \cdots \otimes |\phi\rangle_{A_N}$ if the state $|\phi\rangle^{\otimes N} \otimes |\phi\rangle_B$ is the unique eigenvector of U having the form $|E\rangle_A \otimes |\phi\rangle_B$. This last condition can be verified by focusing on the global Hamiltonian H: if indeed $|\phi\rangle^{\otimes N} \otimes |\phi\rangle_B$ is the unique eigenvector of the global Hamiltonian H with the factorization property $|E\rangle_A \otimes |\phi\rangle_B$ then the same property will hold for U for almost all the values of t. We have shown elsewhere [17] that for "excitation" preserving Hamiltonians the above factorization condition depends only on the geometry of the associated graph. For example, an open chain with A_N being an end qudit has the required property. We can apply this result to the Hamiltonian (15). Therefore for any given network configuration satisfying the topological constraint of Ref. [17] we can conclude that, for all initial pure states $|\phi\rangle$ of the controller, the above iterative procedure will drive A to a unique fixed point. Before determining such fixed point, let us first observe that the same convergence will hold also when assuming the initial states of the controllers to be a general mixed state ω_B. This is a trivial consequence of the pure case scenario which can be obtained by expanding any such mixture into a convolution of pure states $\omega_B = \sum_j q_j |\phi_j\rangle_B\langle\phi_j|$ and applying the Haag criterion.

Since we have now proved that for all choices of the controller state ω_B the channel $\mathcal{E}$ is relaxing, to verify Eq. (17), it is sufficient to show that $(\omega_A)^{\otimes N}$ satisfies Eq. (4). The latter can be easily verified by noticing that each summand of the Hamiltonian (15) commutes with all tensor product operators of the form $\Theta^{\otimes N}$, and therefore

$$[H_A, \Theta^{\otimes N}] = 0. \tag{20}$$

Consequently for ρ_A^* as in Eq. (17) and H_I as in Eq. (16) we can write

$$[H_A + H_I, \rho_A^* \otimes \rho_B] = 0 \Rightarrow [U, \rho_A^* \otimes \rho_B] = 0, \tag{21}$$

which is sufficient to show that ρ_A^* satisfies the invariance condition (4).

IV. BUILDING ENTANGLEMENT IN THE NETWORK

In the preceding section we found a model where independently from the initial state of A, the final state of the network is the *separable* state $\omega_A^{\otimes N}$. Each of the N qudits of the network has been driven into the initial state of the controllers. In this section we show that, keeping H_I as in Eq. (16), there are also Hamiltonians H_A which are capable of building entanglement among the qudits of the network. Although this is no longer a homogenization protocol (the controller state is not transferred to the controlled system) it could have useful applications as a method of state preparation.

Consider for the sake of simplicity $d=2$. In this case the swap interaction of Eq. (15) corresponds (up to a constant) to a Heisenberg coupling. A natural generalization of it is then provided by the anisotropic Heisenberg Hamiltonian

$$H_A = \sum_{k,k'} \frac{J_{kk'}}{2} (\sigma_k^{(x)}\sigma_{k'}^{(x)} + \sigma_k^{(y)}\sigma_{k'}^{(y)} + \Delta \sigma_k^{(z)}\sigma_{k'}^{(z)}), \tag{22}$$

where $\sigma_k^{(x,y,z)}$ represents the Pauli matrix of the kth qubit of A and where $\Delta - 1$ is the anisotropy parameter (the isotropic coupling is obtained for $\Delta = 1$). For this coupling we can use the same argument given in the preceding section to characterize the relaxing properties of the associated map (1) (in particular the factorization property of its eigenvectors depends only on the geometry of the associated graph [17]). In this case however the isotropy is lost and the Hamiltonian has a preferred spatial direction associated with the $\hat{z}$ axis which makes $|0\rangle_B$ and $|1\rangle_B$ special with respect to the other controller pure states. Indeed we can still show that mediated homogenization takes place for input states ω_B which are diagonal in the computational basis, i.e.,

$$\omega_B = p|0\rangle_B\langle 0| + (1-p)|1\rangle_B\langle 1|. \tag{23}$$

This follows by the fact that for such a choice Eq. (21) holds independently from the value of Δ. On the contrary for more general controller states mediated homogenization is lost. As an example, consider

$$\omega_B = p|0\rangle_B\langle 0| + (1-p)|-\rangle_B\langle -|, \tag{24}$$

where $|-\rangle_B \equiv (|0\rangle_B - |1\rangle_B)/\sqrt{2}$ and $p > 0$. In this case Haag's theorem can still be used to ensure relaxing of the map (1) even though computing the fixed point is not simple. For such choice however we have numerically verified that the mediated homogenization does not take place in general. We evaluated the asymptotic limit of the von Neumann entropy of ρ_A^*, verifying that it is no longer a multiple of the von Neumann entropy of the bath state. In Fig. 3 we show an example for a XX chain ($\Delta = 0$).

Of particular interest is the case $p=0$. In this limit $\omega_B = |-\rangle_B\langle -|$ and the relaxing property cannot be established from the theorem by Haag (simply ρ_B is not a convex combination of $|0\rangle_B\langle 0|$). Nevertheless we can use numerical analysis to show that the map (1) is still relaxing [18]. We found that the convergence point is highly mixed. Since this is so much different from the isotropic Heisenberg model with fixed point $|-\rangle^{\otimes N}$ it seemed natural to compute the re-

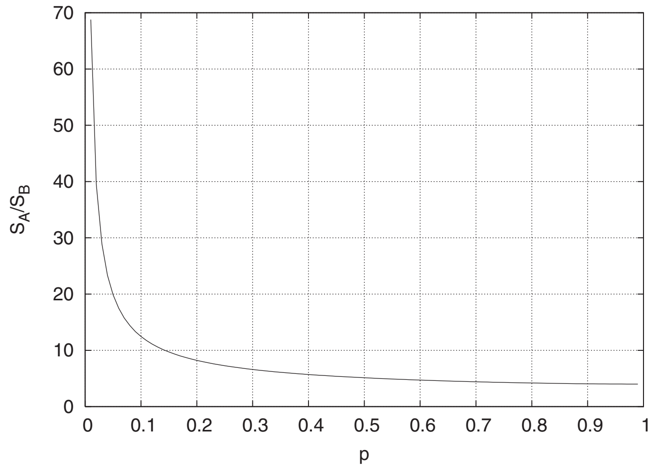

FIG. 3. Ratio R of the "output" entropy S_A of the convergence point and the "input" entropy S_B of the bath state for a XX chain of length $N=4$. The controller state is as in Eq. (24). For all values of p the dynamics is relaxing—for $p>0$ this is a trivial consequence of Haag theorem, for $p=0$ instead it can be directly proved by numerical means [18]. When $p\rightarrow 1$ the state becomes diagonal and the ratio converges to N. For $p\rightarrow 0$ the ratio diverges as the bath state becomes pure but the convergence point remains mixed. This should be compared with the behavior of a mediated homogenization process (e.g., the swap coupling of Sec. III) where R is always equal to N for all p. The parameters for the computation are $J_{kk'}=\delta_{k,k+1}$ and $t=0.5$.

laxing property and convergence point of the anisotropic model for $p=0$ as a function of Δ to see the transition for a XX chain (with highly mixed convergence point) to the Heisenberg chain (pure convergence point). In particular we wanted to check if there are also entangled fixed points. For this purpose we computed the concurrence between the first and second qubit of the chain (say) for intermediate Δ (see Fig. 4). Again, for the given parameters, all examples were relaxing. We found that the convergence point is indeed entangled for some values of Δ. Contrary to the results in the preceding section, the numerical examples of convergence points observed here depend on the parameters of the model. An important open problem is to determine if there exist H_A and ω_B that have a fixed point with interesting applications (e.g., a cluster state).

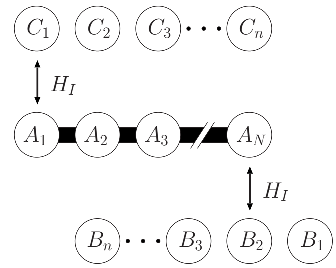

FIG. 5. Setup of the dynamical equilibrium: here the system of Fig. 2 is coupled to two competing baths at different temperatures.

V. DYNAMICAL EQUILIBRIUM

The many-body structure of A presented in Fig. 2 allows us to consider more complicated procedures. For instance, we can analyze *competitive* configurations where the dynamics of the network A is driven by the simultaneous coupling with two independent sets of controllers (the $B_1,\ldots,B_n$ and the $C_1,\ldots,C_n$ of Fig. 5). We can then model the "transport" of excitations through the network by assuming the two sets to be directly coupled with distinct network elements (say A_N for B and A_1 for C) and assuming different "temperature" for the two species of controllers [say $\omega B=p|0\rangle B\langle 0|+(1-p)|1\rangle B\langle 1|$ for the Bs and $\nu_C=q|0\rangle_C\langle 0|+(1-q)|1\rangle_C\langle 1|$ for the Cs]. A similar situation is considered in [19–21] where in the case of a linear chain coupled through Heisenberg and XX interactions the relaxing property was observed *numerically* [22]. Here, the convergence can be derived analytically

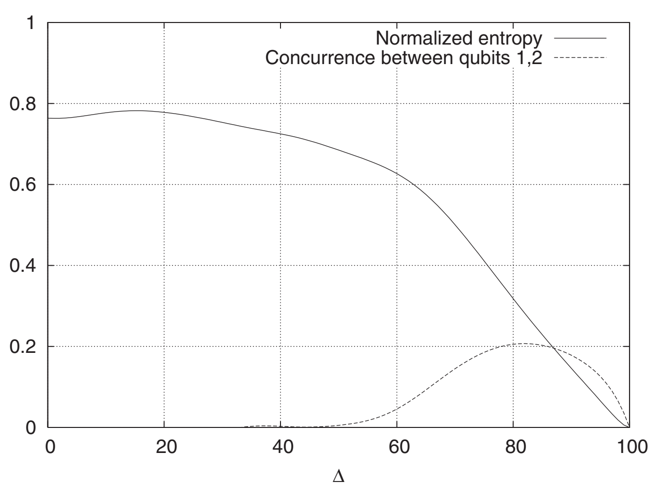

FIG. 4. The entropy S_A of the convergence point and the concurrence between the first two qubits for an anisotropic Heisenberg chain of length $N=4$ as a function of the anisotropy parameter Δ. The bath state is given by $(|0\rangle_B-|1\rangle_B)/\sqrt{2}$. The parameters for the numerics are $J_{kk'}=\delta_{k,k+1}$ and $t=0.5$.

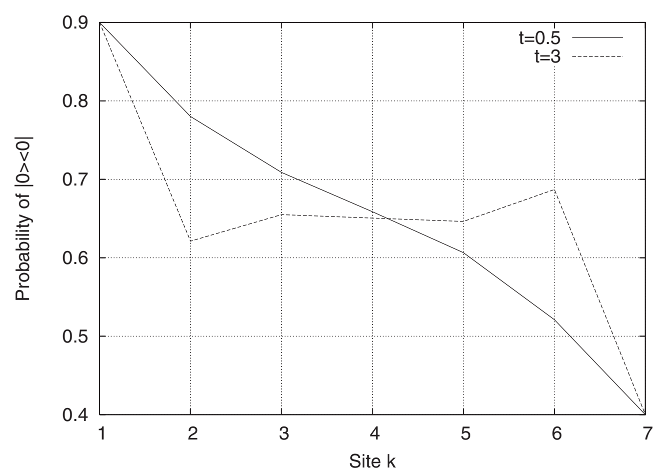

FIG. 6. Probability of finding the kth qubit in the state $|0\rangle\langle 0|$ for the steady state of a Heisenberg spin chain of length $N=5$. The plot includes the systems B (site 1) and C (site 7).The parameters $p=0.9$ and $q=0.4$. We give two examples with different choice of the interaction time t.

for arbitrary chain length as a consequence of the Haag theorem. To verify this it is convenient to treat B and C as a unique controller composed by elements $B_1C_1, B_2C_2, \ldots, B_nC_n$. From the above definitions it then follows that such composite controllers are initialized in the state

$$\omega_B \otimes \nu_C = pq|0\rangle_B\langle 0| \otimes |0\rangle_C\langle 0| + (1-pq)\varrho_{BC}, \qquad (25)$$

with ϱ_{BC} being a density matrix. Therefore, according to the Haag theorem the convergence can be verified by focusing only on the case in which B and C are initialized in $|0\rangle_B\langle 0| \otimes |0\rangle_C\langle 0|$. With this choice however the iterative procedure is equivalent to the "cooling" protocol discussed in Ref. [17] and the convergence is automatically verified. Deriving the exact steady state in this case is however quite complicated so we restrict to numerical analysis. Again its form depends strongly on the parameters, as shown in Fig. 6.

VI. CONCLUSION

We have generalized the homogenization protocols to a scenario where the system is no longer a single qudit. We found that mediated homogenization still takes place on the lattice when the interaction is taken to be isotropic. Anisotropic interactions, on the other hand, do not in general show homogenization. Our numerical results are quite suggestive in this direction but are certainly not conclusive. This suggests many further studies: what is the structure of the fixed points of these systems? How are their entropies related to the bath entropy? What happens when the system is close to a critical point? Can we use this convergence as a way of preparing *useful* states such as cluster states on optical lattices? Finally we looked at transport along chains interconnecting baths at different temperature, where the Haag criterion allowed us to prove the convergence to a dynamical equilibrium. We found that the temperature profile is strongly depending on the parameters of the system, such as the interaction time, and not even monotonic for some times. While at the moment these results are numerical only, it may be possible to obtain an analytic expression for the fixed point in a weak coupling limit by deriving a closed equation for the proper ansatz (cf. [20]). This will be the subject of future investigations.

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