Transverse spatial entanglement in parametric down-conversion

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It is known that the entanglement present in the transverse spatial properties of photon pairs generated by spontaneous parametric down-conversion depends explicitly on the characteristics of the pump laser beam. Here we show that, for pump beams described by a Hermite-Gaussian mode, the entanglement depends on the beam width as well as the order of the beam. The amount of entanglement generally increases with the order of the beam and is less dependent on the beam width for higher orders. We propose a method to measure the spatial entanglement directly, independent of the dimension of the subsystems.

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I. INTRODUCTION

Entangled photon pairs have played a crucial role in recent advances in quantum information and communication, and are a valuable tool for our understanding of entanglement in general. Spontaneous parametric down-conversion (SPDC) is a robust technique for the probabilistic generation of multiphoton entangled states through the nonlinear interaction of a classical pump field with a nonlinear birefringent crystal. To explore entanglement, there are a number of degrees of freedom to choose from, including momentum [1], time-bins [2], polarization [3,4], orbital angular momentum [5], Hermite-Gaussian modes [6,7], and transverse position-momentum [8,9]. Though the majority of work is performed with entangled two-dimensional qubits, it is possible to define higher-dimensional qudits in one or more degrees of freedom [8–11]. It has been shown that higher dimensional systems display certain advantages, including more secure cryptography systems [12–14], greater and more noise-resistant violation of Bell's inequalities [15], and decoherence-free encoding [16].

There has been some recent theoretical [17–23] and experimental work [5,24–30] concerning the presence, detection, and engineering of transverse spatial entanglement in the two-photon state produced by SPDC. It has been known for some time that the transverse spatial profile of the pump beam is transferred to the two-photon state [31,32], and thus the spatial characteristics of the pump field determine the entanglement content of the two-photon state [17–19]. In this paper, we consider Hermite-Gaussian (HG) pump fields, and show that the amount of entanglement depends upon the order $n+m$ of the pump field, and generally increases as $n+m$ increases. In addition, we show that the entanglement depends less on the width of the pump beam as the order of the beam increases. The entanglement content of the spatial degrees of freedom of the two-photon state is calculated using the I-concurrence [33] through the Hermite-Gaussian mode decomposition derived in Ref. [6]. Finally, we propose a method to measure the I-concurrence directly using two copies of the entangled state.

II. QUANTUM STATE

A. State generated by SPDC

Let us briefly discuss the relevant approximations used in obtaining the two-photon quantum state produced by SPDC. Here we will be interested primarily in transverse spatial correlations, and for this reason we will assume that the pump beam and down-converted fields have well-defined polarizations. Assuming paraxial fields, it is straightforward to include the polarization degree of freedom. We also find it convenient to work in the monochromatic approximation, which is further justified experimentally through the use of narrow-band interference filters in the down-converted beams. For a continuous-wave pump laser, the pump field is nearly monochromatic, while for a pulsed pump beam this is not the case. The two-photon state in the case of a pulsed pump laser has been considered in [35]. The advantage to working with monochromatic fields is that the spectral and spatial components of the two-photon state are separable. However, this condition is approximately true for pulsed fields as long as the bandwidths of the fields are narrow ($\Delta\omega \ll \omega$) [34], which is valid even for pulses of a few hundred femtoseconds. Thus, our conclusions concerning the spatial entanglement should be applicable under these conditions, provided that the nonlinear crystal is thin enough, and the pump pulse narrow enough in the frequency domain so that anisotropic effects can be ignored. Let us further assume that the down-converted fields are degenerate, with wavelength twice that of the pump field: $\lambda = 2\lambda_p$. We will also restrict our analysis to situations in which the paraxial approximation is valid, which is nearly always the case. If the pump field is weak enough so that the probability to generate two or more pairs of photons simultaneously is negligible, the quantum state generated by SPDC at the face of the crystal is [31,32,36]

$$|\psi\rangle_{12} = A_1|\text{vac}\rangle + A_2|\psi\rangle, \tag{1}$$

where $|\psi\rangle$ is the two-photon component. Here the coefficients A_1 and A_2 are such that $|A_2| \ll |A_1|$. A_2 depends on the

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type and length of the nonlinear crystal, as well as the intensity of the pump beam. The two-photon component is given by

$$|\psi\rangle = \int\int d\mathbf{q}_1 d\mathbf{q}_2 \Phi(\mathbf{q}_1,\mathbf{q}_2)|\mathbf{q}_1\rangle_1|\mathbf{q}_2\rangle_2, \tag{2}$$

where the kets $|\mathbf{q}_j\rangle$ represent single-photon states in a plane wave mode. The two-dimensional vector $\mathbf{q}_j$ is the transverse component of the wave vector $\mathbf{k}_j$. In the paraxial approximation considered here, $|\mathbf{q}| \ll |\mathbf{k}|$. The normalized function $\Phi(\mathbf{q}_1,\mathbf{q}_2)$ is defined as [32]

$$\Phi(\mathbf{q}_1,\mathbf{q}_2) = \frac{1}{\pi}\sqrt{\frac{2L}{K}} v(\mathbf{q}_1+\mathbf{q}_2)\gamma(\mathbf{q}_1-\mathbf{q}_2), \tag{3}$$

where $v(\mathbf{q})$ is the normalized angular spectrum of the pump beam and $\gamma(\mathbf{q})=\mathrm{sinc}(L|\mathbf{q}|^2/4K)$ is the phase matching function. Here L is the length of the nonlinear crystal in the propagation (z) direction and K is the magnitude of the pump field wave vector. Equations. (2) and (3) show the transfer of the angular spectrum of the pump field to the two-photon state produced by SPDC, as investigated in Refs. [31,32]. As $\Phi(\mathbf{q}_1,\mathbf{q}_2)$ is generally a nonseparable function of $\mathbf{q}_1$ and $\mathbf{q}_2$, this effect is responsible for the entanglement in transverse spatial degrees of freedom of the SPDC photons.

Here we will assume that the nonlinear crystal is thin enough and the pump beam width is large enough so that we may ignore any effects which arise from the birefringence [29,37], including astigmatism and walk-off. For simplicity, we assume that $\Phi(\mathbf{q}_1,\mathbf{q}_2)$ is polarization independent, which is also valid for thin crystals.

The two-photon state (2) is written in terms of continuous plane-wave modes. However, there are a number of discrete bases which one can use to express state (2). In Ref. [6] and below, we employ the Hermite-Gaussian (HG) modes.

B. HG mode expansion of the two-photon state

We will be interested in how the structure of the pump field affects the entanglement present in the two-photon state $|\psi\rangle$. Let us assume that the pump field is given by a Hermite-Gaussian mode, characterized by the angular spectrum $v_{nm}(\mathbf{q})$ given by

$$v_{nm}(\mathbf{q}) = w_0 D_{nm} H_n\left(\frac{w_0 q_x}{\sqrt{2}}\right) H_m\left(\frac{w_0 q_y}{\sqrt{2}}\right)\exp\left(-\frac{w_0^2|\mathbf{q}|^2}{4}\right), \tag{4}$$

where the coefficients D_{nm} are defined as

$$D_{nm} = i^{n+m}[2^{(n+m+1)}\pi n!m!]^{-1/2}, \tag{5}$$

$H_n(x)$ is the nth-order Hermite polynomial [39] and w_0 is the beam waist (assumed to be on the $z=0$ plane). The order of the beam is defined as the sum of indices $m+n$.

In Ref. [6], it was shown that the two-photon quantum state can be written in terms of Hermite-Gaussian modes as

$$|\psi_{nm}\rangle = \sum_{j,k,s,t=0}^{\infty} C_{jkst}^{nm}|v_{jk}\rangle|v_{st}\rangle, \tag{6}$$

where

$$|v_{\alpha\beta}\rangle = \int d\mathbf{q}\, v_{\alpha\beta}(\mathbf{q})|\mathbf{q}\rangle \tag{7}$$

represents a single photon in a HG mode. We note here that $j(k)$ and $s(t)$ are the $x(y)$ indices of the down-converted fields 1 and 2. The coefficients C_{jkst}^{nm} can be calculated with

$$C_{jkst} = \langle v_{st}|\langle v_{jk}|\psi\rangle. \tag{8}$$

In Ref. [6], it was shown that these coefficients are of the form

$$C_{jkst}^{nm} = \sqrt{\frac{\alpha!\beta!}{A\pi}}\left(\frac{1}{2}\right)^{(\alpha+\beta)/2}\frac{\arctan A}{(\alpha/2)!(\beta/2)!}b(j,s,\alpha)b(k,t,\beta) \times \sum_{r=0}^{(\alpha+\beta)/2}\begin{pmatrix}\frac{\alpha+\beta}{2}\\ r\end{pmatrix}\left(\frac{-2}{\sqrt{1+A^2}}\right)^r \mathrm{sinc}(r\arctan A) \tag{9}$$

if $j+s \geqslant n$ and $k+t \geqslant m$, or else $C_{jkst}^{nm}=0$. Here $b(j,s,\alpha)$ is given by [38]

$$b(n,m,k) = \sqrt{\frac{(n+m-k)!k!}{2^{(n+m)}n!m!}}\frac{1}{k!}\frac{d^k}{dt^k}[(1-t)^n(1+t)^m]|_{t=0}, \tag{10}$$

and we have defined $A=L/(Kw_p^2)$, $N=j+s$, $M=k+t$, $\alpha=N-n$, and $\beta=M-m$, and w_p is the waist of the pump beam. The parameter A is just the ratio between the crystal length L and the depth of focus $d_f=2z_0=Kw_p^2$ of the pump beam. Alternatively, A can be seen as the square of the ratio between the width $2/w_p$ of the pump beam angular spectrum $v(\mathbf{q})$ and the width $2\sqrt{K/L}$ of the phase matching function $\gamma(\mathbf{q})$.

For thin nonlinear crystals ($L\sim 1$ mm), it is illustrative to approximate sinc ~ 1 in Eq. (3), in which case C_{jkst}^{nm} simplifies to

$$\tilde{C}_{jkst}^{nm} = \sqrt{2A}b(j,s,N-n)b(k,t,M-m) \times |D_{N-n,M-m}|H_{N-n}(0)H_{M-m}(0), \tag{11}$$

if $j+s \geqslant n$ and $k+t \geqslant m$, otherwise $C_{jkut}^{nm}=0$. However, under this approximation the two-photon state is no longer normalizable. The Hermite polynomials H_{N-n} and H_{M-m} evaluated at $x=y=0$ put in evidence another conservation condition which is not directly apparent in Eq. (9): $N-n$ and $M-m$ must be even, since odd-ordered Hermite polynomials are zero at the origin. Thus, the sum of the $x(y)$ indices of the down converted fields $N=j+s$ $(M=k+t)$ must have the same parity as the $x(y)$ index of the pump field $n(m)$.

Though we have considered the case of a HG pump beam, more complicated pump fields can be accounted for by first expanding the angular spectrum as a linear combination of HG modes, and computing the two-photon state associated to each HG component.

C. Detection amplitude

For a two-photon state, the two-photon detection probability is proportional to the fourth-order correlation function, and can be written as

$$P(\mathbf{r}_1,\mathbf{r}_2)=|\Psi(\mathbf{r}_1,\mathbf{r}_2)|^2, \tag{12}$$

with the detection amplitude defined as

$$\Psi(\mathbf{r}_1,\mathbf{r}_2)=\langle\text{vac}|\mathbf{E}(\mathbf{r}_1)\mathbf{E}(\mathbf{r}_2)|\psi\rangle. \tag{13}$$

In the paraxial approximation, the field operator $\mathbf{E}(\mathbf{r})$ associated with the detection of a photon at position $\mathbf{r}$ with well-defined frequency and polarization is

$$\mathbf{E}(\boldsymbol{\rho},z)=\mathcal{E}e^{ikz}\int e^{i(\mathbf{q}\cdot\boldsymbol{\rho}-q^2z/2k)}\mathbf{a}(\mathbf{q})d\mathbf{q}, \tag{14}$$

where $\mathbf{a}(\mathbf{q})$ is the annihilation operator for a photon with transverse wave vector $\mathbf{q}$. Using the state (2), the detection amplitude is

$$\Psi(\mathbf{r}_1,\mathbf{r}_2)=B_{nm}\mathcal{W}(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2,Z)\Gamma(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,Z), \tag{15}$$

where B_{nm} is a constant, $\Gamma(\boldsymbol{\rho})$ is the Fourier transform of $\gamma(\mathbf{q})$, and $\mathcal{W}(\boldsymbol{\rho},Z)$ is the profile of the pump beam, both propagated from $z=0$ to $z=Z$. Equivalently, using the HG mode representation (6) of the state $|\psi\rangle$, we obtain

$$\Psi(\mathbf{r}_1,\mathbf{r}_2)=B_{nm}\sum_{j,k,s,t=0}^{\infty}C_{jkst}^{nm}u_{jk}(\boldsymbol{\rho}_1,Z)u_{st}(\boldsymbol{\rho}_2,Z). \tag{16}$$

Here $u_{jk}(\boldsymbol{\rho},Z)$ is the HG mode $|v_{jk}\rangle$ in the position representation, at $z=Z$. It is apparent that the modulus of the expansion coefficients C_{jkst}^{nm}, and thus the entanglement, are unchanged under free propagation.

III. ENTANGLEMENT

The two-photon state generated by SPDC can be expressed in terms of an infinite dimensional discrete basis such as the Hermite-Gaussian modes or continuous variables. For a bipartite finite-dimensional discrete system, there are several methods available to quantify the entanglement, such as the Schmidt number [40], the von Neumann entropy of the subsystems [40], or the concurrence [33,41].

Provided the expansion coefficients grow smaller and smaller, one can truncate the infinite number of terms, resulting in a discrete, finite-dimensional system [19,42,43]. Fortunately, in our case, the relative weights $|C_{jkst}^{nm}|^2$ for higher orders (large j,k,s,t) decrease [6], so it is not unreasonable to abbreviate the infinite sums in Eq. (6) with finite sums. Let us assume that the dimensionality of the Hilbert space associated to the down-converted photons is arbitrarily large but finite, and write the state (6) as

$$|\psi_{nm}^T\rangle=\sum_{j,k,s,t=0}^{\mathcal{N}}C_{jkst}^{nm}|v_{jk}\rangle_1|v_{st}\rangle_2, \tag{17}$$

where we assume that the coefficients C_{jkst}^{nm} have been adjusted so that $|\psi_{nm}\rangle$ is normalized. The $\mathcal{N}$ necessary so that the truncated state (17) is a good approximation to the actual state (6) depends on the experimental parameters, such as the order $n+m$, wave number K and width w_p of the pump beam, as well as the length L of the nonlinear crystal. For typical experimental situations, one can achieve $|\langle\psi_{nm}^T|\psi_{nm}\rangle|\sim 0.95$ with $\mathcal{N}\leqslant 20$.

We will now calculate the amount of entanglement present in the two-photon quantum state produced by SPDC. For bipartite pure states composed of d-dimensional systems, an adequate and convenient entanglement quantifier is the I-concurrence, defined as [33,44]

$$\mathcal{C}(\psi)=\sqrt{2(1-\text{tr }\varrho_1^2)}=\sqrt{2(1-\text{tr }\varrho_2^2)}, \tag{18}$$

where ϱ_1 and ϱ_2 are the reduced density operators of the down-converted photons. For a maximally entangled $d\times d$ state $|\psi_d\rangle$, $\mathcal{C}(\psi_d)=\sqrt{2(d-1)/d}$. We note that in the limit $d\to\infty$ and $\mathcal{C}(\psi)\to\sqrt{2}$. For the bipartite state (17), the reduced density matrix for photon 1 is

$$\varrho_1=\text{tr}_2(|\psi_{nm}\rangle\langle\psi_{nm}|)=\sum_{jkdf}^{\mathcal{N}}F_{jkdf}|v_{jk}\rangle_1\,{}_1\langle v_{df}|, \tag{19}$$

where

$$F_{jkdf}\equiv\sum_{st}^{\mathcal{N}}C_{jkst}^{nm}C_{dfst}^{nm}. \tag{20}$$

Above we have used the fact that the coefficients C_{jkst}^{nm} are real [6]. Using Eq. (20) and provided that tr $\varrho_1=1$, the coefficients F_{jkdf} have the following properties:

$$\sum_{jk}^{\mathcal{N}}F_{jkjk}=1, \tag{21a}$$

$$F_{jkjk}\geqslant 0, \tag{21b}$$

$$F_{jkdf}=F_{dfjk}, \tag{21c}$$

while the parity conservation relations imply that $F_{jkdf}=0$ if j and d or k and f have different parities.

The purity of the reduced density operator is

$$\text{tr }\varrho_1^2=\text{tr}\left(\sum_{jkdf}^{\mathcal{N}}\sum_{\alpha\beta\gamma\delta}^{\mathcal{N}}F_{jkdf}F_{\alpha\beta\gamma\delta}|v_jk\rangle\langle v_{df}|v_{\alpha\beta}\rangle\langle v_{\gamma\delta}|\right)=\sum_{jkdf}^{\mathcal{N}}(F_{jkdf})^2, \tag{22}$$

where we used the orthogonality of the HG modes and the symmetry property (21c). We have dropped the subscript "1" on the bras and kets for notational convenience. Using Eq. (22), the I-concurrence (18) for the state (17) is

$$\mathcal{C}(\psi)=\sqrt{2\left[1-\sum_{jkdf}^{\mathcal{N}}(F_{jkdf})^2\right]}. \tag{23}$$

We are now in a position to analyze the amount of transverse spatial entanglement present in the two photon state as a function of different experimental parameters.

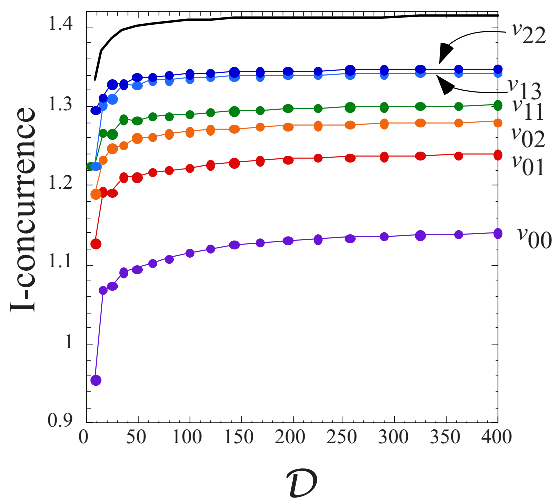

FIG. 1. (Color online) I-concurrence as a function of the number of states $\mathcal{D}=(\mathcal{N}+1)^2$, with $\sqrt{L/2z_0}=0.25$ for different HG pump beams. One can see that, in general, the I-concurrence increases with the order $n+m$ of the pump beam. The solid black line shows I-concurrence for a maximally entangled state of dimension $\mathcal{D}$.

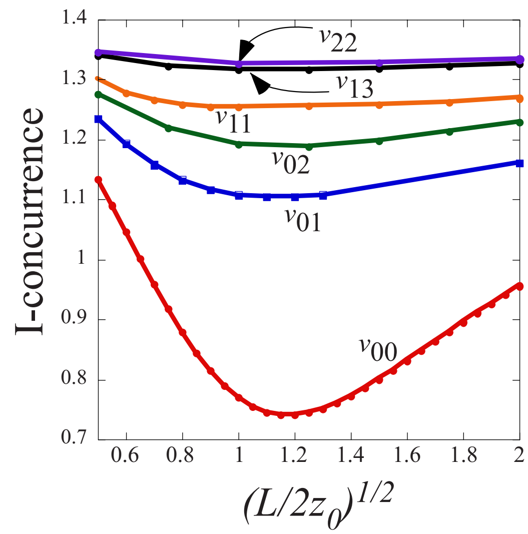

FIG. 2. (Color online) I-concurrence as a function of $\sqrt{L/2z_0}$. Here the down-converted HG modes are truncated at $\mathcal{N}=16$, which accounts for more than 90% of the quantum state.

Figure 1 shows that I-concurrence as a function of the dimension $\mathcal{D}=(\mathcal{N}+1)^2$ of the reduced density matrix ϱ for $0\leqslant\mathcal{N}\leqslant 19$. Shown are results for several Hermite-Gaussian pump beams v_{nm}. In all cases the entanglement generally increases with $\mathcal{D}$, and saturates for large $\mathcal{D}$. This behavior is to be expected, since increasing $\mathcal{D}$ for the truncated state in Eq. (17) is essentially equivalent to considering more and more terms of a Schmidt decomposition, thus increasing the Schmidt number.

It can also be seen that the amount of transverse spatial entanglement increases as the order $(n+m)$ of the pump beam increases. Looking at the largest dimension, we see that the Gaussian pump beam v_{00} generates less entangled states ($\mathcal{C}\approx 1.14$), while the fourth order beams v_{13} and v_{22} generate states with $\mathcal{C}\approx 1.35$. For $\mathcal{N}=19$, the expansion (17) accounts for more than 96% of the state (2) in all cases.

In Ref. [19] it was shown that the Schmidt number, and thus the entanglement, depends on the quantity $\sqrt{A}=\sqrt{L/2z_0}$ for the case of a Gaussian pump beam. Figure 2 shows the I-concurrence as a function of the parameter $\sqrt{L/2z_0}$ for several Hermite-Gaussian pump beams. For the Gaussian (v_{00}) case, it can be seen that the entanglement reaches a minimum when $\sqrt{L/2z_0}\sim 1$, and increases with $\sqrt{L/2z_0}$, as was observed in Ref. [19]. However, for higher-order HG beams, this effect is less present. For example, the entanglement present in a two-photon state created by a v_{11} pump beam varies about 10% as a function of $\sqrt{L/2z_0}$, and for higher order modes the variation is even less. This is due to the fact that the $v_{11}(\mathbf{q})$ function is less similar to the phase matching function $\gamma(\mathbf{q})$, and thus the two photon state is less separable. Indeed, if one approximates the phase matching function $\gamma(\mathbf{q})$ with a Gaussian function and considers that the pump beam is also a Gaussian v_{00}, the two-photon state is separable when $v(\mathbf{q})$ and $\gamma(\mathbf{q})$ have the same width [19]. This is no longer the case for higher order HG pump beams.

A. Direct measurement of spatial entanglement

We now discuss a method to measure transverse spatial entanglement directly using two-photon Hong-Ou-Mandel interference [45] of two copies of the SPDC state. In Ref. [28], the visibility of HOM interference of a single photon pair was used to experimentally infer the Schmidt number. Here we will show that the probability of a coincidence event in separate output ports in our two-copy scheme gives $\mathcal{C}^2$ directly.

It has recently been shown that the I-concurrence can be associated to a physical observable which operates on identical copies of a bipartite pure state [44,46]. In fact, one can show that

$$\mathcal{C}=\sqrt{\langle\psi|\otimes\langle\psi|\mathbf{A}|\psi\rangle\otimes|\psi\rangle}, \tag{24}$$

where $\mathbf{A}$ is the projector onto the antisymmetric subspace. This measurement has been realized in the case of photonic qubits [47], and the application to other physical systems has been discussed in Ref. [48]. Here we show that, using a simple beam splitter, the same scheme can be used to detect spatial entanglement of any dimension.

For photons, one can project onto the antisymmetric space using two-photon Hong-Ou-Mandel interference at a 50/50 nonpolarizing beam splitter [45,49]. Consider the setup shown in Fig. 3, in which one photon from each of the two identically prepared copies of spatially entangled photons are incident on a 50/50 beam splitter (BS). For simplicity, we suppose that the photons are temporally and spectrally indistinguishable. The copies could be prepared using identical SPDC crystal configurations, or using a double-pass crystal configuration, as in Ref. [50]. The probability to produce two pairs, one in each crystal is roughly the same as the probability to produce zero photons in one crystal and two pairs in the other. However, the former can be distinguished from the latter as long as one detects fourfold coincidences at all

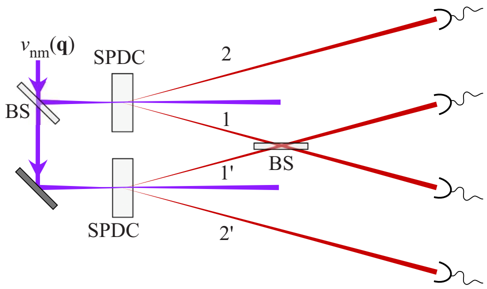

FIG. 3. (Color online) Experimental arrangement to measure spatial entanglement of photon pairs produced from spontaneous parametric down-conversion (SPDC) directly. BS is a 50/50 nonpolarizing beam splitter.

four detectors, since the probability to produce three or more pairs is negligible.

In Refs. [51,52], it was shown that one must take into account the spatial parity of the interfering photons. A momentum state incident on a 50/50 beam splitter becomes

$$|q_x,q_y\rangle_i \rightarrow |q_x,q_y\rangle_1 + i|q_x,-q_y\rangle_2, \tag{25}$$

where the minus sign accounts for the spatial reflection at the beam splitter. Using Eq. (25), a HG state $|v_{jk}\rangle$ as defined in Eq. (7), which has well-defined cartesian symmetry, becomes

$$|v_{jk}\rangle_i \rightarrow |v_{jk}\rangle_1 + i(-1)^k|v_{jk}\rangle_2. \tag{26}$$

Consider now that two photons, one from each copy of state (6), are incident on opposite sides of the BS. Let us concern ourselves with only the coincidence events, in which the input states $|v_{jk}\rangle|v_{\alpha\beta}\rangle$ become

$$|v_{jk}\rangle_1|v_{\alpha\beta}\rangle_{1'} \rightarrow |v_{jk}\rangle_1|v_{\alpha\beta}\rangle_{1'} - (-1)^{k+\beta}|v_{\alpha\beta}\rangle_1|v_{jk}\rangle_{1'}. \tag{27}$$

The final state is

$$|\Psi_A\rangle = \sum_{jkst}\sum_{\alpha\beta\gamma\delta}\{|v_{jk}\rangle_1|v_{\alpha\beta}\rangle_{1'}|v_{st}\rangle_2|v_{\gamma\delta}\rangle_{2'} - (-1)^{k+\beta}|v_{\alpha\beta}\rangle_1|v_{jk}\rangle_{1'}|v_{st}\rangle_2|v_{\gamma\delta}\rangle_{2'}\}. \tag{28}$$

Using the orthonormality of the states $|v_{jk}\rangle$, the probability to detect a coincidence event is

$$\begin{aligned}|\Psi_A|^2 &= 2\sum_{jkst}\sum_{\alpha\beta\gamma\delta}|C_{jkst}|^2|C_{\alpha\beta\gamma\delta}|^2 \\ &\quad - 2\sum_{jkst}\sum_{\alpha\beta\gamma\delta}C_{jkst}C_{\alpha\beta st}C_{\alpha\beta\gamma\delta}C_{jk\gamma\delta} \\ &= 2\left(1-\sum_{jk\alpha\beta}F^2_{jk\alpha\beta}\right),\end{aligned} \tag{29}$$

which, from Eq. (22), is equivalent to twice the linear entropy: $2(1-\mathrm{tr}\,\varrho^2)$. The I-concurrence is then given by $C=\sqrt{|\Psi_A|^2}$. We note that this measurement is independent of the dimension of the system, and in fact, it is not even necessary to know the dimension of the system. However, it is necessary that the state is pure or quasipure [48].

IV. CONCLUSION

We have shown that the order $(n+m)$ and width of the pump beam affect the quantum entanglement of the two-photon state produced by spontaneous parametric down-conversion when the pump beam is described by a Hermite Gaussian mode v_{nm}. In particular, the entanglement tends to increase with the order, and is less dependent on the width of the pump beam for higher-order modes $(n>0,\ m>0)$. These results are valid under certain experimental constraints, in particular, we have ignored effects due to anisotropy in the nonlinear crystal. These results are thus valid for thin crystals and continuous-wave pump beams under weak focusing. The entanglement content of the two-photon state under a wider range of experimental conditions will be considered in a future work [37]. Here we have not considered any limiting factors on the spatial bandwidth of the photons, other than the paraxial approximation. Effects due to finite-size optical systems will be included in a future publication. We have also proposed a dimension-independent method to measure the bipartite I-concurrence directly using two copies of the entangled state.

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