Continuous-variable entanglement sharing in noninertial frames

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We study the distribution of entanglement between modes of a free scalar field from the perspective of observers in uniform acceleration. We consider a two-mode squeezed state of the field from an inertial perspective, and analytically study the degradation of entanglement due to the Unruh effect, in the cases of either one or both observers undergoing uniform acceleration. We find that, for two observers undergoing finite acceleration, the entanglement vanishes between the lowest-frequency modes. The loss of entanglement is precisely explained as a redistribution of the inertial entanglement into multipartite quantum correlations among accessible and inaccessible modes from a noninertial perspective. We show that classical correlations are also lost from the perspective of two accelerated observers but conserved if one of the observers remains inertial.

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I. INTRODUCTION

In the study of most quantum-information tasks such as teleportation and quantum cryptography, nonrelativistic observers share entangled resources to perform their experiments [1]. Apart from a few studies [2–8], most works on quantum information assume a world without gravity where spacetime is flat. But the world is relativistic and any serious theoretical study must take this into account. It is therefore of fundamental interest to revise quantum-information protocols for relativistic settings [9]. It has been shown that relativistic effects on quantum resources not only are quantitatively important but also induce novel qualitative features [3–5,7]. For example, it has been shown that the dynamics of spacetime can generate entanglement [5]. This, in principle, would have a consequence in any entanglement-based protocol performed in curved spacetime. Relativistic effects have also been found to be relevant in a flat spacetime, where the entanglement described by observers in uniform acceleration is observer dependent since it is degraded by the Unruh effect [3,4,7]. In the infinite-acceleration limit, the entanglement vanishes for bosons [3,7] and reaches a nonvanishing minimum for fermions [4]. This degradation of entanglement results in the loss of fidelity of teleportation protocols which involve observers in uniform acceleration [6].

Understanding entanglement in a relativistic framework is not only of interest to quantum information. Entanglement plays an important role in black hole entropy [10] and in the apparent loss of information in black holes [11], one of the most challenging problems in theoretical physics at the moment [12]. Understanding the entanglement between modes of a field close to the horizon of a black hole might help to understand some of the key questions in black hole thermodynamics and their relation to information.

In this paper we interpret the loss of bipartite entanglement between two modes of a scalar field from a noninertial perspective, as an effect of entanglement redistribution. Precisely, we consider the entanglement between two field modes, each described from the perspective of a different observer. Suppose that the two modes are entangled to a given degree from the perspective of two inertial observers. The state will appear less entangled if either one or both the observers move with uniform acceleration [3]. This is because each mode described by an inertial observer corresponds to two entangled modes from the perspective of a noninertial observer [13]. Consequently a two-mode entangled state described from the inertial perspective corresponds to a three-mode state when described from the perspective of one inertial observer and one in uniform acceleration, and to a four-mode state if both observers are accelerated. Physical observers moving with uniform acceleration have access only to one of the noninertial modes. Thus, when describing the state (which involves tracing over the unaccessible modes) the observers find that some of the correlations are lost.

This phenomenon, stemming from the Unruh effect [13], was first studied from the quantum-information perspective for bosonic scalar fields [3] (considering one inertial observer and the other one undergoing uniform acceleration) and later for fermionic Dirac fields [4]. Although entanglement of particle number states is in both cases degraded as a function of the acceleration, there are important differences in the results. For example, in the infinite-acceleration limit, the entanglement reaches a nonvanishing minimum value for

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fermions, while it completely disappears in the bosonic case. For photon-helicity-entangled states, instead, the correlations are not degraded at all [8]. The loss of entanglement was explained in the fermionic case in the light of the entanglement-sharing framework as an effect of the redistribution of entanglement among all, accessible and inaccessible, modes. Although the loss of entanglement was first studied for scalar fields (considering a state from an inertial perspective which is maximally entangled in a two-qubit space, $|\psi\rangle \sim |00\rangle + |11\rangle$), entanglement sharing was not analyzed in that instance, due to the difficulty of computing entanglement in such a hybrid qubit–continuous-variable system. Fortunately, the theory of continuous-variable entanglement has been developed in recent times, allowing for the exact, quantitative study of bipartite entanglement and its distribution in the special class of Gaussian states [14], which includes, among others, squeezed, coherent, and thermal states of harmonic oscillators.

Here, we consider a free scalar field which is, from an inertial perspective, in a two-mode squeezed state. This choice of the state is motivated by different observations. First, the two-mode squeezed state is the paradigmatic entangled state of a continuous variable system, approximating to an arbitrarily good extent the Einstein-Podolsky-Rosen (EPR) pair [15]. Second, the state can be produced in the laboratory and exploited for any current realization of bipartite quantum information with continuous variables [16]. Third, it belongs to the class of Gaussian states, which admit an exact description of their classical and quantum correlations. Since the Unruh transformations [13] are Gaussian themselves (i.e., they preserve the Gaussian character of the state), it is possible to characterize analytically the redistribution of correlations due to relativistic effects. Finally, the two-mode squeezed state has a special role in quantum field theory. It is possible to define particle states (necessary in any entanglement discussion) when the spacetime has at least two asymptotically flat regions [5,17]. In this case, particle states commonly correspond to multimode squeezed states in which several field modes are in a pairwise squeezed entangled state. The state we consider in our entanglement discussion is the simplest multimode squeezed state in which only two modes are entangled.

A first investigation of the degradation of entanglement in a two-mode squeezed state due to the Unruh effect has been recently reported [7]. The entanglement degradation (quantified by the logarithmic negativity [18]) was analyzed when one of the observers is accelerated and found to be more drastic when the entanglement described from the inertial perspective is stronger, resulting in a vanishing entanglement, from a noninertial perspective, in the infinite-acceleration limit.

We perform an extensive study of both quantum (entanglement) and classical correlations of the two-mode squeezed state from a noninertial perspective. Our work aims at a conclusive understanding and characterization of the relativistic effects on continuous-variable correlations described by observers in uniform acceleration. Therefore, we evaluate not only the bipartite entanglement as degraded by the Unruh thermalization, but, remarkably, the multipartite entanglement which arises among all Rindler modes. Our analysis is possible thanks to recent analytical results on entanglement sharing and the quantification of multipartite entanglement in Gaussian states. This analysis relies on the *contangle*, which is a computable measure of entanglement [19]. The contangle for mixed states is not fully equivalent to the negativity. Therefore, in the case of a single accelerated observer, our results will evidence significant differences from the results presented in Ref. [7]. The main result we find in this case is that, in the infinite-acceleration limit, all the bipartite entanglement described by inertial observers is exactly redistributed into genuine tripartite correlations between the modes described by one inertial and two noninertial observers (one real and one fictitious, or virtual), as a consequence of the monogamy constraints on entanglement distribution [19–21]. We also analyze total correlations, finding that the classical correlations are invariant under acceleration when one observer is accelerated.

Furthermore, we present an original analysis of the Unruh effect on continuous-variable entanglement when both observers undergo uniform acceleration. This analysis yields a series of significant results. First, the bipartite entanglement described by noninertial observers may vanish completely at finite acceleration even when the state contains an infinite amount of entanglement from the point of view of inertial observers. Second, the acceleration induces a redistribution of entanglement, such that the modes described from a noninertial perspective are correlated via a genuine four-partite entanglement. This entanglement increases unboundedly with the acceleration, easily surpassing the bipartite entanglement described from an inertial perspective. Third, classical correlations are also degraded as a function of the acceleration. The degradation is of at most one unit with respect to the case of a single noninertial observer. Moreover, we study the dependence of the bipartite entanglement on the frequency of the modes described by the noninertial observers, finding that with increasing acceleration the range of entangled frequencies gets narrower and narrower, becoming empty in the limit of infinite acceleration.

Our results on one hand are an interesting application of continuous-variable quantum-information techniques (commonly confined to quantum optics or light-matter interfaces) to a relativistic setting, and on the other hand provide a deeper understanding of the characterization of the inherent relativistic effects on the distribution of information. This may lead to a better understanding of the behavior of information in the presence of a black hole [22].

The paper is organized as follows. In Sec. II A we introduce the basic tools of quantum information with Gaussian states of continuous-variable systems and we discuss the mechanism of entanglement sharing. In Sec. II B we describe the Unruh effect and its consequences on the entanglement between two field modes. In Sec. III, we study distributed entanglement between modes of a free scalar field when one observer is accelerated. The case when both observers are accelerated, resulting in a four-partite entangled state, is studied in Sec. IV. Both Secs. III and IV include an analysis of the dependence of classical correlations under acceleration of the observers. Finally, in Sec. V we draw our concluding remarks and compare our results to those obtained in the case of Dirac fields [4].

II. PRELIMINARY TOOLBOX

A. Gaussian states and Gaussian entanglement measures

Entanglement in continuous-variable (CV) systems is encoded in the form of Einsteiy-Podolsky-Rosen correlations [15]. Let us consider the quadratures of a two-mode radiation field, where mode $k=i,j$ is described by the ladder operators $\hat{a}_k,\hat{a}_k^\dagger$ satisfying the bosonic commutation relation $[\hat{a}_k,\hat{a}_k^\dagger]=1$. An arbitrarily increasing degree of entanglement can be encoded in a two-mode squeezed state $|\psi^{sq}\rangle_{i,j}=U_{i,j}(r)(|0\rangle_i\otimes|0\rangle_j)$ with increasing squeezing factor $r\in\mathbb{R}$, where the (phase-free) two-mode squeezing operator is given by

$$U_{i,j}(r)=\exp\left(\frac{r}{2}(\hat{a}_i^\dagger\hat{a}_j^\dagger-\hat{a}_i\hat{a}_j)\right), \tag{1}$$

and $|0\rangle_k$ denotes the vacuum state in the Fock space of mode k. In the limit of infinite squeezing ($r\to\infty$), the state approaches the ideal EPR state [15], which is, simultaneously, an eigenstate of total momentum and relative position of the two subsystems. Therefore, the state contains infinite entanglement. The EPR state is unnormalizable and unphysical. The two-mode squeezed state is an arbitrarily good approximation of it with increasing squeezing, and therefore represents a key resource for practical implementations of CV quantum information protocols [16]. Mathematically, squeezed states belong to the class of *Gaussian states* of CV systems, i.e., states with Gaussian characteristic functions and quasiprobability distributions, whose structural and informational properties have been intensively studied in recent times [14].

1. Covariance matrix formalism

In view of the subsequent analysis, it is sufficient to recall that Gaussian states of N modes are completely described in phase space (up to local unitaries) by the real, symmetric covariance matrix (CM) $\boldsymbol{\sigma}$, whose entries are $\sigma_{ij}=(1/2)\times\langle\{\hat{X}_i,\hat{X}_j\}\rangle-\langle\hat{X}_i\rangle\langle\hat{X}_j\rangle$. Here $\hat{X}=\{\hat{x}_1,\hat{p}_1,\ldots,\hat{x}_N,\hat{p}_N\}$ is the vector of the field quadrature operators, whose canonical commutation relations can be expressed in matrix form, $[\hat{X}_i,\hat{X}_j]=2i\Omega_{ij}$, with the symplectic form $\Omega=\oplus_{i=1}^n\omega$ and $\omega=\delta_{ij-1}-\delta_{ij+1}, i,j=1,2$. The CM $\boldsymbol{\sigma}$ must satisfy the Robertson-Schrödinger uncertainty relation [23]

$$\boldsymbol{\sigma}+i\Omega\geqslant 0, \tag{2}$$

to describe a physical state. Throughout the paper, $\boldsymbol{\sigma}$ will be used indifferently to indicate the CM of a Gaussian state or the state itself.

Unitary Gaussian operations U amount, in phase space, to symplectic transformations S (which preserve the symplectic form $\Omega=S^T\Omega S$) acting "by congruence" on the CM (i.e., so that $\boldsymbol{\sigma}\mapsto S\boldsymbol{\sigma}S^T$). For instance, the two-mode squeezing operator Eq. (1) corresponds to the symplectic transformation

$$S_{i,j}(r)=\begin{pmatrix}\cosh r & 0 & \sinh r & 0\\ 0 & \cosh r & 0 & -\sinh r\\ \sinh r & 0 & \cosh r & 0\\ 0 & -\sinh r & 0 & \cosh r\end{pmatrix}, \tag{3}$$

where the matrix is understood to act on the pair of modes i and j. A two-mode squeezed state with squeezing degree r [24] will be thus described by a CM

$$\boldsymbol{\sigma}^{sq}_{i,j}(r)=S_{i,j}(r)\mathbb{I}_4S^T_{i,j}(r)=\begin{pmatrix}\cosh 2r & 0 & \sinh 2r & 0\\ 0 & \cosh 2r & 0 & -\sinh 2r\\ \sinh 2r & 0 & \cosh 2r & 0\\ 0 & -\sinh 2r & 0 & \cosh 2r\end{pmatrix}, \tag{4}$$

where we have used that the CM of an N-mode vacuum is the $2N\times 2N$ identity matrix $\mathbb{I}_{2N}$.

2. Qualifying and quantifying entanglement

Concerning the characterization of bipartite entanglement, the positive partial transpose (PPT) criterion states that a Gaussian CM $\boldsymbol{\sigma}$ is separable (with respect to a $1\times N$ bipartition) if and only if the partially transposed CM $\tilde{\boldsymbol{\sigma}}$ satisfies the uncertainty principle given by Eq. (2) [25,26]. The tilde denotes the partial transposition, implemented by reversing time in the subspace of only one subsystem of a bipartite composite CV system [25]. An ensuing computable measure of CV entanglement is the *logarithmic negativity* [18] $E_\mathcal{N}\equiv\log\|\tilde{\varrho}\|_1$, where $\|\cdot\|_1$ denotes the trace norm. This measure is an upper bound to the *distillable entanglement* of the state ϱ. The logarithmic negativity is used in Ref. [7] to quantify the degradation of two-mode Gaussian entanglement due to one accelerated observer.

We employ a different measure of bipartite entanglement, the *contangle* [19], which is an entanglement monotone under Gaussian local operations and classical communication (GLOCC), that belongs to the family of "Gaussian entanglement measures" [27]. The principal motivation for this choice is that our main focus is to study the effects of the Unruh thermalization mechanism on the distribution of entanglement among field modes described from a noninertial perspective. In this setting, the contangle is the best measure to enable a mathematical treatment of distributed CV entanglement as emerging from the fundamental monogamy constraints [19–21]. The contangle τ is defined for pure states as the square of the logarithmic negativity and it is extended to mixed states via the Gaussian convex roof [27,28], that is, as the minimum of the average pure-state entanglement over all decompositions of the mixed state in ensembles of pure Gaussian states. If $\boldsymbol{\sigma}_{i|j}$ is the CM of a (generally mixed) bipartite Gaussian state where subsystem i comprises one mode only, then the contangle τ can be computed as [19]

$$\tau(\boldsymbol{\sigma}_{i|j}) \equiv \tau(\boldsymbol{\sigma}_{i|j}^{opt}) = g(m_{i|j}^2), \quad g(x) = \operatorname{arcsinh}^2(\sqrt{x-1}), \quad (5)$$

where $\boldsymbol{\sigma}_{i|j}^{opt}$ corresponds to a pure Gaussian state, and $m_{i|j} \equiv m(\boldsymbol{\sigma}_{i|j}^{opt}) = \sqrt{\mathrm{Det}\boldsymbol{\sigma}_i^{opt}} = \sqrt{\mathrm{Det}\boldsymbol{\sigma}_j^{opt}}$, with $\boldsymbol{\sigma}_{i(j)}^{opt}$ the reduced CM of subsystem i (j), obtained by tracing over the degrees of freedom of subsystem j (i). The CM $\boldsymbol{\sigma}_{i|j}^{opt}$ denotes the pure bipartite Gaussian state which minimizes $m(\boldsymbol{\sigma}_{i|j}^{p})$ among all pure-state CMs $\boldsymbol{\sigma}_{i|j}^{p}$ such that $\boldsymbol{\sigma}_{i|j}^{p} \leq \boldsymbol{\sigma}_{i|j}$. If $\boldsymbol{\sigma}_{i|j}$ is a pure state, then $\boldsymbol{\sigma}_{i|j}^{opt} = \boldsymbol{\sigma}_{i|j}$, while for a mixed Gaussian state Eq. (5) is mathematically equivalent to constructing the Gaussian convex roof. For a separable state, $m(\boldsymbol{\sigma}_{i|j}^{opt}) = 1$ and the entanglement vanishes. The contangle τ is completely equivalent to the Gaussian entanglement of formation [28], which quantifies the cost of creating a given mixed, entangled Gaussian state out of an ensemble of pure, entangled Gaussian states. Notice also that in general the Gaussian entanglement measures are inequivalent to the negativities, in that they may induce opposite ordering on the set of entangled, nonsymmetric two-mode Gaussian states [27]: this will be explicitly unfolded in the following analysis.

3. Entropy and mutual information

In a bipartite setting, another important correlation measure is the so-called *mutual information* quantifying the total (classical and quantum) correlations between two parties. The mutual information of a state $\varrho_{A|B}$ of a bipartite system is defined as

$$I(\varrho_{A|B}) = S_V(\varrho_A) + S_V(\varrho_B) - S_V(\varrho_{A|B}), \quad (6)$$

where ϱ_A (ϱ_B) is the reduced state of subsystem A (B) and S_V denotes the von Neumann entropy, defined for a quantum state ϱ as $S_V(\varrho) = -\mathrm{Tr}\, \varrho \log \varrho$. If $\varrho_{A|B}$ is a pure quantum state $[S_V(\varrho_{A|B}) = 0]$, the von Neumann entropy of its reduced states $S_V(\varrho_A) = S_V(\varrho_B)$ quantifies the entanglement between the two parties [29]. Being $I(\varrho_{A|B}) = 2S_V(\varrho_A) = 2S_V(\varrho_B)$ in this case, one says that the pure state also contains some classical correlations, equal in content to the quantum part, $S_V(\varrho_A) = S_V(\varrho_B)$. In mixed states, the distinction between classical and quantum correlations cannot be considered an accomplished task yet [30].

For an arbitrary bipartite (pure or mixed) Gaussian state, the von Neumann entropy and hence the mutual information can be easily computed in terms of the symplectic spectra of the CM of the global state, and of the reduced CMs of both subsystems. In the case of a two-mode state with global CM $\boldsymbol{\sigma}_{A|B}$, the mutual information yields [31,32],

$$I(\boldsymbol{\sigma}_{A|B}) = f(\sqrt{\mathrm{Det}\, \boldsymbol{\sigma}_A}) + f(\sqrt{\mathrm{Det}\, \boldsymbol{\sigma}_B}) - f(\eta_{A|B}^{-}) - f(\eta_{A|B}^{+}), \quad (7)$$

where

$$f(x) \equiv \frac{x+1}{2} \log\left(\frac{x+1}{2}\right) - \frac{x-1}{2} \log\left(\frac{x-1}{2}\right), \quad (8)$$

and $\{\eta_{A|B}^{-}, \eta_{A|B}^{+}\}$ are the symplectic eigenvalues of $\boldsymbol{\sigma}_{A|B}$ (i.e., the orthogonal eigenvalues of the matrix $|i\Omega \boldsymbol{\sigma}_{A|B}|$).

4. Distributed quantum correlations and multipartite entanglement

Quantifying entanglement in multipartite systems is generally very involved. A way to determine the existence of multipartite correlations in a state is by exploring the entanglement distributed between multipartite systems. Unlike classical correlations, entanglement is *monogamous*, meaning that it cannot be freely shared among multiple subsystems of a composite quantum system [21]. This fundamental constraint on entanglement sharing has been mathematically demonstrated, so far, for arbitrary systems of qubits within the discrete-variable scenario [33,34], for a special case of two qubits and an infinite-dimensional system [35], and for all N-mode Gaussian states within the CV scenario [19,20].

In the general case of a state distributed among N parties (each owning a single qubit, or a single mode, respectively), the monogamy constraint takes the form of the Coffman-Kundu-Wootters inequality [33],

$$E_{S_i|(S_1,\ldots,S_{i-1},S_{i+1},\ldots,S_N)} \geq \sum_{j\neq i}^{N} E_{S_i|S_j}, \quad (9)$$

where the global system is multipartitioned into subsystems S_k ($k=1,\ldots,N$), each owned by a corresponding party, and E is a proper measure of bipartite entanglement. The left-hand side of inequality (9) quantifies the bipartite entanglement between a probe subsystem S_i and the remaining subsystems taken as a whole. The right-hand side quantifies the total bipartite entanglement between S_i and each one of the other subsystems $S_{j\neq i}$ in the respective reduced states. The nonnegative difference between these two entanglements, minimized over all choices of the probe subsystem, is referred to as the *residual multipartite entanglement*. It quantifies the purely quantum correlations that are not encoded in pairwise form, so it includes all manifestations of genuine K-partite entanglement, involving K subsystems at a time, with $2 < K \leq N$. In the simplest nontrivial instance of $N=3$, the residual entanglement has the meaning of the genuine tripartite entanglement shared by the three subsystems [33]. Such a quantity has been proven to be a tripartite entanglement monotone for pure three-mode Gaussian states, when bipartite entanglement is quantified by the contangle [19].

B. Entanglement in noninertial frames: The Unruh effect

To study entanglement from the point of view of parties in uniform acceleration, it is necessary to consider that field quantizations in different coordinates are inequivalent. While an inertial observer concludes that the field is in the vacuum state, from the perspective of an observer in uniform acceleration the field is described as a thermal distribution of particles, with the effective temperature being proportional to his or her acceleration. This is known as the Unruh effect [13], and it has important consequences on the entanglement between (bosonic and/or fermionic) field modes and its distribution properties [3,4]. We will study such consequences in the case of a bosonic field in a state that corresponds to a two-mode squeezed state from an inertial perspective (see also [7]). Let us first discuss how the Unruh effect arises.

Consider an observer moving in the (t,z) plane $(c=1)$ with uniform acceleration $\aleph$. Rindler coordinates (τ,ζ) are appropriate for describing the viewpoint of a uniformly accelerated observer. Two different sets of Rindler coordinates, which differ from each other by an overall change in sign, are necessary to partially cover Minkowski space,

$$\aleph t = e^{\aleph\zeta}\sinh(\aleph\tau), \quad \aleph z = e^{\aleph\zeta}\cosh(\aleph\tau),$$

$$\aleph t = -e^{\aleph\zeta}\sinh(\aleph\tau), \quad \aleph z = -e^{\aleph\zeta}\cosh(\aleph\tau).$$

These sets of coordinates define two Rindler regions (respectively I and II) that are causally disconnected from each other. A particle undergoing eternal uniform acceleration remains constrained to either Rindler region I or II and has no access to the opposite region.

Now consider a free quantum scalar field in a flat background. The quantization of a scalar field in the Minkowski coordinates is not equivalent to its quantization in Rindler coordinates. However, the vacuum state of a given field mode described by an inertial observer can be expressed as a two-mode squeezed state [24] from the Rindler perspective [13,17]

$$|0\rangle_{\rho_M} = \frac{1}{\cosh r}\sum_{n=0}^{\infty}\tanh^n r|n\rangle_{\rho_I}|n\rangle_{\rho_{II}} = U(r)|n\rangle_{\rho_I}|n\rangle_{\rho_{II}}, \quad (10)$$

where

$$\cosh r = (1-e^{-2\pi|\omega_\rho|/\aleph})^{-1/2}, \quad (11)$$

and $U(r)$ is the two-mode squeezing operator introduced in Eq. (1). Each Minkowski mode of frequency $|\omega_\rho|$ has a Rindler mode expansion given by Eq. (10). The relation between higher-energy states can be found using Eq. (10) and the Bogoliubov transformation between the creation and annihilation operators,

$$\hat{a}_\rho = \cosh r\hat{b}_{\rho_I} - \sinh r\hat{b}^\dagger_{\rho_{II}},$$

where $\hat{a}_\rho$ is the annihilation operator in Minkowski space for mode ρ and $\hat{b}_{\rho_I}$ and $\hat{b}_{\rho_{II}}$ are the annihilation operators for the same mode in the two Rindler regions [13]. A Rindler observer moving in region I needs to trace over the modes in region II since he has no access to the information in this causally disconnected region. Therefore, while a Minkowski observer concludes that the field mode ρ is in the vacuum $|0\rangle_{\rho_M}$, the state from the perspective of an observer in uniform acceleration $\aleph$, constrained to region I, is

$$|0\rangle\langle 0|_{\rho_M} \to \frac{1}{\cosh^2 r}\sum_{n=0}^{\infty}\tanh^{2n} r|n\rangle\langle n|_{\rho_I}, \quad (12)$$

which is a thermal state with temperature $T=\aleph/2\pi k_B$ where k_B is Boltzmann's constant.

III. DISTRIBUTED GAUSSIAN ENTANGLEMENT DUE TO ONE ACCELERATED OBSERVER

From the perspective of inertial observers, we consider a scalar field which is in a two-mode squeezed state with mode frequencies α and ρ and squeezing parameter s, as in [7]. This state, which is the simplest multimode squeezed state (of relevance in quantum field theory [17]), allows for the exact quantification of entanglement in all partitions of the system from the inertial and noninertial perspective. We can define the two-mode squeezed state, described from an inertial perspective, via its CM [see Eq. (4)],

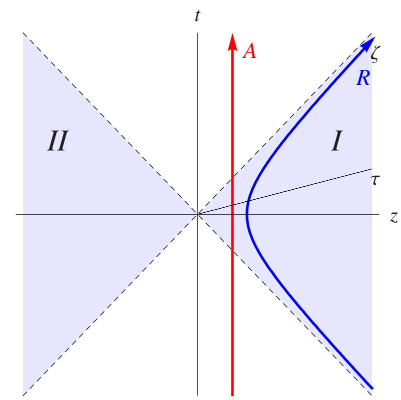

FIG. 1. (Color online) Sketch of the world lines for the inertial observer Alice and the accelerated observer Rob. The set (z,t) denotes Minkowski coordinates, while the set (ζ,τ) denotes Rindler coordinates. The causally disconnected Rindler regions I and II are evidenced.

$$\boldsymbol{\sigma}^P_{AR}(s) = S_{\alpha_M,\rho_M}(s)\mathbb{I}_4 S^T_{\alpha_M,\rho_M}(s), \quad (13)$$

where $\mathbb{I}_4$ is the CM of the vacuum $|0\rangle_{\alpha_M}\otimes|0\rangle_{\rho_M}$.

If an observer (Rob) undergoes uniform acceleration $\aleph_R$, the state corresponding to the mode ρ [36] must be described in Rindler coordinates (see Fig. 1), so that the Minkowski vacuum is given by $|0\rangle_{\rho_M} = U_{\rho_I,\rho_{II}}(r)(|0\rangle_{\rho_I}\otimes|0\rangle_{\rho_{II}})$, with $U(r)$ given by Eq. (1). That is, due to the fact that Rob is in uniform acceleration, the description of the state from his perspective must include a further two-mode squeezing transformation, with squeezing r proportional to Rob's acceleration $\aleph_R$ via Eq. (11). As a consequence of this transformation, the original two-mode entanglement in the state Eq. (13) described by Alice (always inertial) and Rob from an inertial perspective becomes distributed among the modes described by Alice, the accelerated Rob moving in Rindler region I, and a virtual anti-Rob $(\bar{R})$ theoretically able to describe the mode ρ_{II} in the complementary Rindler region II. Our aim is to investigate the distribution of entanglement induced by the purely relativistic effect of Rob's acceleration. It is clear that the three-mode state described by Alice, Rob, and anti-Rob is obtained from the vacuum by the application of Gaussian unitary operations only; therefore, it is a pure Gaussian state. Its CM, according to the above description, is (see also [7])

$$\boldsymbol{\sigma}_{AR\bar{R}}(r,s) = [\mathbb{I}_{\alpha_M}\oplus S_{\rho_I,\rho_{II}}(r)][S_{\alpha_M,\rho_I}(s)\oplus\mathbb{I}_{\rho_{II}}] \times \mathbb{I}_6[S^T_{\alpha_M,\rho_I}(s)\oplus\mathbb{I}_{\rho_{II}}][\mathbb{I}_{\alpha_M}\oplus S^T_{\rho_I,\rho_{II}}(r)], \quad (14)$$

where the symplectic transformations S are given by Eq. (3), and $\mathbb{I}_6$ is the CM of the vacuum $|0\rangle_{\alpha_M}\otimes|0\rangle_{\rho_I}\otimes|0\rangle_{\rho_{II}}$. Explicitly,

$$\boldsymbol{\sigma}_{AR\bar{R}}=\begin{pmatrix}\boldsymbol{\sigma}_A & \boldsymbol{\varepsilon}_{AR} & \boldsymbol{\varepsilon}_{A\bar{R}}\\ \boldsymbol{\varepsilon}_{AR}^T & \boldsymbol{\sigma}_R & \boldsymbol{\varepsilon}_{R\bar{R}}\\ \boldsymbol{\varepsilon}_{A\bar{R}}^T & \boldsymbol{\varepsilon}_{R\bar{R}}^T & \boldsymbol{\sigma}_{\bar{R}}\end{pmatrix}, \quad (15)$$

where

$$\boldsymbol{\sigma}_A=\cosh(2s)\mathbb{I}_2,$$

$$\boldsymbol{\sigma}_R=[\cosh(2s)\cosh^2(r)+\sinh^2(r)]\mathbb{I}_2,$$

$$\boldsymbol{\sigma}_{\bar{R}}=[\cosh^2(r)+\cosh(2s)\sinh^2(r)]\mathbb{I}_2,$$

$$\boldsymbol{\varepsilon}_{AR}=[\cosh^2(r)+\cosh(2s)\sinh^2(r)]Z_2,$$

$$\boldsymbol{\varepsilon}_{A\bar{R}}=[\sinh(r)\sinh(2s)]\mathbb{I}_2,$$

$$\boldsymbol{\varepsilon}_{R\bar{R}}=[\cosh^2(s)\sinh(2r)]Z_2,$$

with $Z_2=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

As pointed out in Ref. [3], the regime of very high acceleration ($r\gg 0$) can be interpreted as Alice and Rob moving close to the horizon of a Schwarzschild black hole. While Alice falls into the black hole, Rob barely escapes the fall by accelerating away from it with uniform acceleration parameter r.

A. Bipartite entanglement

The contangle $\tau(\boldsymbol{\sigma}^P_{A|R})$, quantifying the bipartite entanglement described by two inertial observers, is equal to $4s^2$, as can be straightforwardly found by inserting $m^P_{A|R}=\cosh(2s)$ in Eq. (5).

Let us now compute the bipartite entanglement in the various 1×1 and 1×2 partitions of the state $\boldsymbol{\sigma}_{AR\bar{R}}$. The 1×2 contangles are immediately obtained from the determinants of the reduced single-mode states of the globally pure state $\boldsymbol{\sigma}_{AR\bar{R}}$, Eq. (15), yielding [37]

$$m_{A|(R\bar{R})}=\sqrt{\mathrm{Det}\boldsymbol{\sigma}_A}=\cosh(2s),$$

$$m_{R|(A\bar{R})}=\sqrt{\mathrm{Det}\boldsymbol{\sigma}_R}=\cosh(2s)\cosh^2(r)+\sinh^2(r),$$

$$m_{\bar{R}|(AR)}=\sqrt{\mathrm{Det}\boldsymbol{\sigma}_{\bar{R}}}=\cosh^2(r)+\cosh(2s)\sinh^2(r). \quad (16)$$

For any nonzero value of the two squeezing parameters s and r (i.e., entanglement from the point of view of inertial observers and Rob's acceleration, respectively), each single party is in an entangled state with the block of the remaining two parties, with respect to all possible global splitting of the modes. This classifies the state $\boldsymbol{\sigma}_{AR\bar{R}}$ as *fully inseparable* [38]: it contains therefore genuine tripartite entanglement, which will be precisely quantified in the next section. Notice

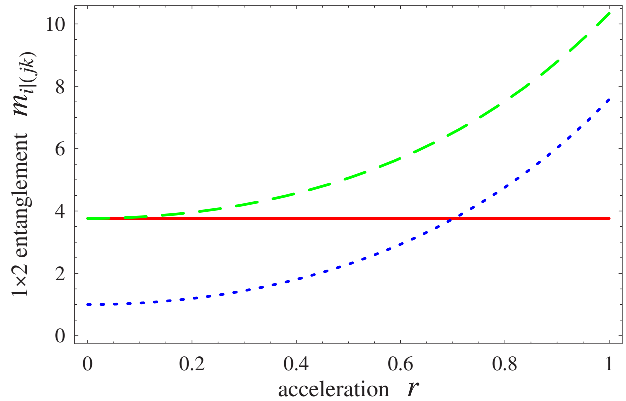

FIG. 2. (Color online) Plot, as a function of the acceleration parameter r, of the bipartite entanglement between the mode described by one observer and the group of modes described by the other two, as expressed by the single-mode determinants $m_{i(jk)}$ defined in Eq. (16). The inertial entanglement is kept fixed at $s=1$. The solid red line represents $m_{A|(R\bar{R})}$, the dashed green line corresponds to $m_{R|(A\bar{R})}$, while the dotted blue line depicts $m_{\bar{R}|(AR)}$.

also that $m_{A|(R\bar{R})}=m^P_{A|R}$, i.e., all the inertial entanglement is distributed, from a noninertial perspective, between modes described by Alice and the group {Rob, anti-Rob}, as expected, since the coordinate transformation $S_{\rho_I,\rho_{II}}(r)$ is a local unitary operation with respect to the considered bipartition, which preserves entanglement by definition. In the following, we will always assume $s\neq 0$ to rule out trivial circumstances.

Interestingly, as already pointed out in Ref. [7], the mode described by Alice is not directly entangled with the mode described by anti-Rob, because the reduced state $\boldsymbol{\sigma}_{A|\bar{R}}$ is separable by inspection, since Det $\boldsymbol{\varepsilon}_{A\bar{R}}\geq 0$. Actually, we can further explore this point by noticing that the mode described by anti-Rob has the *minimum* possible bipartite entanglement with the group of modes described by Alice and Rob. This follows on recalling that, in any pure three-mode Gaussian state $\boldsymbol{\sigma}_{123}$, the local single-mode determinants have to satisfy a triangle inequality [39]

$$|m_1-m_2|+1\leq m_3\leq m_1+m_2-1, \quad (17)$$

with $m_i\equiv\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}_i}$. In our case, identifying mode 1 with Alice, mode 2 with Rob, and mode 3 with anti-Rob, Eq. (16) shows that the state $\boldsymbol{\sigma}_{AR\bar{R}}$ saturates the leftmost side of the triangle inequality (17),

$$m_{\bar{R}|(AR)}=m_{R|(A\bar{R})}-m_{A|(R\bar{R})}+1.$$

In other words, the mixedness of anti-Rob's mode, which is directly related to its entanglement with the other two modes, is the smallest possible one. The values of the entanglement parameters $m_{i|(jk)}$ from Eq. (16) are plotted in Fig. 2 as a function of the acceleration r, for a fixed degree of initial squeezing s.

On the other hand, the PPT criterion states that the reduced two-mode states $\boldsymbol{\sigma}_{A|R}$ and $\boldsymbol{\sigma}_{R|\bar{R}}$ are both entangled. To compute the contangle in those partitions, we first observe

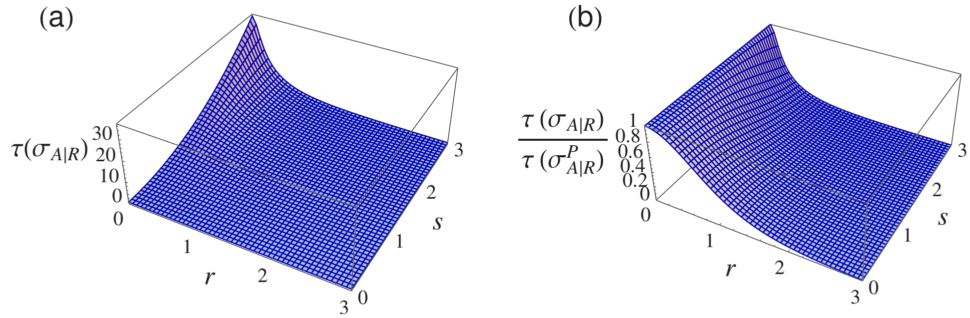

FIG. 3. (Color online) Bipartite entanglement described by Alice and the noninertial observer Rob, who moves with uniform acceleration parametrized by the effective squeezing r. From an inertial perspective, the field is in a two-mode squeezed state with squeezing degree s. (a) depicts the contangle $\tau(\boldsymbol{\sigma}_{A|R})$, given by Eqs. (5) and (18), as a function of r and s. In (b) the same quantity is normalized to the original contangle as seen by inertial observers, $\tau(\boldsymbol{\sigma}^P_{A|R})=4s^2$. Notice in (a) how the bipartite contangle is an increasing function of the entanglement s, while it decreases with increase in Rob's acceleration r, vanishing in the limit $r\rightarrow\infty$. This degradation is faster for higher s, as clearly visible in (b).

that all the two-mode reductions of $\boldsymbol{\sigma}_{AR\bar{R}}$ belong to the special class of mixed Gaussian states of maximal entanglement at given marginal mixednesses (GMEMMS) [40]. This is a curious coincidence because, when considering entanglement of Dirac fields from a noninertial perspective [4], and describing the effective three-qubit states described by the three observers, also in that case all two-qubit reduced states belong to the corresponding family of MEMMS mixed two-qubit states of maximal entanglement at fixed marginal mixednesses (MEMMS) [41]. Coming back to the CV case, this observation is useful as we know that for two-mode GMEMMS the Gaussian entanglement measures, including the contangle, are computable in closed form [27],

$$m_{A|R}=\frac{2\sinh^2(r)+[\cosh(2r)+3]\cosh(2s)}{2\cosh(2s)\sinh^2(r)+\cosh(2r)+3}, \tag{18}$$

$$m_{R|\bar{R}}=\cosh(2r). \tag{19}$$

Let us first comment on the quantum correlations created between the two Rindler regions I and II, given by Eq. (19). Note that the entanglement in the mixed state $\boldsymbol{\sigma}_{R\bar{R}}$ is exactly equal, in content to that of a pure two-mode squeezed state with squeezing r, regardless of the inertial Alice-Rob entanglement quantified by s. This provides a clearcut interpretation of the Unruh mechanism, in which the acceleration alone is responsible for the creation of entanglement between the accessible degrees of freedom described by Rob, and the inaccessible ones described by the virtual anti-Rob. By comparison with Ref. [7], we remark that, if the logarithmic negativity is used as an entanglement measure, this insightful picture is no longer true, as in that case the entanglement described by Rob and anti-Rob depends on s as well. While this is not surprising given the aforementioned inequivalence between negativities and Gaussian entanglement measures in quantifying quantum correlation of nonsymmetric mixed Gaussian states [27], it gives an indication that the negativity is probably not the best quantifier to capture the transformation of quantum information due to relativistic effects.

The proper quantification of Gaussian entanglement shows indeed that the quantum correlations are regulated by two competing squeezing degrees. One one hand, the resource parameter s regulates the entanglement $\tau(\boldsymbol{\sigma}^P_{A|R})=4s^2$ described by inertial observers. On the other hand, the acceleration parameter r regulates the uprising entanglement $\tau(\boldsymbol{\sigma}_{R|\bar{R}})=4s^2$ between the modes described by the uniformly accelerated Rob and by his *alter ego* anti-Rob. The latter entanglement, obviously, increases, to the detriment of the entanglement $\tau(\boldsymbol{\sigma}_{A|R})=g(m^2_{A|R})$ described by Alice and Rob from the noninertial perspective. Equation (18) shows in fact that $\tau(\boldsymbol{\sigma}_{A|R})$ is increasing with s and decreasing with r, as pictorially depicted in Fig. 3. Interestingly, the rate at which this bipartite entanglement degrades with r, $|\partial\tau(\boldsymbol{\sigma}_{A|R})/\partial r|$, increases with s: for higher s Alice and Rob describe the field as more entangled (from the inertial perspective, which corresponds to $r=0$), but it drops faster when the acceleration (r) comes into play. The same behavior is observed for the negativity [7]. For any inertial entanglement s, no quantum correlations are left in the infinite-acceleration limit ($r\rightarrow\infty$), when the state $\boldsymbol{\sigma}_{A|R}$ becomes asymptotically separable.

It is instructive to compare these results to the analysis of entanglement when the field (for $r=0$) is in a two-qubit Bell state $\sqrt{\frac{1}{2}}(|0\rangle_{\alpha_M}|0\rangle_{\rho_M}+|1\rangle_{\alpha_M}|1\rangle_{\rho_M})$, where $|1\rangle$ stands for the single-boson Fock state [3]. When one observer is accelerated, the state belongs to a three-partite Hilbert space with dimension $2\times\infty\times\infty$. The free entanglement in the state is degraded with increasing acceleration and vanishes in the infinite-acceleration limit. Figure 4 plots the entanglement between modes described by Alice and the noninertial Rob in such a qubit-CV setting [3], compared with the fully CV scenario considered in this paper. When the field described from the inertial perspective is in a two-mode squeezed Gaussian state with $s>1/2$, the entanglement is always stronger than the entanglement in the Bell-state case. We also observe that, even for $s<1/2$, the degradation of entanglement with acceleration is slower for the Gaussian state. The exploitation of all the infinitely many degrees of freedom available in the Hilbert space, therefore, results in an improved robustness of the entanglement against the thermalization induced by the Unruh effect.

In this context, we can pose the question of how much entanglement, at most, can Alice and the noninertial Rob

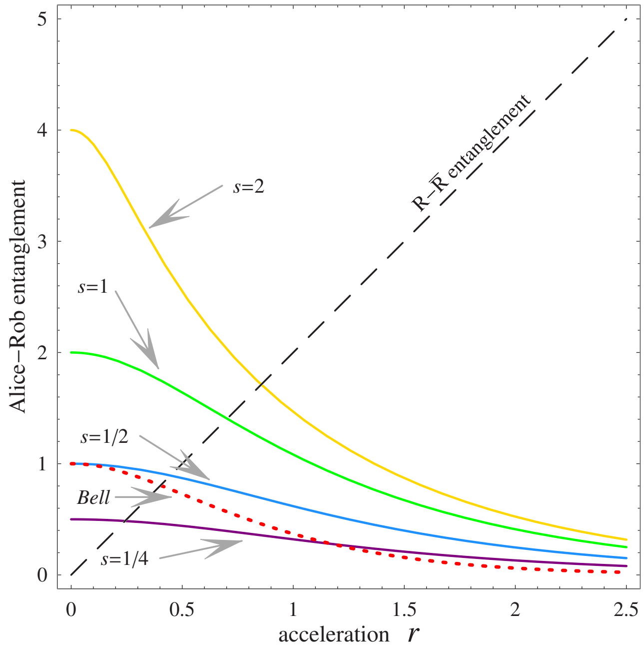

FIG. 4. (Color online) Bipartite entanglement between modes described by Alice and the noninertial Rob moving with uniform acceleration parametrized by r. The dotted red curve depicts the dependence of the logarithmic negativity between modes described by Alice and Rob in the instance of a state which corresponds to a two-qubit Bell state from the inertial perspective as computed in Ref. [3]. The other solid curves correspond to $\sqrt{\tau(\boldsymbol{\sigma}_{A|R})}$ (the square root of the contangle is taken to provide a fair dimensional comparison) as computed in this paper [see Eq. (18)], in the instance of an entangled two-mode squeezed state described from the perspective of two inertial observers, with different squeezing parameters $s=0.25,0.5,1,2$ (referring to the purple, blue, green, and gold curves, respectively). As a further comparison, the entanglement described by Rob and anti-Rob, given by $\sqrt{\tau(\boldsymbol{\sigma}_{R|\bar{R}})}=2r$ [see Eq. (19)] independently of s, is plotted as well (dashed black diagonal line).

hope to maintain, given that Rob is moving with a finite, known acceleration r. Assuming that from an inertial perspective the state is a perfect EPR state, we find

$$\lim_{s\to\infty} m_{A|R} = 1 + 2/\sinh^2(r), \tag{20}$$

meaning that the maximum entanglement left by the Unruh thermalization, out of an initial unlimited entanglement, approaches asymptotically

$$\tau_r^{\max}(\boldsymbol{\sigma}_{A|R}) = \operatorname{arcsinh}^2\left(\frac{2\cosh(r)}{\sinh^2(r)}\right). \tag{21}$$

Only for zero acceleration, $r=0$, does this maximum entanglement diverge. For any nonzero acceleration, the quantity $\tau_r^{\max}(\boldsymbol{\sigma}_{A|R})$ is finite and rapidly degrades with increasing r. This provides an upper bound on the effective quantum correlations and thus on the efficiency of any conceivable quantum-information protocol that Alice and the noninertial Rob may implement. For example, if Rob travels with a modest acceleration given by $r=0.5$, no more than 8 ebits of entanglement are left between the modes described by Alice and Rob, even if the state contained an infinite amount of entanglement from the point of view of inertial observers. This apparent "loss" of quantum information will be precisely understood in the next section, where we will show that the inertial bipartite entanglement does not disappear, but is redistributed into tripartite correlations among Alice, Rob, and anti-Rob.

B. Tripartite entanglement

A proper measure of genuine tripartite entanglement is available for any three-mode Gaussian state [19,39]. The measure, known as the "residual contangle," emerges from the monogamy inequality (9) and is an entanglement monotone under tripartite GLOCC for pure states. The residual contangle of a three-mode (i, j, and k) Gaussian state $\boldsymbol{\sigma}$ is defined as [19]

$$\tau(\boldsymbol{\sigma}_{i|j|k}) \equiv \min_{(i,j,k)}[\tau(\boldsymbol{\sigma}_{i|(jk)}) - \tau(\boldsymbol{\sigma}_{i|j}) - \tau(\boldsymbol{\sigma}_{i|k})], \tag{22}$$

where (i,j,k) denotes all the permutations of the three mode indices. For pure states, the minimum in Eq. (22) is always attained by the decomposition realized with respect to the probe mode i with smallest local determinant Det $\boldsymbol{\sigma}_i=m^2_{i|(jk)}$.

We can promptly apply this definition to compute the distributed tripartite entanglement in the state $\boldsymbol{\sigma}_{AR\bar{R}}$. From Eq. (16), we find that $m_{\bar{R}|(AR)}<m_{A|(R\bar{R})}$ for $r<r^*$, with

$$r^* = \operatorname{arccosh}\sqrt{\tanh^2(s)+1},$$

while $m_{R|(A\bar{R})}$ is always bigger than the other two quantities. Using Eqs. (5), (16), (18), (19), and (22) together with $\tau(\boldsymbol{\sigma}_{A|\bar{R}})=0$, we find that the residual contangle is given by

$$\begin{aligned}\tau(\boldsymbol{\sigma}_{A|R|\bar{R}}) &= \begin{cases} g(m^2_{\bar{R}|(AR)}) - g(m^2_{R|\bar{R}}), & r<r^*,\\ g(m^2_{A|(R\bar{R})}) - g(m^2_{A|R}) & \text{otherwise},\end{cases}\\ &= \begin{cases} -4r^2 + \operatorname{arcsinh}^2\sqrt{[\cosh^2(r)+\cosh(2s)\sinh^2(r)]^2-1}, & r<r^*,\\ 4s^2 - \operatorname{arcsinh}^2\sqrt{\dfrac{\{2\sinh^2(r)+[\cosh(2r)+3]\cosh(2s)\}^2}{[2\cosh(2s)\sinh^2(r)+\cosh(2r)+3]^2}-1} & \text{otherwise}.\end{cases}\end{aligned} \tag{23}$$

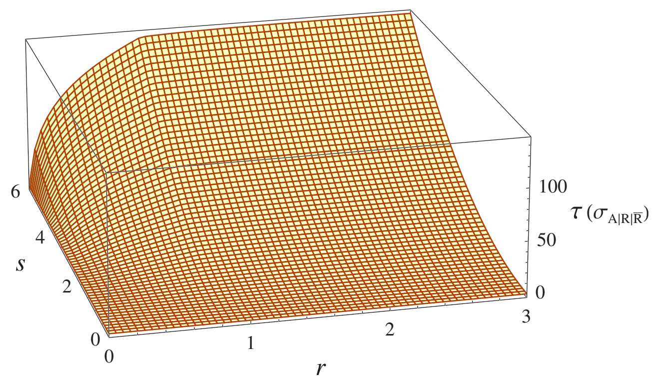

FIG. 5. (Color online) Genuine tripartite entanglement, as quantified by the residual contangle Eq. (23), among the inertial Alice, Rob in Rindler region I, and anti-Rob in Rindler region II, plotted as a function of the initial squeezing s and of Rob's acceleration r. The tripartite entanglement increases with r, and for $r\to\infty$ it approaches the original entanglement content $4s^2$ between modes described by Alice and Rob from the inertial perspective.

The tripartite entanglement is plotted in Fig. 5 as a function of r and s. Very remarkably, for any initial squeezing s, it increases with increasing acceleration r. In the limit of infinite acceleration, the bipartite entanglement between modes described by Alice and Rob vanishes so we have that

$$\lim_{r\to\infty} \tau(\boldsymbol{\sigma}_{A|R|\bar{R}}) = \tau(\boldsymbol{\sigma}_{A|(R\bar{R})}) = \tau(\boldsymbol{\sigma}^P_{A|R}) = 4s^2. \quad (24)$$

Precisely, the genuine tripartite entanglement tends asymptotically to the two-mode squeezed entanglement described by inertial observers.

We have now all the elements necessary to fully understand the Unruh effect on CV entanglement of bosonic particles, when a single observer is accelerated. Due to the fact that Rob is accelerated, from his perspective, (1) there is bipartite entanglement between the two modes in the two distinct Rindler regions, and this entanglement is a function only of the acceleration; (2) the bipartite entanglement described by inertial observers is redistributed into a genuine tripartite entanglement among the modes described by Alice, Rob, and anti-Rob. Therefore, as a consequence of the monogamy of entanglement, the entanglement between the two modes described by Alice and Rob is degraded.

In fact, there is no bipartite entanglement between the modes described by Alice and anti-Rob. This is very different from the distribution of entanglement of Dirac fields from a noninertial perspective [4], where the fermionic statistics does not allow the creation of maximal entanglement between the two Rindler regions. Therefore, the entanglement between modes described by Alice and Rob is never fully degraded. As a result of the monogamy constraints on entanglement sharing [33], the mode described by Alice becomes entangled with the mode described by anti-Rob, and the entanglement in the resulting three-qubit system is distributed in couplewise correlations, and a genuine tripartite entanglement is never created in that case [4].

In the next section, we will show how in the bosonic case the picture radically changes when both observers undergo uniform acceleration, in which case the relativistic effects are even more surprising.

C. Mutual information

It is interesting to compute the total (classical and quantum) correlations between modes described by Alice and the noninertial Rob, encoded in the reduced (mixed) two-mode state $\boldsymbol{\sigma}_{A|R}$ of Eq. (15), using the mutual information $I(\boldsymbol{\sigma}_{A|R})$, Eq. (7). The symplectic spectrum of such a state is constituted by $\eta^-_{A|R}=1$ and $\eta^+_{A|R}=\sqrt{\text{Det}\boldsymbol{\sigma}_{\bar{R}}}$. Since it belongs to the class of GMEMMS, it is in particular a mixed state of partial minimum uncertainty, which saturates inequality (2) [40]. Therefore, the mutual information reads

$$I(\boldsymbol{\sigma}_{A|R}) = f(\sqrt{\text{Det}\,\boldsymbol{\sigma}_A}) + f(\sqrt{\text{Det}\,\boldsymbol{\sigma}_R}) - f(\sqrt{\text{Det}\,\boldsymbol{\sigma}_{\bar{R}}}). \quad (25)$$

Explicitly,

$$\begin{aligned} I(\boldsymbol{\sigma}_{A|R}) = {} & \log[\cosh^2(s)\sinh^2(r)]\sinh^2(r)\cosh^2(s) \\ & + \log[\cosh^2(s)]\cosh^2(s) \\ & + \log[\cosh^2(r)\cosh^2(s)]\cosh^2(r)\cosh^2(s) \\ & - \log[\sinh^2(s)]\sinh^2(s) \\ & - \frac{1}{2}\log\left\{\frac{1}{2}[\cosh(2s)\cosh^2(r)+\sinh^2(r)-1]\right\} \\ & \times[\cosh(2s)\cosh^2(r)+\sinh^2(r)-1] \\ & - \frac{1}{2}\log\left\{\frac{1}{2}[\cosh^2(r)+\cosh(2s)\sinh^2(r)+1]\right\} \\ & \times[\cosh^2(r)+\cosh(2s)\sinh^2(r)+1]. \end{aligned}$$

The mutual information of Eq. (25) is plotted in Fig. 6(a) as a function of the squeezing degrees s (corresponding to the entanglement described from the inertial perspective) and r (reflecting Rob's acceleration). It is interesting to compare the mutual information with the original two-mode squeezed entanglement described between the inertial observers. In this case, it is more appropriate to quantify the entanglement in terms of the entropy of entanglement, $E_V(\boldsymbol{\sigma}^P_{A|R})$, defined as the von Neumann entropy of each reduced single-mode CM, $E_V(\boldsymbol{\sigma}^P_{A|R})\equiv S_V(\boldsymbol{\sigma}^P_A)\equiv S_V(\boldsymbol{\sigma}^P_B)$. That is,

$$E_V(\boldsymbol{\sigma}^P_{A|R}) = f(\cosh 2s), \quad (26)$$

with $f(x)$ given by Eq. (8). From the perspective of inertial observers ($r=0$), the state is pure, $\boldsymbol{\sigma}_{A|R}\equiv\boldsymbol{\sigma}^P_{A|R}$ and the mutual information is equal to twice the entropy of entanglement of Eq. (26), meaning that the two modes described by inertial observers are correlated in both quantum and classical descriptions to the same degree. When Rob is under acceleration ($r\neq 0$), the entanglement with the modes described by Alice is degraded by the Unruh effect (see Fig. 3), but the classical correlations are left untouched. In the limit $r\to\infty$, all entanglement is destroyed and the remaining mutual information $I(\boldsymbol{\sigma}_{A|R})$, quantifying classical correlations only, saturates to $E_V(\boldsymbol{\sigma}^P_{A|R})$ from Eq. (27). For any $s>0$ the mutual information of Eq. (25), once normalized by such entropy of entanglement [see Fig. 6(b)], ranges between 2 (1 normal-

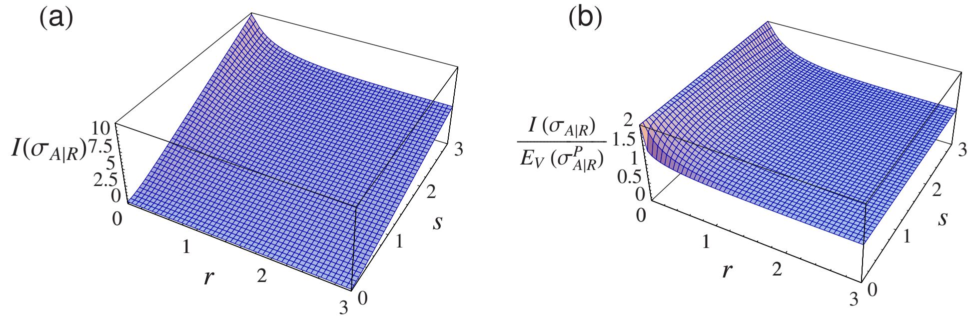

FIG. 6. (Color online) Total correlations between modes described by Alice and the noninertial observer Rob, moving with acceleration given by the effective squeezing parameter r. From an inertial perspective, the field is in a two-mode squeezed state with squeezing degree s. (a) depicts the dependence of the mutual information $I(\boldsymbol{\sigma}_{A|R})$, given by Eq. (25), as a function of r and s. In (b) the same quantity is normalized to the entropy of entanglement as perceived by inertial observers, $E_V(\boldsymbol{\sigma}^P_{A|R})$, Eq. (26). Notice in (a) how the mutual information is an increasing function of the inertial entanglement s; at variance with the entanglement (see Fig. 3), it saturates to a nonzero value in the limit of infinite acceleration. From (b), one clearly sees that this asymptotic value is exactly equal to the entropy of entanglement described by inertial observers.

ized unit of entanglement plus 1 normalized unit of classical correlations) at $r=0$, and 1 (all classical correlations and zero entanglement) at $r\to\infty$. The same behavior is found for classical correlations in the case of entangled states of the field which are bosonic two-qubit Bell states in an inertial perspective [3].

IV. DISTRIBUTED GAUSSIAN ENTANGLEMENT DUE TO BOTH ACCELERATED OBSERVERS

A natural question arises as to whether the mechanism of degradation or, to be precise, distribution of entanglement due to the Unruh effect is qualitatively modified according to the number of accelerated observers. One might guess that, when both observers travel with uniform acceleration, basically the same features as unveiled above for the case of a single noninertial observer will be manifest, with a merely quantitative rescaling of the relevant figures of merit (such as the bipartite entanglement degradation rate). However, we will now show that this is *not* the case.

We consider here two noninertial observers, with different names for ease of clarity and to avoid confusion with the previous picture. Leo and Nadia both travel with uniform accelerations $\aleph_L$ and $\aleph_N$, respectively, and describe the state of a scalar field in Rindler coordinates from their noninertial perspective (see Fig. 7). As in the previous instance, we consider that from the perspective of inertial observers only two field modes, of frequencies λ and ν, are entangled in a pure two-mode squeezed state $\boldsymbol{\sigma}^P_{LN}(s)$ of the form Eq. (4), with squeezing parameter s as before. Due to the acceleration of both observers, the entanglement is redistributed among modes described by four observers: Leo, Nadia (living in Rindler region I), and anti-Leo and anti-Nadia (living in Rindler region II). These four (some real and some virtual) parties will describe modes λ_I, ν_I, λ_{II}, and ν_{II}, respectively. By the same argument of Sec. III, the four observers will describe a pure four-mode Gaussian state with CM given by [42]

$$\boldsymbol{\sigma}_{\bar{L}LN\bar{N}}(s,l,n) = S_{\lambda_I,\lambda_{II}}(l)S_{\nu_I,\nu_{II}}(n)S_{\lambda_I,\nu_I}(s)\mathbb{I}_8 \times S^T_{\lambda_I,\nu_I}(s)S^T_{\nu_I,\nu_{II}}(n)S^T_{\lambda_I,\lambda_{II}}(l), \tag{27}$$

where the symplectic transformations S are given by Eq. (3), $\mathbb{I}_8$ is the CM of the vacuum $|0\rangle_{\lambda_{II}}\otimes|0\rangle_{\lambda_I}\otimes|0\rangle_{\nu_I}\otimes|0\rangle_{\nu_{II}}$, and l and n are the squeezing parameters associated with the respective accelerations $\aleph_L$ and $\aleph_N$ of Leo and Nadia [see Eq. (11)]. Explicitly,

$$\boldsymbol{\sigma}_{\bar{L}LN\bar{N}} = \begin{pmatrix} \boldsymbol{\sigma}_{\bar{L}} & \boldsymbol{\varepsilon}_{\bar{L}L} & \boldsymbol{\varepsilon}_{\bar{L}N} & \boldsymbol{\varepsilon}_{\bar{L}\bar{N}} \\ \boldsymbol{\varepsilon}^T_{\bar{L}L} & \boldsymbol{\sigma}_L & \boldsymbol{\varepsilon}_{LN} & \boldsymbol{\varepsilon}_{L\bar{N}} \\ \boldsymbol{\varepsilon}^T_{\bar{L}N} & \boldsymbol{\varepsilon}^T_{LN} & \boldsymbol{\sigma}_N & \boldsymbol{\varepsilon}_{N\bar{N}} \\ \boldsymbol{\varepsilon}^T_{\bar{L}\bar{N}} & \boldsymbol{\varepsilon}^T_{L\bar{n}} & \boldsymbol{\varepsilon}^T_{N\bar{N}} & \boldsymbol{\sigma}_{\bar{N}} \end{pmatrix}, \tag{28}$$

where:

$$\boldsymbol{\sigma}_{\bar{X}} = [\cosh^2(x) + \cosh(2s)\sinh^2(x)]\mathbb{I}_2,$$

$$\boldsymbol{\sigma}_X = [\cosh^2(x)\cosh(2s) + \sinh^2(x)]\mathbb{I}_2,$$

$$\boldsymbol{\varepsilon}_{\bar{X}X} = \boldsymbol{\varepsilon}_{X\bar{X}} = [\cosh^2(s)\sinh(2x)]Z_2,$$

$$\boldsymbol{\varepsilon}_{\bar{X}Y} = \boldsymbol{\varepsilon}_{Y\bar{X}} = [\cosh(y)\sinh(2s)\sinh(x)]\mathbb{I}_2,$$

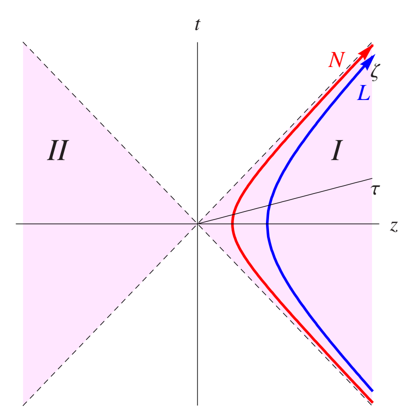

FIG. 7. (Color online) Sketch of the world lines for the two noninertial observers Leo and Nadia. The set (z,t) denotes Minkowski coordinates, while the set (ζ,τ) denotes Rindler coordinates. The causally disconnected Rindler regions I and II are shown.

$$\boldsymbol{\varepsilon}_{\bar{X}\bar{Y}} = [\sinh(2s)\sinh(x)\sinh(y)]Z_2,$$

$$\boldsymbol{\varepsilon}_{XY} = [\cosh(x)\cosh(y)\sinh(2s)]Z_2,$$

with $Z_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$; $X, Y = \{L, N\}$ with $X \neq Y$, and accordingly for the lower-case symbols $x, y = \{l, n\}$.

The very high-acceleration regime $(l, n \gg 0)$ can now be interpreted as Leo and Nadia both escaping the fall into the black hole by accelerating away from it with acceleration $\aleph_L$ and $\aleph_N$, respectively. Their entanglement will be degraded since part of the information is lost through the horizon into the black hole. Their acceleration makes part of the information unavailable to them. We will show that this loss involves both quantum and classical information.

A. Bipartite entanglement

We first recall that the original contangle $\tau(\boldsymbol{\sigma}^P_{L|N}) = 4s^2$ described by two inertial observers is preserved under the form of bipartite four-mode entanglement $\tau(\boldsymbol{\sigma}_{(\bar{L}L)|(N\bar{N})})$ between the two real and the two virtual observers, as the two Rindler change of coordinates amount to local unitary operations with respect to the $(\bar{L}L)|(N\bar{N})$ bipartition. The computation of the bipartite contangle in the various 1×1 partitions of the state $\boldsymbol{\sigma}_{\bar{L}LN\bar{N}}$ is still possible in closed form thanks to the results of Ref. [27]. From Eqs. (5) and (28), we find

$$m_{L|\bar{N}} = m_{N|\bar{L}} = m_{\bar{L}|\bar{N}} = 1, \tag{29}$$

$$m_{L|\bar{L}} = \cosh(2l), \quad m_{N|\bar{N}} = \cosh(2n), \tag{30}$$

$$m_{L|N} = \begin{cases} 1, & \tanh(s) \leq \sinh(l)\sinh(n), \\ \dfrac{2\cosh(2l)\cosh(2n)\cosh^2(s) + 3\cosh(2s) - 4\sinh(l)\sinh(n)\sinh(2s) - 1}{2\{[\cosh(2l) + \cosh(2n)]\cosh^2(s) - 2\sinh^2(s) + 2\sinh(l)\sinh(n)\sinh(2s)\}} & \text{otherwise}. \end{cases} \tag{31}$$

Let us first comment on the similarities with the setting of an inertial Alice and a noninertial Rob. In the case of two accelerated observers, Eq. (29) entails (we recall that $m=1$ means separability) that the mode described by Leo (Nadia) never gets entangled with the mode described by anti-Nadia (anti-Leo). Naturally, there is no bipartite entanglement generated between the modes described by the two virtual observers $\bar{L}$ and $\bar{N}$. Another similarity found in Eq. (30) is that the reduced two-mode state $\boldsymbol{\sigma}_{X\bar{X}}$ assigned to each observer $X = \{L, N\}$ and her or his respective virtual counterpart $\bar{X}$ is exactly of the same form as $\boldsymbol{\sigma}_{R\bar{R}}$. Therefore, due to the fact that Leo and Nadia are accelerated, from their perspective we find again that a bipartite contangle is present between the mode described by each observer in region I and the corresponding causally disconnected mode described by the respective *alter ego* virtual observer in region II; this entanglement is a function of the corresponding acceleration $x = \{l, n\}$ only. The two entanglements corresponding to each observer-antiobserver pair are mutually independent, and for each the $X|\bar{X}$ entanglement content is again the same as that of a pure, two-mode squeezed state created with squeezing parameter x.

The only entanglement that is physically accessible to the noninertial observers is encoded in the two modes λ_I and ν_I corresponding to Rindler regions I of Leo and Nadia. These two modes are left in the state $\boldsymbol{\sigma}_{LN}$, which is not a GMEMMS (like the state $\boldsymbol{\sigma}_{AR}$ in Sec. III) but a nonsymmetric thermal squeezed state [40], for which the Gaussian entanglement measures are available as well [27]. The contangle of such a state is in fact given by Eq. (31). Here we find a first significant qualitative difference with the case of a single accelerated observer: a state entangled from an inertial perspective can become disentangled for two noninertial observers, both traveling with *finite* acceleration. Equation (31) shows that there is a trade-off between the amount of entanglement (s) described from an inertial perspective and the acceleration parameters of both parties (l and n). If the observers are highly accelerated [namely, if $\sinh(l)\sinh(n)$ exceeds $\tanh(s)$], the entanglement in the state $\boldsymbol{\sigma}_{LN}$ vanishes, or better said, becomes physically inaccessible to the noninertial observers. Even in the ideal case, where the state contains infinite entanglement (corresponding to $s \to \infty$) from the perspective of inertial observers, the entanglement *completely* vanishes from the perspective of one inertial and one noninertial observer if $\sinh(l)\sinh(n) \geq 1$. We find here another important difference from the Dirac case where entanglement never vanishes for two noninertial observers [4]. Conversely, for any nonzero, arbitrarily small acceleration parameters l and n, there is a threshold on the entanglement s such that, if the entanglement is smaller than the threshold, it vanishes when described from the perspective of one inertial and one noninertial observer. With only one noninertial observer, instead (Sec. III), any infinitesimal entanglement will survive for arbitrarily large acceleration, vanishing only in the infinite-acceleration limit.

To provide a better comparison between the two settings, let us address the following question. Can the entanglement degradation observed by Leo and Nadia (with acceleration parameters l and n, respectively) be observed by an inertial Alice and a noninertial Rob traveling with some effective acceleration r^{eff}? We will look for a value of r^{eff} such that the reduced state $\boldsymbol{\sigma}_{AR}$ of the three-mode state in Eq. (15) is as entangled as the reduced state $\boldsymbol{\sigma}_{LN}$ of the four-mode state in Eq. (28). The problem can be straightforwardly solved by equating the corresponding contangles Eq. (18) and Eq. (31), to obtain

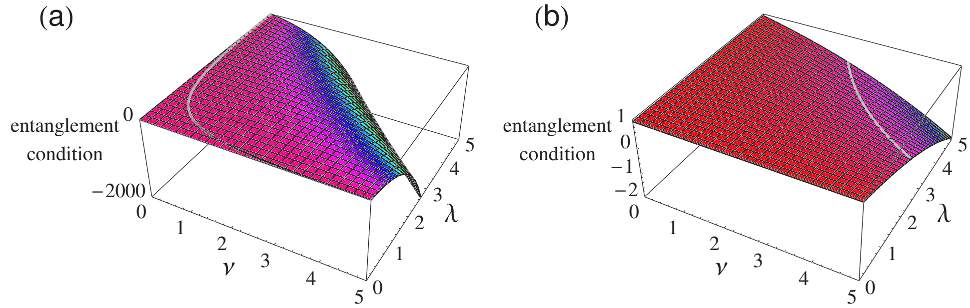

FIG. 8. (Color online) Entanglement condition (33) for different frequency modes, assuming that Leo and Nadia have the same acceleration $\aleph=2\pi$ (a) and (b) 10π. Entanglement is present only in the frequency range where the plotted surfaces assume negative values, and vanishes for frequencies where the plots become positive; the threshold [saturation of (33)] is highlighted with a black line. Only modes whose frequencies are sufficiently high exhibit bipartite entanglement. For higher accelerations of the observers, the range of entangled frequency modes gets narrower, and in the infinite-acceleration limit the bipartite entanglement between all frequency modes vanishes.

$$r^{eff}=\begin{cases}\operatorname{arccosh}\left(\dfrac{\cosh(l)\cosh(n)\sinh(s)}{\sinh(s)-\cosh(s)\sinh(l)\sinh(n)}\right), & \tanh(s)>\sinh(l)\sinh(n),\\ \infty & \text{otherwise}.\end{cases} \qquad (32)$$

Clearly, for very high acceleration parameters l and n (or, equivalently, very small inertial entanglement s), the information loss due to the noninertiality of both observers is only matched by an infinite effective acceleration in the case of a single noninertial observer. In the regime in which entanglement does not completely vanish, the effective acceleration of Rob in the equivalent single-noninertial-observer setting is a function of the inertial entanglement s, as well as of the accelerations of Leo and Nadia.

1. Entanglement between different frequency modes

The condition on the acceleration parameters l and n for which the entanglement of the maximally entangled state ($s\to\infty$) vanishes, from Eq. (31), corresponds to the condition $e^{\pi\Omega_L}+e^{\pi\Omega_N}-e^{\pi(\Omega_L+\Omega_N)}\geqslant 0$ where $\Omega_L=2\lambda/(\aleph_L)$ and $\Omega_N=2\nu/(\aleph_N)$. Here we recall that $\aleph_{L,N}$ are the proper accelerations of the two noninertial observers and λ,ν the frequencies of the respective modes [see Eq. (11)]. We assume now that Leo and Nadia have the same acceleration,

$$\aleph_L=\aleph_N\equiv\aleph,$$

and ask the question, given their acceleration, which frequency modes would they describe as entangled? This provides a deeper understanding of the effect of the Unruh thermalization on the distribution of CV correlations.

Our results immediately show that in this context the entanglement *vanishes* between field modes such that

$$e^{(2\pi/\aleph)\lambda}+e^{(2\pi/\aleph)\mu}-e^{(2\pi/\aleph)(\lambda+\nu)}\geqslant 0. \qquad (33)$$

This means that if the field is, from the inertial perspective, in a two-mode squeezed state with frequencies satisfying Eq. (33), the accelerated Leo and Nadia would describe the field as in a two-mode disentangled (separable) state. We have thus a practical condition to determine which modes would be entangled from Leo and Nadia's noninertial perspective, depending on their frequency.

In Fig. 8 we plot the condition on entanglement for different frequency modes. The modes become disentangled when the graph takes positive values. We see that only modes with the highest frequencies exhibit bipartite entanglement for a given acceleration $\aleph$ of the observers. The larger the acceleration, the fewer modes remain entangled, as expected. In the limit of infinite acceleration $\lambda/(\aleph_L),\nu/(\aleph_N)\to\infty$, the set of entangled modes becomes empty. In the high-acceleration regime, where Alice and Rob escape the fall into a black hole, only very high-frequency modes remain entangled.

Considering once more equally accelerated observers, $\aleph_L=\aleph_N\equiv\aleph$ with finite $\aleph$, it is straightforward to compute the contangle of the modes that do remain entangled, in the case of a state that is maximally (infinitely) entangled from the inertial perspective. From Eq. (31), we have

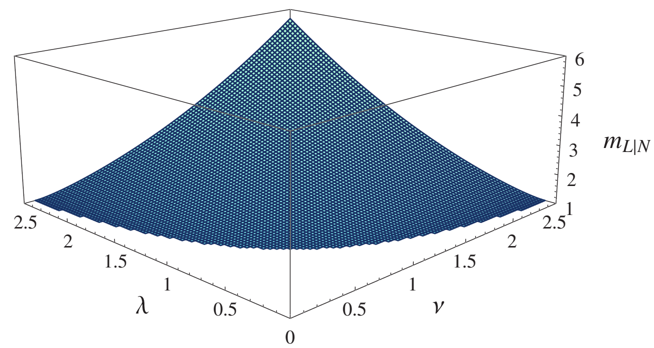

FIG. 9. (Color online) Entanglement between different frequency modes assuming that Leo and Nadia have the same acceleration $\aleph=2\pi$.

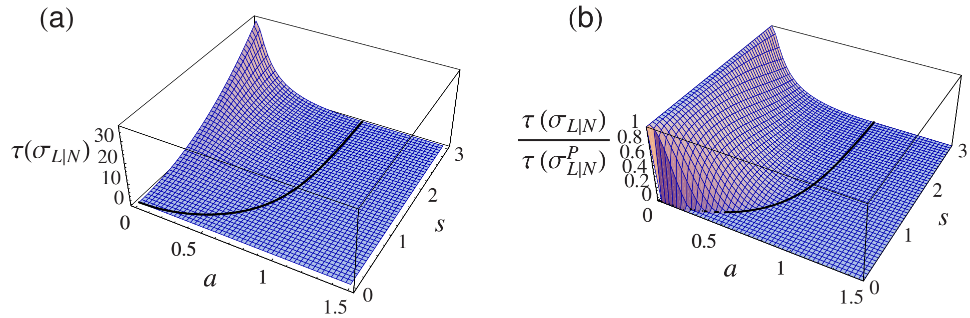

FIG. 10. (Color online) Bipartite entanglement between the modes described by the two noninertial observers Leo and Nadia, both traveling with uniform acceleration given by the effective squeezing parameter a. From an inertial perspective, the field is in a two-mode squeezed state with squeezing degree s. (a) depicts the contangle $\tau(\boldsymbol{\sigma}_{L|N})$, given by Eqs. (5) and (37), as a function of a and s. In (b) the same quantity is normalized to the inertial contangle, $\tau(\boldsymbol{\sigma}^P_{L|N})=4s^2$. Notice in (a) how the bipartite contangle is an increasing function of the inertial entanglement s, while it decreases with increasing acceleration a. This degradation is faster for higher s, as clearly visible in (b). At variance with the case of only one accelerated observer (Fig. 3), in this case the bipartite entanglement can be completely destroyed at finite acceleration. The black line depicts the threshold acceleration $a^*(s)$, Eq. (36), such that for $a \geqslant a^*(s)$ the bipartite entanglement described by the two noninertial observers is exactly zero.

$$m_{L|N}(s \to \infty) = \frac{\cosh(2l)\cosh(2n) - 4\sinh(l)\sinh(n) + 3}{2[\sinh(l)+\sinh(n)]^2}. \tag{34}$$

In Fig. 9 we plot the entanglement between the modes, Eq. (34), as a function of their frequency λ and ν [using Eq. (11)] when Leo and Nadia have acceleration $\aleph=2\pi$. We see that, consistently with the previous analysis, at fixed acceleration, the entanglement is larger for higher frequencies. In the infinite-acceleration limit, as already remarked, entanglement vanishes for all frequency modes.

2. Equal acceleration parameters

We proceed now to analyze the bipartite entanglement in the field from the noninertial perspective, at fixed mode frequencies. For simplicity, we restrict our attention to the case where Leo and Nadia's trajectories have the same acceleration parameter

$$l = n \equiv a. \tag{35}$$

This means that $\lambda/\aleph_L=\nu/\aleph_N$. While the following results do not rely on this assumption, it is particularly useful in order to provide a pictorial representation of entanglement in the four-mode state $\boldsymbol{\sigma}_{L\bar{L}N\bar{N}}$, which is now parametrized only by the two competing squeezing degrees, the inertial quantum correlations (s) and the acceleration parameter of both observers (a). In this case, the acceleration parameter a^*, for which the entanglement between the modes described by Leo and Nadia vanishes, is

$$a^*(s) = \text{arcsinh}[\sqrt{\tanh(s)}], \tag{36}$$

where we used Eq. (31). The contangle in the state $\boldsymbol{\sigma}_{LN}$ is therefore given by

$$m_{L|N} = \begin{cases} 1, & a \geqslant a^*(s), \\ \dfrac{2\cosh^2(2a)\cosh^2(s) + 3\cosh(2s) - 4\sinh^2(a)\sinh(2s) - 1}{4[\cosh^2(a)+e^{2s}\sinh^2(a)]} & \text{otherwise}, \end{cases} \tag{37}$$

which we plot in Fig. 10. The entanglement increases with s and decreasing with a with a stronger rate of degradation for increasing s. The main difference from Fig. 3 is that entanglement here completely vanishes at finite acceleration. Even if the state contains infinite entanglement as described by inertial observers, entanglement vanishes at $a \geqslant \text{arcsinh}(1) \approx 0.8814$.

B. Residual multipartite entanglement

It is straightforward to show that the four-mode state $\boldsymbol{\sigma}_{L\bar{L}N\bar{N}}$ of Eq. (28) is fully inseparable, which means that it contains multipartite entanglement distributed among all the four parties involved. This follows from the observation that the determinant of each reduced one- and two-mode CM obtainable from $\boldsymbol{\sigma}_{L\bar{L}N\bar{N}}$ is strictly bigger than 1 for any nonzero squeezings. This in addition to the global purity of the state means that there is entanglement across all global bipartitions of the four modes. We now aim to provide a quantitative characterization of such multipartite entanglement.

For the sake of simplicity, we focus once more on the case of two observers with equal acceleration parameter a. The state under consideration is obtained from Eq. (28) via the prescription Eq. (35). The entanglement properties of this four-mode pure Gaussian state have been investigated in detail by some of us [43]. We showed, in particular, that the entanglement sharing structure in such state is infinitely *promiscuous*. The state admits the coexistence of an unlimited, genuine four-partite entanglement, together with an accordingly unlimited bipartite entanglement in the reduced two-mode states of two pair of parties, here referred to as {Leo, anti-Leo} and {Nadia, anti-Nadia}. Both four-partite and bipartite correlations increase with a. We will now review the study of multipartite entanglement of these four-partite Gaussian states shown in Ref. [43], with the particular aim of showing the effects of relativistic acceleration in the distribution of quantum information.

1. Monogamy inequality

We begin by verifying that the state $\boldsymbol{\sigma}_{\bar{L}LN\bar{N}}$ satisfies the fundamental monogamy inequality (9) for the entanglement distributed among the four parties (each one describing a single mode). To this end, we compute the pure-state contangle between one probe mode and the remaining three modes. From Eq. (5) we find

$$\begin{aligned} m_{\bar{L}|(LN\bar{N})} &= m_{\bar{N}|(NL\bar{L})} = \cosh^2 a + \cosh(2s)\sinh^2 a, \\ m_{L|(\bar{L}N\bar{N})} &= m_{N|(\bar{N}L\bar{L})} = \sinh^2 a + \cosh(2s)\cosh^2 a. \end{aligned} \tag{38}$$

Thanks to the explicit expressions Eqs. (29), (31), and (38) for the bipartite entanglements, proving monogamy reduces to showing that $\min\{g(m^2_{\bar{L}|(LN\bar{N})}) - g(m^2_{\bar{L}|L}), g(m^2_{L|(\bar{L}N\bar{N})}) - g(m^2_{\bar{L}|L}) - g(m^2_{L|N})\}$ is nonnegative. One can verify that the first quantity always achieves the minimum; therefore we define

$$\tau^{res}(\boldsymbol{\sigma}_{\bar{L}LN\bar{N}}) \equiv \tau(\boldsymbol{\sigma}_{\bar{L}|(LN\bar{N})}) - \tau(\boldsymbol{\sigma}_{\bar{L}|L}) = \text{arcsinh}^2\{\sqrt{[\cosh^2 a + \cosh(2s)\sinh^2 a]^2 - 1}\} - 4a^2 > 0. \tag{39}$$

The residual contangle τ^{res} is positive as $\cosh(2s) > 1$ for $s > 0$, and it quantifies precisely the multipartite correlations that cannot be stored in bipartite form. Those quantum correlations, however, can be either tripartite, involving three of the four modes, and/or genuinely four-partite among all of them. We can now quantitatively estimate to what extent such correlations are encoded in some tripartite form: as an anticipation, we will find them negligible in the limit of high acceleration.

2. Tripartite entanglement

Let us first observe that in the tripartitions $\bar{L}|L|\bar{N}$ and $\bar{L}|N|\bar{N}$ the tripartite entanglement is exactly zero. This is because the mode described by anti-Nadia is not entangled with the modes described by the pair {Leo, anti-Leo}, and the mode described by anti-Leo is not entangled with the modes described by the pair {Nadia, anti-Nadia}. The corresponding three-mode states are then said to be biseparable [38]. The only tripartite entanglement present, if any, is equal in content (due to the symmetry of the state) for the tripartitions $\bar{L}|L|N$ and $L|N|\bar{N}$. It is properly quantified by the residual tripartite contangle $\tau(\boldsymbol{\sigma}_{\bar{L}|L|N})$ emerging from the corresponding mixed-state three-mode monogamy inequality, via Eq. (22). In Ref. [43] an upper bound on $\tau(\boldsymbol{\sigma}_{\bar{L}|L|N})$ was obtained. Its derivation is recalled in the Appendix for the sake of completeness. From Eq. (A6) we have

$$\begin{aligned} \tau(\boldsymbol{\sigma}_{\bar{L}|L|N}) &\leq \tau^{bound}(\boldsymbol{\sigma}_{\bar{L}|L|N}) \\ &\equiv \min\Bigg\{ g\Bigg[\Bigg(\cosh^2 a + \frac{1+\text{sech}^2 a \tanh^2 s}{1-\text{sech}^2 a \tanh^2 s}\sinh^2 a\Bigg)^2\Bigg] - 4a^2, g\Bigg[\Bigg(\frac{1+\text{sech}^2 a \tanh^2 s}{1-\text{sech}^2 a \tanh^2 s}\Bigg)^2\Bigg] - g(m^2_{L|N})\Bigg\}, \end{aligned} \tag{40}$$

with $m_{L|N}$ obtainable by substituting Eq. (35) in Eq. (31).

The upper bound $\tau^{bound}(\boldsymbol{\sigma}_{\bar{L}|L|N})$ is of course always nonnegative (as a consequence of monogamy); it decreases with increasing acceleration a, and vanishes in the limit $a \to \infty$. Therefore, in the regime of increasingly high a, eventually approaching infinity, any form of tripartite entanglement among any three modes in the state $\boldsymbol{\sigma}_{\bar{L}LN\bar{N}}$ is negligible (exactly zero in the limit of infinite acceleration).

3. Genuine four-partite entanglement

The above analysis of the tripartite contribution to multipartite entanglement shows that, in the regime of high acceleration a, the residual entanglement τ^{res} determined by Eq. (39) is stored entirely in the form of four-partite quantum correlations. Therefore, the residual entanglement in this case is a good measure of *genuine* four-partite entanglement among the four Rindler spacetime modes. It is now straightforward to see that $\tau^{res}(\boldsymbol{\sigma}_{\bar{L}LN\bar{N}})$ is itself an *increasing* function of a for any value of s (see Fig. 11), and it *diverges* in the limit $a \to \infty$.

The four-mode state Eq. (39) obtained with an arbitrarily large acceleration a, consequently, exhibits a coexistence of unlimited genuine four-partite entanglement, and pairwise bipartite entanglement in the reduced two-mode states $\boldsymbol{\sigma}_{L|\bar{L}}$ and $\boldsymbol{\sigma}_{N|\bar{N}}$. This peculiar distribution of CV entanglement in the considered Gaussian state has been defined as *infinitely promiscuous* in Ref. [43]. The properties of such entangled states are discussed in Ref. [43] in a practical optical setting.

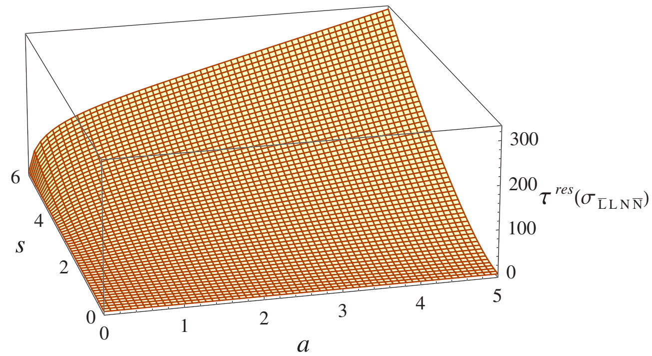

FIG. 11. (Color online) Residual contangle, Eq. (39), not stored in bipartite form, distributed among the modes described by the noninertial observers Rob and Nadia in Rindler region I and their virtual counterparts anti-Leo and anti-Nadia in Rindler region II, as quantified by the residual contangle Eq. (23), among the inertial Alice, plotted as a function of the initial squeezing s and of the acceleration a of both observers. In the regime of high acceleration ($a\to\infty$), the displayed residual entanglement is completely distributed in the form of genuine four-partite quantum correlations. This four-partite entanglement is monotonically increasing with increasing acceleration a, and diverges as a approaches infinity.

It is interesting to note that, in the relativistic analysis we present here, the genuine four-partite entanglement increases without bound with the observers' acceleration. This is in fact in strong contrast with the case of an inertial observer and an accelerating one (Sec. III), where we find that, in the infinite-acceleration limit, the genuine tripartite entanglement saturates at $4s^2$ (i.e., the original entanglement encoded between the two inertial observers).

In the scenario considered here, the fact that Leo and Nadia are accelerated is perceived as an *ex novo* entanglement (function of the acceleration) across the Rindler horizon, interlinking independently both mode pairs described by the corresponding observers. The information loss is such that, even if the state contains infinite entanglement when described by inertial observers, it contains no quantum correlations when described by two observers traveling at finite acceleration. If one considers even higher acceleration of the observers, it is basically the (inaccessible) entanglement modes in the Rindler region I and modes in the Rindler region II which are redistributed into genuine four-partite form. The tripartite correlations tend to vanish as a consequence of the thermalization, which destroys the inertial bipartite entanglement. The multipartite entanglement, obviously, increases infinitely with acceleration because the entanglement between causally disconnected modes increases without bound with acceleration. It is remarkable that such promiscuous distribution of entanglement can occur without violating the fundamental monogamy constraints on entanglement sharing [19,20].

To give a simple example, suppose the bipartite entanglement contained in the state described by inertial observers is given by $4s^2=16$ for $s=2$. If both observers accelerate such that the two-mode squeezed state described from the inertial perspective has squeezing parameter $a=7$, the four-partite entanglement [given by Eq. (39)] is 81.2 ebits, more than five times the inertial bipartite entanglement. At the same time, a bipartite entanglement of $4a^2=196$ is present, from a noninertial perspective, between modes in region I and modes in region II. A final caution needs to be stated. The above results suggest that unbounded entanglement is created by merely the observers' motion. This requires, of course, an unlimited energy needed to fuel their spaceships, let alone all the technicalities of realizing such a situation in practice. Unfortunately, this entanglement is mostly inaccessible, as both Leo and Nadia are confined in their respective Rindler region I. The only entanglement resource they are left with is the degraded two-mode thermal squeezed state.

C. Mutual information

It is very interesting to evaluate the mutual information $I(\boldsymbol{\sigma}_{L|N})$ between the states described by Leo and Nadia, both moving with acceleration parameter a.

In this case the symplectic spectrum of the reduced (mixed) two-mode CM $\boldsymbol{\sigma}_{L|N}$ of Eq. (28) is degenerate [40], yielding $\eta^{-}_{L|N}=\eta^{+}_{L|N}=(\mathrm{Det}\,\boldsymbol{\sigma}_{L|N})$. From Eq. (7), the mutual information then reads

$$I(\boldsymbol{\sigma}_{L|N}) = f(\sqrt{\mathrm{Det}\boldsymbol{\sigma}_L}) + f(\sqrt{\mathrm{Det}\boldsymbol{\sigma}_N}) - 2f[(\mathrm{Det}\boldsymbol{\sigma}_{L|N})^{1/4}]. \tag{41}$$

Explicitly,

$$\begin{aligned}
I(\boldsymbol{\sigma}_{L|N}) = {} & 2\cosh^2(a)\cosh^2(s)\log[\cosh^2(a)\cosh^2(s)] \\
& - [\cosh(2s)\cosh^2(a) + \sinh^2(a) \\
& - 1]\log\left\{\frac{1}{2}[\cosh(2s)\cosh^2(a) + \sinh^2(a) - 1]\right\} \\
& + \frac{1}{2}\{[2\cosh(2s)\sinh^2(2a) + \cosh(4a) + 3]^{1/2} \\
& - 2\}\log\{[2\cosh(2s)\sinh^2(2a) + \cosh(4a) + 3]^{1/2} \\
& - 2\} - \frac{1}{2}\{[2\cosh(2s)\sinh^2(2a) + \cosh(4a) + 3]^{1/2} \\
& + 2\}\log\{[2\cosh(2s)\sinh^2(2a) + \cosh(4a) + 3]^{1/2} \\
& + 2\} + \log(16).
\end{aligned}$$

We plot in Fig. 12 the mutual information both directly, and normalized to the inertial entropy of entanglement, which is equal to Eq. (26),

$$E_V(\boldsymbol{\sigma}^{P}_{L|N}) = f(\cosh 2s), \tag{42}$$

with $f(x)$ given by Eq. (8). We immediately notice another interesting effect. Not only is the entanglement completely destroyed at finite acceleration, but also classical correlations are degraded [see Fig. 12(b)]. This is very different from the case with a single noninertial observer where classical correlations remain invariant.

The asymptotic state described by Leo and Nadia, in the infinite-acceleration limit $a\to\infty$, contains indeed some residual classical correlations (whose amount is an increasing function of the squeezing s). But these correlations are *always* smaller than the classical correlations described from

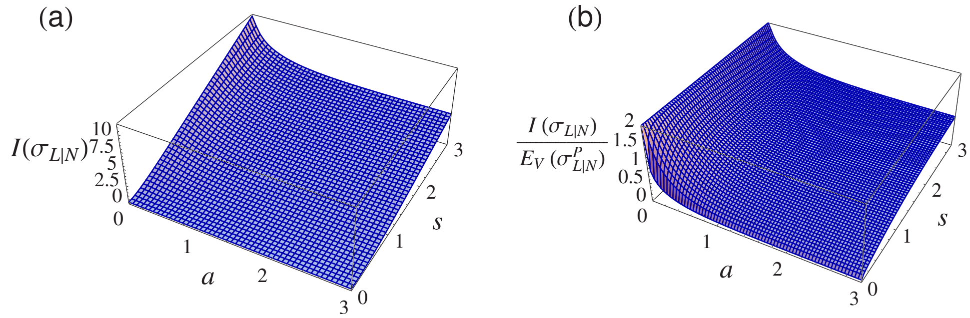

FIG. 12. (Color online) Total correlations between the modes described by the two noninertial observers Leo and Nadia, traveling with equal, uniform acceleration given by the effective squeezing parameter a. From the inertial perspective, the field is in a two-mode squeezed state with squeezing degree s. (a) shows the mutual information $I(\boldsymbol{\sigma}_{L|N})$, given by Eq. (41), as a function of a and s. In (b) the same quantity is normalized to the entropy of entanglement perceived by inertial observers, $E_V(\boldsymbol{\sigma}^P_{L|N})$, Eq. (42). Notice in (a) how the mutual information is an increasing function of the squeezing parameter s and saturates to a nonzero value in the limit of infinite acceleration; in contrast, the entanglement vanishes at finite acceleration (see Fig. 10). (b) shows that this asymptotic value is smaller than the entropy of entanglement described by inertial observers (which is equal to the classical correlations described by the inertial observers). Therefore, classical correlations are also degraded when both observers are accelerated, in contrast to the case where only one observer is in uniform acceleration (see Fig. 10).

the inertial perspective, given by Eq. (42). Classical correlations are robust against the effects of the double acceleration only when the classical correlations in the state described by inertial observers are infinite (corresponding to infinite entanglement from the point of view of inertial observers, $s \to \infty$). The entanglement, however, is always fragile, since we have seen that it is completely destroyed at a finite, relatively small acceleration parameter a.

Another interesting fact is that, comparing Figs. 6(a) and 12(a), one sees that in both cases (either one or two noninertial observers) the mutual information between the two "real" observers is a function of the acceleration parameter and of the initial squeezing. In the case of both accelerated observers, however, the mutual information is always smaller, as we have just discussed. It is interesting to study the difference between them (where we set for ease of comparison equal acceleration parameters, $r=a$, where r regulates Rob's acceleration when Alice is inertial, and a is related to the acceleration of both Leo and Nadia in the present situation),

$$D(a,s) = I(\boldsymbol{\sigma}_{A|R})\big|_{r=a} - I(\boldsymbol{\sigma}_{L|N}). \tag{43}$$

The quantity $D(a,s)$ is plotted in Fig. 13: surprisingly, it is strictly bounded. It increases with both s and a, but in the asymptotic limit of infinite inertial entanglement, $D(a,s \to \infty)$ saturates exactly to 1 (as can be checked analytically) for any $a>0$. We remark that both mutual informations $I(\boldsymbol{\sigma}_{A|R})$ and $I(\boldsymbol{\sigma}_{L|N})$ diverge in this limit: yet their difference is finite and equal to 1. Clearly, the small deficit of the mutual information seen when both observers are accelerated is reflected as loss of classical correlations, as plotted in Fig. 12(b). Mysteriously, the Unruh thermalization affects classical correlations when both observers are accelerated: however, it degrades at most one absolute unit of classical correlations. This means that in the case when both Leo and Nadia escape the fall into a black hole, not only is their entanglement degraded but *there is also a loss of classical information* [22].

V. DISCUSSION AND OUTLOOK

We presented a thorough study of classical and quantum correlations between modes of a scalar field described by observers in uniform acceleration. By considering the state of the field in the simplest multimode squeezed state possible (the two-mode case) from the perspective of inertial observers, we were able to investigate in detail the entanglement in all partitions of the system from noninertial perspectives, specifically when one observer is in uniform acceleration and when both of them are accelerated. We found that in both settings the accessible entanglement is degraded with the observers' acceleration and we explained this degradation as an effect of redistribution of the entanglement in the state described from an inertial perspective.

Our main results can be summarized as follows. When one of the observers is accelerated, the entanglement lost

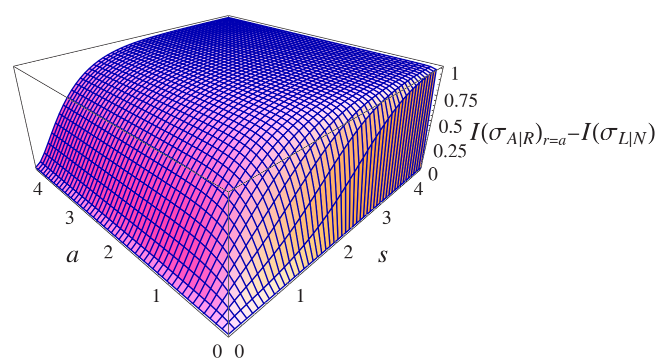

FIG. 13. (Color online) Plot, as a function of the acceleration parameter a and the squeezing parameter s, of the difference between the mutual information described by the inertial Alice and the noninertial Rob, and the mutual information described by the uniformly accelerating Leo and Nadia, as given by Eq. (43).

between the modes described by him and the inertial observer is re-distributed into tripartite correlations. No entanglement is generated between the mode described by the inertial observer and the modes in the causally disconnected region II. This shows that indeed the behavior for bosonic fields is very different from the Dirac case, where the entanglement lost from the perspective of noninertial observers is redistributed not into tripartite correlations but into bipartite correlations between the mode described by the inertial observer and the mode in region II. The analysis of the mutual information shows that in this case classical correlations are conserved independently of the acceleration. The situation changes drastically when considering both observers to be accelerated. In this case the entanglement lost from the noninertial perspective is redistributed into mainly four-partite correlations although some tripartite correlations exist for finite acceleration. The surprising result here (though expected in the framework of distributed entanglement, as the additional fourth mode comes into play) is that entanglement vanishes completely at a finite acceleration. This is also drastically different from the results in the Dirac case, where entanglement remains positive for all accelerations (as a direct consequence of the restricted Hilbert space in that instance). Another surprising result in this case is that we find that classical correlations are no longer invariant to acceleration but are also degraded to some extent. We analyzed the entanglement between the modes of the field described by two accelerated observers as a function of their frequencies, and found that for a fixed acceleration high-frequency modes remain entangled while lower-frequency modes disentangle. In the limit of infinitely accelerated observers, the field modes are in a separable state for any pair of frequencies.

The tools developed in this paper can be used to investigate the problem of information loss in black holes [22]. There is a correspondence between the Rindler-Minkowski frames and the Schwarzschild-Kruskal frames [13,17] that allows us to study the loss (and redistribution) of quantum and classical correlations for observers describing entangled modes outside the black hole, extending and reinterpreting the results presented in Sec. IV of this paper. In that case the degradation of correlations can be understood as essentially being due to the Hawking effect [12].

Furthermore, all our results can in principle be corroborated experimentally in a quantum-optics setting. The role of the acceleration in the description of the field can be reproduced by the effects of a nonlinear crystal through the mechanism of parametric down-conversion [24]. The results of Sec. IV, for instance, can be applied to study the efficient generation and entanglement characterization in four-mode Gaussian states of light beams [43]. In such a setting, each of the modes can be really accessed and manipulated, and the different types of entanglement can be described by true observers and employed as a resource for bipartite and/or multipartite transmission and processing of CV quantum information [14,16].

We are currently interested in the study of classical and quantum correlations in general multimode squeezed states which involve several modes being pairwise entangled [17]. The study of entanglement in these states will provide a deeper understanding of quantum information in quantum field theory in curved spacetimes.

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APPENDIX: UPPER BOUND ON THE MIXED-STATE TRIPARTITE ENTANGLEMENT IN PRESENCE OF TWO ACCELERATED OBSERVERS

We are interested in computing the residual tripartite contangle, Eq. (22), distributed among modes described by observers anti-Leo, Leo, and Nadia in the reduced mixed three-mode state $\boldsymbol{\sigma}_{\bar{L}LN}$ obtained from Eq. (28) (with $l=n\equiv a$) by tracing over the degrees of freedom of anti-Nadia. To quantify such tripartite entanglement exactly, it is necessary to compute the three-mode bipartite contangle between one mode and the block of the two other modes. This requires solving the nontrivial optimization problem of Eq. (5) over all possible pure three-mode Gaussian states. However, from the definition itself Eq. (5), the bipartite contangle $\tau(\boldsymbol{\sigma}_{i|(jk)})$ (with i,j,k a permutation of $\bar{L},L,N$) is bounded from above by the corresponding bipartite contangle $\tau(\boldsymbol{\sigma}^p_{i|(jk)})$ in any pure, three-mode Gaussian state with CM $\boldsymbol{\sigma}^p_{i|(jk)}\leqslant\boldsymbol{\sigma}_{i|(jk)}$. As an ansatz we can choose pure three-mode Gaussian states whose CM $\boldsymbol{\sigma}^p_{\bar{L}LN}$ has the same matrix structure as our mixed state $\boldsymbol{\sigma}_{\bar{L}LN}$ (in particular, zero correlations between position and momentum operators, and diagonal subblocks proportional to the identity), and restrict the optimization to such class of states. This task is accomplished by choosing a pure state given by the following CM [43]:

$$\boldsymbol{\gamma}^p_{\bar{L}LN}=S_{\lambda_I,\lambda_{II}}(a)S_{\lambda_I,\nu_I}(t)\mathbb{I}_6(t)S^T_{\lambda_I,\nu_I}S_{\lambda_I,\nu_I}(t)S_{\lambda_I,\lambda_{II}}(a),\quad\text{(A1)}$$

where we have adopted the notation of Eq. (27), and

$$t=\frac{1}{2}\operatorname{arccosh}\left(\frac{1+\operatorname{sech}^2 a\tanh^2 s}{1-\operatorname{sech}^2 a\tanh^2 s}\right).$$

We have then

$$\tau(\boldsymbol{\sigma}_{i|(jk)})\leqslant g[(m^{\gamma}_{i|(jk)})^2],\quad\text{(A2)}$$

where m^{γ} is meant to determine entanglement in the state $\boldsymbol{\gamma}^p$, Eq. (A1), via Eq. (5). The bipartite entanglement properties of the state $\boldsymbol{\gamma}^p$ can be determined analogously to what was done in Sec. III. We find

$$m^{\gamma}_{N|(\bar{L}L)}=\frac{1+\operatorname{sech}^2 a\tanh^2 s}{1-\operatorname{sech}^2 a\tanh^2 s},\quad\text{(A3)}$$

$$m^{\gamma}_{\bar{L}|(LN)}=\cosh^2 a+m^{\gamma}_{N|(\bar{L}L)}\sinh^2 a,\quad\text{(A4)}$$

$$m^{\gamma}_{L|(\bar{L}N)} = \sinh^2 a + m^{\gamma}_{\bar{N}|(\bar{L}L)} \cosh^2 a. \tag{A5}$$

Equations (22) and (A2) thus lead to

$$\tau(\boldsymbol{\sigma}_{\bar{L}|L|N}) \leq \min\{g[(m^{\gamma}_{\bar{L}|(LN)})^2] - g(m^2_{\bar{L}|L}), g[(m^{\gamma}_{N|(\bar{L}L)})^2] - g(m^2_{L|N})\}, \tag{A6}$$

where the two-mode entanglements m without the superscript γ refer to the reductions of the mixed state $\boldsymbol{\sigma}_{\bar{L}|L|N}$ and are listed in Eqs. (29)–(32). In Eq. (A6) the quantity $g[(m^{\gamma}_{L|(\bar{L}N)})^2] - g(m^2_{\bar{L}|L}) - g(m^2_{L|N})$ is not included in the minimization, being always larger than the other two terms. Numerical investigations in the space of all pure three-mode Gaussian states seem to confirm that the upper bound of Eq. (A6) is actually tight (meaning that the three-mode contangle is globally minimized on the state $\boldsymbol{\gamma}^p$), but this statement can be left as a conjecture since it is not required for the subsequent analysis of Sec. IV B.

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