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Electromagnetic field quantization in a magnetodielectric medium with external charges

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The electromagnetic field inside a cubic cavity filled with a linear magnetodielectric medium and in the presence of external charges is quantized by modeling the magnetodielectric medium with two independent quantum fields. The electric and magnetic polarization densities of the medium are defined in terms of the ladder operators of the medium and eigenmodes of the cavity. The Maxwell and constitutive equations of the medium together with the equation of motion of the charged particles have been obtained from the Heisenberg equations using a minimal coupling scheme. The spontaneous emission of a two-level atom embedded in a magnetodielectric medium is calculated in terms of the electric and magnetic susceptibilities of the medium and the Green function of the cubic cavity, as an application of the model.

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I. INTRODUCTION

The usual approach to quantum optics in nondispersive dielectric media is to introduce the medium using its linear or nonlinear susceptibilities. In one procedure, the macroscopic fields are used to build an effective Lagrangian density whose Euler-Lagrange equations are identical to the macroscopic Maxwell's equations in the dielectric medium, where the constitutive equations couple the polarization field to the electric field [1–7]. However, attempts to add dispersion to this effective scheme have run into difficulties [7,8]. The reason for this is that inclusion of dispersion leads to a temporally nonlocal relationship between the electric field and the displacement field, and the effective Lagrangian in this case is also nonlocal in time and cannot be used directly in a quantization scheme. Another problem is the inclusion of losses into the system. It is well known that the dissipative nature of a medium is an immediate consequence of its dispersive character, and vice versa, according to the Kramers-Kronig relations [9]. This suggests that in order to quantize the electromagnetic field in a dielectric medium in a way that is consistent with the Kramers-Kronig relations, one has to introduce the medium into the formalism explicitly. This should be done in such a way that the interaction between light and matter will generate both dispersion and damping of the light field. In an attempt to overcome these problems, by taking the polarization of the medium as a dissipative quantum system and based on the Hopfield model of a dielectric medium [10,11], a canonical quantization of the electromagnetic field inside a dispersive and absorptive dielectric medium can be presented, where its polarization is modeled by a collection of interacting matter fields [12,13]. The absorptive character of the medium is modeled through the interaction of the matter fields with a reservoir consisting of a continuum of the Klein-Gordon fields. In this model, eigenoperators for coupled systems are calculated and the electromagnetic field has been expressed in terms of them. The dielectric function is derived in terms of a coupling function which couples the polarization of the medium to the reservoir.

One approach to studying a quantum dissipative system tries to relate dissipation to the interaction between the system and a heat bath containing a collection of harmonic oscillators [14–26]. In this method, the whole system is composed of two parts, the main system and a bath, which interacts with the main system and causes the dissipation of energy on it. In order to quantize the electromagnetic field in the presence of an absorptive magnetodielectric medium, it is reasonable to take the electromagnetic field as the main quantum dissipative system and the medium as the heat bath. In this point of view, the polarizability of the medium is a result of the properties of the heat bath and, accordingly, the polarizability should be defined in terms of the dynamical variables of the medium. In this method, in contrast to the damped polarization model, the polarizability and absorptivity of the medium are not independent of each other [12,13]. Furthermore, if the medium is magnetizable as well as polarizable, the medium can be modeled with two independent collections of harmonic oscillators, so that one of these collections describes the electric properties and the other describes the magnetic properties of the medium. This scheme proposes a consistent quantization of the electromagnetic field in the presence of an absorptive magnetodielectric [27].

In the present paper, we generalize the quantization scheme presented in [27] to the case where there are some external charges in the medium. We take a cubic cavity filled with a linear magnetodielectric medium, containing some external charges, as the medium. As a simple application of the method, we study the spontaneous decay of an initially excited two-level atom in a dispersing and absorbing magnetodielectric medium.

II. QUANTUM DYNAMICS

Quantum electrodynamics in a linear magnetodielectric medium can be accomplished by modeling the medium with two independent quantum fields that interact with the electromagnetic field. One of these quantum fields, namely, the E quantum field, describes the polarizability character of the medium and interacts with a displacement field through a

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minimal coupling term. The other quantum field, the M quantum field, describes the magnetizability character of the medium and interacts with the magnetic field through a dipole interaction term. The Heisenberg equations for the electromagnetic field and the E and M quantum fields give not only the Maxwell equations but also the constitutive equations of the medium and the equations of motion of external charges.

For this purpose let us consider an ideal rectangular cavity with sides L_1, L_2, and L_3, along the coordinate axes x, y, and z, respectively, filled with a magnetodielectric medium. The vector potential of the electromagnetic field in the Coulomb gauge can be expanded as [29]

$$\vec{A}(\vec{r},t) = \sum_{\vec{n}} \sum_{\lambda=1}^{2} \sqrt{\frac{4\hbar}{\varepsilon_0 V \omega_{\vec{n}}}} [a_{\vec{n}\lambda}(t) + a^{\dagger}_{\vec{n}\lambda}(t)] \vec{u}_{\vec{n}\lambda}(\vec{r}), \quad (1)$$

where

$$\vec{u}_{\vec{n}\lambda}(\vec{r}) = e_x(\vec{n},\lambda) f_1(\vec{n},\vec{r})\hat{x} + e_y(\vec{n},\lambda) f_2(\vec{n},\vec{r})\hat{y} + e_z(\vec{n},\lambda) f_3(\vec{n},\vec{r})\hat{z},$$

$$f_1(\vec{n},\vec{r}) = \cos\frac{n_1\pi x}{L_1}\sin\frac{n_2\pi y}{L_2}\sin\frac{n_3\pi z}{L_3},$$

$$f_2(\vec{n},\vec{r}) = \sin\frac{n_1\pi x}{L_1}\cos\frac{n_2\pi y}{L_2}\sin\frac{n_3\pi z}{L_3},$$

$$f_3(\vec{n},\vec{r}) = \sin\frac{n_1\pi x}{L_1}\sin\frac{n_2\pi y}{L_2}\cos\frac{n_3\pi z}{L_3}, \quad (2)$$

the mode vector $\vec{n}$ is a triplet of natural numbers (n_1,n_2,n_3), $\Sigma_{\vec{n}}$ means $\Sigma_{n_1,n_2,n_3=1}^{\infty}$, $V=L_1L_2L_3$ is the volume of the cavity, ε_0 is the permittivity of the vacuum, $\omega_{\vec{n}} = c\sqrt{n_1^2\pi^2/L_1^2 + n_2^2\pi^2/L_2^2 + n_3^2\pi^2/L_3^2}$ is the frequency corresponding to the mode $\vec{n}$, and $\vec{e}(\vec{n},\lambda)$ $(\lambda=1,2)$ are polarization vectors that satisfy

$$\vec{e}(\vec{n},\lambda)\cdot\vec{e}(\vec{n},\lambda') = \delta_{\lambda\lambda'},$$

$$\vec{e}(\vec{n},\lambda)\cdot\vec{k}_{\vec{n}} = 0, \quad (3)$$

where $\vec{k}_{\vec{n}} = (n_1\pi/L_1)\vec{i} + (n_2\pi/L_2)\vec{j} + (n_3\pi/L_3)\vec{k}$ is the wave vector. The operators $a_{\vec{n}\lambda}(t)$ and $a^{\dagger}_{\vec{n}\lambda}(t)$ are annihilation and creation operators of the electromagnetic field and satisfy the following equal time commutation rules:

$$[a_{\vec{n}\lambda}(t), a^{\dagger}_{\vec{m}\lambda'}(t)] = \delta_{\vec{n},\vec{m}}\delta_{\lambda\lambda'}. \quad (4)$$

Let $\vec{F}(\vec{r},t)$ be an arbitrary vector field; the transverse component $\vec{F}^{\perp}(\vec{r},t)$ and the longitudinal component $\vec{F}^{\parallel}(\vec{r},t)$ of $\vec{F}(\vec{r},t)$ are defined as

$$\vec{F}^{\perp}(\vec{r},t) = \vec{F}(\vec{r},t) + \int_V d^3r'\vec{\nabla}'\cdot\vec{F}(\vec{r}',t)\vec{\nabla}G(\vec{r},\vec{r}'), \quad (5)$$

$$\vec{F}^{\parallel}(\vec{r},t) = -\int_V d^3r'\vec{\nabla}'\cdot\vec{F}(\vec{r}',t)\vec{\nabla}G(\vec{r},\vec{r}'), \quad (6)$$

where $G(\vec{r},\vec{r}')$ is the Green function of the cubic cavity,

$$G(\vec{r},\vec{r}') = \frac{8}{V}\sum_{\vec{n}}\frac{1}{|\vec{k}_{\vec{n}}|^2}\sin\frac{n_1\pi x}{L_1}\sin\frac{n_2\pi y}{L_2}\sin\frac{n_3\pi z}{L_3} \times\sin\frac{n_1\pi x'}{L_1}\sin\frac{n_2\pi y'}{L_2}\sin\frac{n_3\pi z'}{L_3}. \quad (7)$$

In the presence of external charges the displacement field is not purely transverse. The transverse component of the displacement field can be expanded in terms of the cavity modes as

$$\vec{D}^{\perp}(\vec{r},t) = -i\varepsilon_0\sum_{\vec{n}}\sum_{\lambda=1}^{2}\sqrt{\frac{4\omega_{\vec{n}}\hbar}{\varepsilon_0 V}}[a^{\dagger}_{\vec{n}\lambda}(t) - a_{\vec{n}\lambda}(t)]\vec{u}_{\vec{n}\lambda}(\vec{r}). \quad (8)$$

The commutation relations between the components of the vector potential $\vec{A}$ and the transverse components of the displacement field, $\vec{D}^{\perp}$, can be obtained from (4) as follows:

$$[A_l(\vec{r},t), -D_j^{\perp}(\vec{r}',t)] = \imath\hbar\,\delta_{lj}^{\perp}(\vec{r}-\vec{r}'), \quad (9)$$

where $\delta_{lj}^{\perp}(\vec{r}-\vec{r}')$ is the transverse delta function defined in terms of the eigenvector fields $\vec{u}_{\vec{n}\lambda}$,

$$\delta_{lj}^{\perp}(\vec{r}-\vec{r}') = \frac{8}{V}\sum_{\vec{n}}\sum_{\lambda=1}^{2}u^{l}_{\vec{n}\lambda}(\vec{r})u^{j}_{\vec{n}\lambda}(\vec{r}') = \frac{8}{V}\sum_{\vec{n}}\left(\delta_{lj} - \frac{k_{\vec{n}l}k_{\vec{n}j}}{|\vec{k}_{\vec{n}}|^2}\right)f_l(\vec{n},\vec{r})f_j(\vec{n},\vec{r}'). \quad (10)$$

The Hamiltonian of the electromagnetic field inside a magnetodielectric medium in the normal ordering form is

$$H_F(t) = \int_V d^3r{:}\left(\frac{\vec{D}^{\perp}\cdot\vec{D}^{\perp}}{2\varepsilon_0} + \frac{(\vec{\nabla}\times\vec{A})^2}{2\mu_0}\right){:} = \sum_{\vec{n}}\sum_{\lambda=1}^{2}\hbar\omega_{\vec{n}}a^{\dagger}_{\vec{n}\lambda}(t)a_{\vec{n}\lambda}(t), \quad (11)$$

where μ_0 is the magnetic permittivity of the vacuum and denotes the normal ordering operator. By modeling the magnetodielectric medium with E and M quantum fields, we can write the Hamiltonian of the medium as the sum of the Hamiltonians of the E and M quantum fields,

$$H_d = H_e + H_m,$$

$$H_e(t) = \sum_{\vec{n}}\sum_{\nu=1}^{3}\int_{-\infty}^{+\infty}d^3k\,\hbar\omega_{\vec{k}}d^{\dagger}_{\vec{n}\nu}(\vec{k},t)d_{\vec{n}\nu}(\vec{k},t),$$

$$H_m(t) = \sum_{\vec{n}} \sum_{\nu=1}^{3} \int_{-\infty}^{+\infty} d^3k\, \hbar\omega_{\vec{k}} b^{\dagger}_{\vec{n}\nu}(\vec{k},t) b_{\vec{n}\nu}(\vec{k},t), \tag{12}$$

where $\omega_{\vec{k}}$ is the dispersion relation of the medium. The quantum dynamics of a dissipative harmonic oscillator interacting with an absorptive environment can be investigated by modeling the environment with a continuum of harmonic oscillators [14–18]. In the case of quantization of an electromagnetic field in the presence of a magnetodielectric medium, the electromagnetic field is the main dissipative system and the medium plays the role of the absorptive environment. The Hamiltonian (11) shows that the electromagnetic field contains a numerable set of harmonic oscillators labeled by $\vec{n},\lambda$. Therefore, to each harmonic oscillator of the electromagnetic field labeled by $\vec{n},\lambda$, a continuum of oscillators should correspond. In the present scheme, for each harmonic oscillator of the electromagnetic field labeled by $\vec{n},\lambda$, we have two continuous sets of harmonic oscillators defined by the ladder operators $d_{\vec{n}\nu}(\vec{k},t),d^{\dagger}_{\vec{n}\nu}(\vec{k},t)$ and $b_{\vec{n}\nu}(\vec{k},t),b^{\dagger}_{\vec{n}\nu}(\vec{k},t)$, which describe the electric and magnetic properties of the medium, respectively, and satisfy the equal time commutation relations

$$[d_{\vec{n}\nu}(\vec{k},t),d^{\dagger}_{\vec{m}\nu'}(\vec{k}',t)] = \delta_{\vec{n},\vec{m}}\delta_{\nu\nu'}\delta(\vec{k}-\vec{k}'),$$

$$[b_{\vec{n}\nu}(\vec{k},t),b^{\dagger}_{\vec{m}\nu'}(\vec{k}',t)] = \delta_{\vec{n},\vec{m}}\delta_{\nu\nu'}\delta(\vec{k}-\vec{k}'). \tag{13}$$

If we want to extend the Huttner and Barnett model [12] to a magnetodielectric medium, we have to take a vector field $\vec{X}$ as the electric polarization density of the medium and a heat bath B interacting with $\vec{X}$ in order to take into account the absorption due to the dispersion induced by the electrical properties of the medium. Similarly, we must take a vector field $\vec{Y}$ as the magnetic polarization density and a heat bath $\tilde{B}$ (independent of B) interacting with $\vec{Y}$ in order to take into account the absorption due to the dispersion induced by the magnetic properties of the medium. Following the Huttner and Barnett method, we find that a Fano diagonalization [11] process for the sum of the Hamiltonians related to the electric polarization $\vec{X}$, the heat bath B, and their interaction leads to what we have called H_e in Eq. (12). Also, a Fano diagonalization process for the sum of the Hamiltonians related to the magnetic polarization $\vec{Y}$, the heat bath $\tilde{B}$, and their interaction leads to what we have called H_m in Eq. (12). If we use the Huttner and Barnett approach inside a rectangular cavity, the labels $\vec{n}$ and ν will appear in the expansion of the electric polarization field $\vec{X}$ in terms of the eigenmodes of the cavity. On the other hand, the heat bath B interacting with the polarization field $\vec{X}$ consists of a continuous set of harmonic oscillators labeled by a continuous parameter ω. It should be noted that the three labels $\vec{n}$, ν, and ω remain in H_e after the diagonalization process.

For a linear medium, the electric polarization density operator can be written as a linear combination of ladder operators $d_{\vec{n}\nu}(\vec{k},t)$ and $d^{\dagger}_{\vec{n}\nu}(\vec{k},t)$:

$$\vec{P}(\vec{r},t) = \sqrt{\frac{8}{V}} \sum_{\vec{n}} \sum_{\nu=1}^{3} \int d^3\vec{k} [f(\omega_{\vec{k}},\vec{r}) d_{\vec{n}\nu}(\vec{k},t) + \text{H.c.}] \vec{v}_{\vec{n}\nu}(\vec{r}) \tag{14}$$

with

$$\vec{v}_{\vec{n}\nu}(\vec{r}) = \vec{u}_{\vec{n}\nu}(\vec{r}), \quad \nu = 1,2,$$

$$\vec{v}_{\vec{n}3}(\vec{r}) = \hat{k}_{\vec{n}x} f_1(\vec{n},\vec{r})\hat{x} + \hat{k}_{\vec{n}y} f_2(\vec{n},\vec{r})\hat{y} + \hat{k}_{\vec{n}z} f_3(\vec{n},\vec{r})\hat{z}, \quad \hat{k}_{\vec{n}} = \frac{\vec{k}_{\vec{n}}}{|\vec{k}_{\vec{n}}|}. \tag{15}$$

The function $f(\omega_{\vec{k}},\vec{r})$ is the coupling function of the electromagnetic field and the E quantum field. Also, for a linearly magnetizable medium, we can express the magnetic polarization density operator of the medium as a linear combination of the ladder operators of the M quantum field [27],

$$\vec{M}(\vec{r},t) = \sqrt{\frac{8}{V}} \sum_{\vec{n}} \sum_{\nu=1}^{3} \int d^3\vec{k} [g(\omega_{\vec{k}},\vec{r}) b_{\vec{n}\nu}(\vec{k},t) + \text{H.c.}] \vec{s}_{\vec{n}\nu}(\vec{r}), \tag{16}$$

where the function $g(\omega_{\vec{k}},\vec{r})$ couples the electromagnetic field to the M quantum field and

$$\vec{s}_{\vec{n}\nu}(\vec{r}) = \frac{\vec{\nabla}\times\vec{u}_{\vec{n}\nu}}{|\vec{k}_{\vec{n}}|}, \quad \nu = 1,2,$$

$$\vec{s}_{\vec{n}3}(\vec{r}) = \hat{k}_{\vec{n}x} g_1(\vec{n},\vec{r})\hat{x} + \hat{k}_{\vec{n}y} g_2(\vec{n},\vec{r})\hat{y} + \hat{k}_{\vec{n}z} g_3(\vec{n},\vec{r})\hat{z},$$

$$g_1(\vec{n},\vec{r}) = \sin\frac{n_1\pi x}{L_1}\cos\frac{n_2\pi y}{L_2}\cos\frac{n_3\pi z}{L_3},$$

$$g_2(\vec{n},\vec{r}) = \cos\frac{n_1\pi x}{L_1}\sin\frac{n_2\pi y}{L_2}\cos\frac{n_3\pi z}{L_3},$$

$$g_3(\vec{n},\vec{r}) = \cos\frac{n_1\pi x}{L_1}\cos\frac{n_2\pi y}{L_2}\sin\frac{n_3\pi z}{L_3}. \tag{17}$$

Now let the total Hamiltonian, i.e., the electromagnetic field, the E and M quantum fields, and the external charges, be

$$\begin{aligned}\tilde{H}(t) = {} & \int_V d^3r \left(\frac{[\vec{D}^{\perp}(\vec{r},t) - \vec{P}(\vec{r},t)]^2}{2\varepsilon_0} + \frac{(\vec{\nabla}\times\vec{A})^2(\vec{r},t)}{2\mu_0} - \vec{\nabla}\times\vec{A}(\vec{r},t)\cdot\vec{M}(\vec{r},t) \right) + H_e + H_m \\ & + \sum_{\alpha=1}^{N} \frac{[\vec{p}_\alpha - q_\alpha\vec{A}(\vec{r}_\alpha,t)]^2}{2m_\alpha} + \frac{1}{2\varepsilon_0}\sum_{\alpha\neq\beta} q_\alpha q_\beta G(\vec{r}_\alpha,\vec{r}_\beta) \\ & - \frac{1}{\varepsilon_0}\sum_{\alpha=1}^{N} q_\alpha \int_V d^3r' G(\vec{r}_\alpha,\vec{r}')\vec{\nabla}'\cdot\vec{P}(\vec{r}',t),\end{aligned} \tag{18}$$

where q_α, m_α, $\vec{p}_\alpha$, and $\vec{r}_\alpha$ are the charge, mass, linear momentum, and position of the αth particle, respectively. The function G is the Green function given by (7). Using (9), the Heisenberg equation for $\vec{A}$ and $\vec{D}^{\perp}$ can be obtained as

$$\frac{\partial\vec{A}(\vec{r},t)}{\partial t}=\frac{\imath}{\hbar}[\tilde{H},\vec{A}(\vec{r},t)]=-\frac{\vec{D}^{\perp}(\vec{r},t)-\vec{P}^{\perp}(\vec{r},t)}{\varepsilon_0},\qquad(19)$$

$$\frac{\partial\vec{D}^{\perp}(\vec{r},t)}{\partial t}=\frac{\imath}{\hbar}[\tilde{H},\vec{D}^{\perp}(\vec{r},t)]=\frac{\vec{\nabla}\times\vec{\nabla}\times\vec{A}(\vec{r},t)}{\mu_0}-\vec{\nabla}\times\vec{M}^{\perp}(\vec{r},t)-\vec{J}^{\perp}(\vec{r},t),\qquad(20)$$

where

$$J_i^{\perp}(\vec{r},t)=\sum_{\alpha=1}^{N}\sum_{l=1}^{3}q_\alpha\frac{d\vec{r}_\alpha}{dt}\delta_{li}^{\perp}(\vec{r}_\alpha(t)-\vec{r})\qquad(21)$$

is the transverse component of the external current density. If we define the transverse electrical field $\vec{E}^{\perp}$, induction $\vec{B}$, and magnetic field $\vec{H}$ as

$$\vec{E}^{\perp}=-\frac{\partial\vec{A}}{\partial t},\quad\vec{B}=\vec{\nabla}\times\vec{A},\quad\vec{H}=\frac{\vec{B}}{\mu_0}-\vec{M},\qquad(22)$$

then (19) and (20) can be rewritten as

$$\vec{D}^{\perp}=\varepsilon_0\vec{E}^{\perp}+\vec{P}^{\perp},\qquad(23)$$

$$\frac{\partial\vec{D}^{\perp}}{\partial t}+\vec{J}^{\perp}=\vec{\nabla}\times\vec{H},\qquad(24)$$

as expected. In the presence of external charges, the longitudinal components of the electrical and displacement fields can be written as

$$\vec{E}^{\parallel}(\vec{r},t)=-\frac{1}{\varepsilon_0}\sum_{\alpha=1}^{N}q_\alpha\vec{\nabla}_rG(\vec{r},\vec{r}_\alpha(t))-\frac{\vec{P}^{\parallel}}{\varepsilon_0},\qquad(25)$$

$$\vec{D}^{\parallel}(\vec{r},t)=\varepsilon_0\vec{E}^{\parallel}+\vec{P}^{\parallel}=-\sum_{\alpha=1}^{N}q_\alpha\vec{\nabla}_rG(\vec{r},\vec{r}_\alpha(t)),\qquad(26)$$

where $\vec{P}^{\parallel}$ is the longitudinal component of the electric polarization density. If we apply the Heisenberg equation to the ladder operators of the medium, we obtain the macroscopic constitutive equations of the medium, which relate the electric and magnetic polarization densities to the electric and magnetic fields, respectively. Using the commutation relations (13), we find from the Heisenberg equations

$$\dot{d}_{\vec{n}\nu}(\vec{k},t)=\frac{\imath}{\hbar}[\tilde{H},d_{\vec{n}\nu}(\vec{k},t)]=-\imath\omega_{\vec{k}}d_{\vec{n}\nu}(\vec{k},t)+\frac{\imath}{\hbar}\sqrt{\frac{8}{V}}\int_V d^3\vec{r}'f^*(\omega_{\vec{k}},\vec{r}')\vec{E}(\vec{r}',t)\cdot\vec{v}_{\vec{n}\nu}(\vec{r}'),\qquad(27)$$

$$\dot{b}_{\vec{n}\nu}(\vec{k},t)=\frac{\imath}{\hbar}[\tilde{H},b_{\vec{n}\nu}(\vec{k},t)]=-\imath\omega_{\vec{k}}b_{\vec{n}\nu}(\vec{k},t)+\frac{\imath}{\hbar}\sqrt{\frac{8}{V}}\int_V d^3r'g^*(\omega_{\vec{k}},\vec{r}')\vec{B}(\vec{r}',t)\cdot\vec{s}_{\vec{n}\nu}(\vec{r}'),\qquad(28)$$

with the following formal solutions:

$$d_{\vec{n}\nu}(\vec{k},t)=d_{\vec{n}\nu}(\vec{k},0)e^{-\imath\omega_{\vec{k}}t}+\frac{\imath}{\hbar}\sqrt{\frac{8}{V}}\int_0^t dt'e^{-\imath\omega_{\vec{k}}(t-t')}\times\int_V d^3r'f^*(\omega_{\vec{k}},\vec{r}')\vec{E}(\vec{r}',t')\cdot\vec{v}_{\vec{n}\nu}(\vec{r}'),\qquad(29)$$

$$b_{\vec{n}\nu}(\vec{k},t)=b_{\vec{n}\nu}(\vec{k},0)e^{-\imath\omega_{\vec{k}}t}+\frac{\imath}{\hbar}\sqrt{\frac{8}{V}}\int_0^t dt'e^{-\imath\omega_{\vec{k}}(t-t')}\times\int_V d^3r'g^*(\omega_{\vec{k}},\vec{r}')\vec{B}(\vec{r}',t')\cdot\vec{s}_{\vec{n}\nu}(\vec{r}').\qquad(30)$$

By substituting (29) in (14) and using the completeness relations

$$\sum_{\vec{m}}\sum_{\nu=1}^{3}v_{\vec{m}\nu}^{\alpha}(\vec{r})v_{\vec{m}\nu}^{\beta}(\vec{r}')=\sum_{\vec{m}}\sum_{\nu=1}^{3}s_{\vec{m}\nu}^{\alpha}(\vec{r})s_{\vec{m}\nu}^{\beta}(\vec{r}')=\frac{V}{8}\delta_{\alpha\beta}\delta(\vec{r}-\vec{r}'),\qquad(31)$$

we can find the constitutive equation

$$\vec{P}(\vec{r},t)=\vec{P}_N(\vec{r},t)+\varepsilon_0\int_0^{|t|}dt'\chi_e(\vec{r},|t|-t')\vec{E}(\vec{r},\pm t')\qquad(32)$$

for the medium, which relates the electric polarization density to the total electric field, where the upper (lower) sign corresponds to $t>0$ ($t<0$), respectively, and $\vec{E}=\vec{E}^{\perp}+\vec{E}^{\parallel}$ is the total electrical field with $\vec{E}^{\perp}$ and $\vec{E}^{\parallel}$ defined in (22) and (25). The memory function

$$\chi_e(\vec{r},t)=\begin{cases}\dfrac{8\pi}{\hbar\varepsilon_0}\displaystyle\int_0^\infty d|\vec{k}||\vec{k}|^2|f(\omega_{\vec{k}},\vec{r})|^2\sin\omega_{\vec{k}}t, & t>0,\\ 0, & t\le 0,\end{cases}\qquad(33)$$

is the electric susceptibility and in the frequency domain satisfies the following relations:

$$\mathrm{Im}[\underline{\chi}_e(\vec{r},\omega)]=\frac{4\pi^2}{\hbar\varepsilon_0}|f(\omega,\vec{r})|^2\sum_i\frac{|\vec{k}_i|^2}{|(d\omega/d|\vec{k}|)(|\vec{k}|=|\vec{k}_i|)|},\qquad(34)$$

$$\mathrm{Re}[\underline{\chi}_e(\vec{r},\omega)]=\frac{8\pi}{\hbar\varepsilon_0}\int_0^\infty d|\vec{k}||\vec{k}|^2|f(\omega_{\vec{k}},\vec{r})|^2\frac{\omega_{\vec{k}}}{\omega_{\vec{k}}^2-\omega^2},\qquad(35)$$

where

$$\underline{\chi}_e(\vec{r},\omega) = \int_0^\infty dt\, \chi_e(\vec{r},t) e^{\imath\omega t} \tag{36}$$

is the Fourier transform of the electric susceptibility and the $|\vec{k}_i|$ are the roots of the algebraic equation $\omega=\omega_{\vec{k}} \equiv \omega(|\vec{k}|)$. A feature of the present quantization method is its flexibility in choosing a dispersion relation and a coupling function such that they satisfy the relation (34). In other words, the dispersion relation and the coupling function $f(\omega_{\vec{k}},\vec{r})$ are two free parameters of this theory up to the constraint relation (34). Various choices of $\omega_{\vec{k}}$ and $f(\omega_{\vec{k}},\vec{r})$ satisfying (34) do not change the commutation relations between the electromagnetic field operators and lead to equivalent expressions for the field operators and physical observables. As can be seen from the relation (34), in order to have a finite value for susceptibility for any frequency ω, we should assume that the denominator of the fraction in (34) is nonzero. So we assume that the dispersion relation is a monotonic function of $|\vec{k}|$. For simplicity and simple calculations we take a linear dispersion relation $\omega=c|\vec{k}|$, which leads to the following relation for the electric susceptibility:

$$\chi_e(\vec{r},t) = \begin{cases} \dfrac{8\pi}{\hbar c^3 \varepsilon_0} \displaystyle\int_0^\infty d\omega\, \omega^2 |f(\omega,\vec{r})|^2 \sin \omega t, & t>0, \\ 0, & t \le 0. \end{cases} \tag{37}$$

The operator $\vec{P}_N(\vec{r},t)$ in (32) is the noise electric polarization density

$$\vec{P}_N(\vec{r},t) = \sqrt{\frac{8}{V}} \sum_{\vec{n}} \sum_{\nu=1}^{3} \int d^3\vec{k} [f(\omega_{\vec{k}},\vec{r}) d_{\vec{n}\nu}(\vec{k},0) e^{-\imath\omega_{\vec{k}} t} + \text{H.c.}] \vec{v}_{\vec{n}\nu}(\vec{r}). \tag{38}$$

Similarly, by substituting (30) in (16), we obtain the following expression for $\vec{M}(\vec{r},t)$:

$$\vec{M}(\vec{r},t) = \vec{M}_N(\vec{r},t) + \frac{1}{\mu_0} \int_0^{|t|} dt'\, \chi_m(\vec{r},|t|-t') \vec{B}(\vec{r},\pm t'), \tag{39}$$

where χ_m is the magnetic susceptibility of the medium,

$$\chi_m(\vec{r},t) = \frac{8\pi\mu_0}{\hbar c^3} \int_0^\infty d\omega\, \omega^2 |g(\omega,\vec{r})|^2 \sin \omega t, \quad t>0,$$

$$\chi_m(\vec{r},t) = 0, \quad t \le 0, \tag{40}$$

and the operator $M_N(\vec{r},t)$ is the noise magnetic polarization density,

$$\vec{M}_N(\vec{r},t) = \sqrt{\frac{8}{V}} \sum_{\vec{n}} \sum_{\nu=1}^{3} \int d^3\vec{k} [g(\omega_{\vec{k}},\vec{r}) b_{\vec{n}\nu}(\vec{k},0) \times e^{-\imath\omega_{\vec{k}} t} + \text{H.c.}] \vec{s}_{\vec{n}\nu}(\vec{r}). \tag{41}$$

If we are given a definite $\chi_e(\vec{r},t)$ and $\chi_m(\vec{r},t)$, then we can invert the relations (37) and (43) and obtain the corresponding coupling functions f and g as

$$|f(\omega,\vec{r})|^2 = \frac{\hbar c^3 \varepsilon_0}{4\pi^2\omega^2} \int_0^\infty dt\, \chi_e(\vec{r},t) \sin \omega t, \quad \omega>0,$$

$$|f(\omega,\vec{r})|^2 = 0, \quad \omega=0; \tag{42}$$

$$|g(\omega,\vec{r})|^2 = \frac{\hbar c^3}{4\pi^2 \mu_0 \omega^2} \int_0^\infty dt\, \chi_m(\vec{r},t) \sin \omega t, \quad \omega>0,$$

$$|g(\omega,\vec{r})|^2 = 0, \quad \omega=0. \tag{43}$$

Therefore the constitutive equations (23), (32), and (41), together with the Maxwell equations, can be obtained directly from the Heisenberg equations applied to the electromagnetic field and the quantum fields E and M.

It is clear from (38) and (41) that the explicit forms of the noise polarization densities are known. Also, because the coupling functions f,g are common factors in the noise densities $\vec{P}_N,\vec{M}_N$ and susceptibilities χ_e,χ_m, it is clear that the strengths of the noise fields are dependent on the strength of χ_e,χ_m, which describe the dissipative character of a magnetodielectric medium. In particular, when the medium tends to a nonabsorbing one, the noise polarizations tend to zero, and this quantization scheme is reduced to the usual quantization in this medium [27].

Finally, using the total Hamiltonian (18) and the commutation relations

$$[\vec{r}_\alpha,\vec{p}_\beta] = \imath\hbar\,\delta_{\alpha\beta} I, \tag{44}$$

it is easy to show that the Heisenberg equations of motion for charged particles are

$$m_\alpha \ddot{\vec{r}}_\alpha = q_\alpha \vec{E}(\vec{r}_\alpha,t) + \frac{q_\alpha}{2}[\dot{\vec{r}}_\alpha \times \vec{B}(\vec{r}_\alpha,t) + \vec{B}(\vec{r}_\alpha,t) \times \dot{\vec{r}}_\alpha],$$

$$\alpha = 1,2,\ldots,N. \tag{45}$$

III. SPONTANEOUS EMISSION OF AN EXCITED TWO-LEVEL ATOM IN THE PRESENCE OF A MAGNETODIELECTRIC MEDIUM

In this section, as a simple application, we use the quantization scheme in the previous section to calculate the spontaneous decay rate of an initially excited two-level atom embedded in a magnetodielectric medium. For this purpose, let us consider a one-electron atom with position $\vec{R}$, which interacts with the electromagnetic field in the presence of a magnetodielectric medium. If we restrict our attention to the electric-dipole approximation [29,30], then the Hamiltonian (18) can be approximated as

$$\tilde{H} = H_0 + H',$$

$$H_0 = H_F + H_e + H_m + H_{\text{at}},$$

$$H' = \int_V d^3r' \left(-\frac{\vec{D}^\perp \cdot \vec{P}}{\varepsilon_0} - \vec{\nabla} \times \vec{A} \cdot \vec{M} \right) - e\vec{r} \cdot \vec{E}_D(\vec{R},t), \tag{46}$$

where

$$H_{\rm at} = \frac{\vec{p}^2}{2m} + \frac{e}{\varepsilon_0} \sum_{q_\alpha \neq e} q_\alpha G(\vec{r},\vec{r}_\alpha),$$

$$\vec{E}_D(\vec{R},t) = \vec{E}^\perp(\vec{R},t) - \frac{\vec{P}^\parallel(\vec{R},t)}{\varepsilon_0} = \frac{\vec{D}^\perp(\vec{R},t)}{\varepsilon_0} - \frac{\vec{P}(\vec{R},t)}{\varepsilon_0}, \tag{47}$$

and e, m, $\vec{r}$, and $\vec{p}$ are the charge, mass, position, and momentum of the atomic electron. In Hamiltonian (46) we have ignored contributions from $(e^2/2m)\vec{A}^2(\vec{R},t)$ and $(1/2\varepsilon_0)\int_V d^3r\, \vec{P}^2(\vec{r},t)$, because these terms do not affect the decay rate or level shifts in the dipole approximation and therefore will not be of concern here. For reality we assume that the atom is localized in a free-space region V_0 without polarization density. Then, for a two-level atom with upper state $|2\rangle$, lower state $|1\rangle$, and transition frequency ω_0, the Hamiltonian (46) can be written as [28]

$$\tilde{H} = \hbar\omega_0 \sigma^\dagger \sigma + H_F + H_e + H_m + H_{\rm at} + \int_\Omega d^3r' \left(-\frac{\vec{D}^\perp \cdot \vec{P}}{\varepsilon_0} - \vec{\nabla} \times \vec{A} \cdot \vec{M} \right) - \frac{e}{\varepsilon_0}(\vec{r}_{12}\sigma + \vec{r}^{\,*}_{12}\sigma^\dagger) \cdot \vec{D}^\perp(\vec{R},t), \tag{48}$$

where $\Omega = V - V_0$ is the region containing the magnetodielectric medium, $\sigma = |1\rangle\langle 2|$ and $\sigma^\dagger = |2\rangle\langle 1|$ are Pauli operators of the two-level atom, and $\vec{r}_{12} = \langle 1|\vec{r}|2\rangle$ is its transition dipole momentum. It is remarkable that the expansions (14) and (16) for the electric and magnetic polarization densities are independent of the shape of the magnetodielectric medium and depend only on the shape of the cavity containing it. The shape of the medium affects the coupling functions $f(\omega,\vec{r})$ and $g(\omega,\vec{r})$ and the time dependence of the ladder operators of the medium, such that, for a given medium with definite susceptibilities χ_e and χ_m, the dynamics of the total system leads to the correct constitutive equations (32) and (39) where the coupling functions are given by (42) and (43). This is just as the expansion (1) for the vector potential of the vacuum field, where the presence or absence of the external charges does not change the form of the expansion, and the presence of the external charges merely affects the time dependence of the ladder operators $a_{\vec{n},\lambda}$ and $a^\dagger_{\vec{n},\lambda}$.

To study the spontaneous decay of an initially excited atom, we may look for the wave function of the total system in the framework of the Weisskopf-Wigner theory [29,31],

$$|\psi(t)\rangle = c(t)|2\rangle|0\rangle_F|0\rangle_e|0\rangle_m + \sum_{\vec{n}} \sum_{\lambda=1}^{2} F_{\vec{n}\lambda}(t)|1\rangle|\vec{n},\lambda\rangle_F|0\rangle_e|0\rangle_m + \sum_{\vec{m}} \sum_{\nu=1}^{3} \int d^3\vec{k}\, D_{\vec{m}\nu}(\vec{k},t)|1\rangle|0\rangle_F|\vec{m},\vec{k},\nu\rangle_e|0\rangle_m + \sum_{\vec{m}} \sum_{\nu=1}^{3} \int d^3\vec{k}\, M_{\vec{m}\nu}(\vec{k},t)|1\rangle|0\rangle_F|0\rangle_e|\vec{m},\vec{k},\nu\rangle_m, \tag{49}$$

where $|0\rangle_F$, $|0\rangle_e$, and $|0\rangle_m$ are the vacuum state of the electromagnetic field and the E and M quantum fields, respectively. The coefficients $c(t)$, $F_{\vec{n}\lambda}(t)$, $D_{\vec{m}\nu}(\vec{k},t)$, and $M_{\vec{m}\nu}(\vec{k},t)$ are to be specified by the Schrödinger equation

$$i\hbar \frac{\partial|\psi(t)\rangle}{\partial t} = H|\psi(t)\rangle \tag{50}$$

for initial conditions $c(0)=1$, $F_{\vec{n}\lambda}(0) = D_{\vec{m}\nu}(\vec{k},0) = M_{\vec{m}\nu}(\vec{k},0) = 0$. Substituting $|\psi(t)\rangle$ from (49) in (50) and using the expansions (8), (14), and (16) and applying the rotating wave approximation [29,30], we find the following coupled differential equations for the coefficients of the wave function (49):

$$i\hbar\dot{M}_{\vec{m}\nu}(\vec{k},t) = \hbar\omega_{\vec{k}} M_{\vec{m}\nu}(\vec{k},t) - \sum_{\vec{n}} \sum_{\lambda=1}^{2} [L^{\vec{m}\nu}_{\vec{n}\lambda}(\omega_{\vec{k}})]^* F_{\vec{n}\lambda}(t), \tag{51}$$

$$i\hbar\dot{D}_{\vec{m}\nu}(\vec{k},t) = \hbar\omega_{\vec{k}} D_{\vec{m}\nu}(\vec{k},t) + \sum_{\vec{n}} \sum_{\lambda=1}^{2} [Q^{\vec{m}\nu}_{\vec{n}\lambda}(\omega_{\vec{k}})]^* F_{\vec{n}\lambda}(t), \tag{52}$$

$$i\hbar\dot{F}_{\vec{n}\lambda}(t) = \hbar\omega_{\vec{n}} F_{\vec{n}\lambda}(t) - \sum_{\vec{m}} \sum_{\nu=1}^{3} \int d^3\vec{k}\, L^{\vec{m}\nu}_{\vec{n}\lambda}(\omega_{\vec{k}}) M_{\vec{m}\nu}(\vec{k},t) + \sum_{\vec{m}} \sum_{\nu=1}^{3} \int d^3\vec{k}\, Q^{\vec{m}\nu}_{\vec{n}\lambda}(\omega_{\vec{k}}) D_{\vec{m}\nu}(\vec{k},t) + \left(ie\sqrt{\frac{4\hbar\omega_{\vec{n}}}{\varepsilon_0 V}}\, \vec{r}_{12} \cdot \vec{u}_{\vec{n}\lambda}(\vec{R}) \right) c(t), \tag{53}$$

$$i\hbar\dot{c}(t) = \hbar\omega_0 c(t) - \left(ie \sum_{\vec{n}} \sum_{\lambda=1}^{2} \sqrt{\frac{4\hbar\omega_{\vec{n}}}{\varepsilon_0 V}}\, \vec{r}^{\,*}_{12} \cdot \vec{u}_{\vec{n}\lambda}(\vec{R}) \right) F_{\vec{n}\lambda}(t), \tag{54}$$

where

$$Q^{\vec{m}\nu}_{\vec{n}\lambda}(\omega_{\vec{k}}) = i\sqrt{\frac{32\hbar\omega_{\vec{n}}}{\varepsilon_0 V^2}} \int_{V-V_0} d^3r'\, f(\omega_{\vec{k}},\vec{r}')\, \vec{u}_{\vec{n}\lambda}(\vec{r}') \cdot \vec{v}_{\vec{m}\nu}(\vec{r}'),$$

$$L_{\vec{n}\lambda}^{\vec{m}\nu}(\omega_{\vec{k}}) = \sqrt{\frac{32\hbar\mu_0\omega_{\vec{n}}}{V^2}}\int_{V-V_0} d^3r'\, g(\omega_{\vec{k}},\vec{r}\,')\vec{s}_{\vec{n}\lambda}(\vec{r}\,')\cdot\vec{s}_{\vec{m}\nu}(\vec{r}\,'). \tag{55}$$

One can solve these coupled differential equations by Laplace transformation. Let $\tilde{f}(\rho)$ denote the Laplace transformation of $f(t)$; then, taking the Laplace transformation of Eqs. (51)–(53) and then their combination and using the completeness relations (31), we find

$$[\imath\hbar\rho - \hbar\omega_{\vec{n}}]\tilde{F}_{\vec{n}\lambda}(\rho) = \hbar\omega_{\vec{n}}\sum_{\vec{n}'}\sum_{\lambda'=1}^{2} W_{\vec{n}\lambda}^{\vec{n}'\lambda'}\tilde{F}_{\vec{n}'\lambda'}(\rho) + \left(\imath e\sqrt{\frac{4\hbar\omega_{\vec{n}}}{\varepsilon_0 V}}\vec{r}_{12}\cdot\vec{u}_{\vec{n}\lambda}(\vec{R})\right)\tilde{c}(\rho), \tag{56}$$

where we have applied the initial conditions $F_{\vec{n}\lambda}(0) = D_{\vec{m}\nu}(\vec{k},0) = M_{\vec{m}\nu}(\vec{k},0) = 0$ and

$$W_{\vec{n}\lambda}^{\vec{n}'\lambda'}(\imath\rho) = \frac{8}{V}\sqrt{\frac{\omega_{\vec{n}'}}{\omega_{\vec{n}}}}\left(\int_{\Omega} d^3r[Z_e(\imath\rho,\vec{r})\vec{u}_{\vec{n}\lambda}(\vec{r})\cdot\vec{u}_{\vec{n}'\lambda'}(\vec{r}) + Z_m(\imath\rho,\vec{r})\vec{s}_{\vec{n}\lambda}(\vec{r})\cdot\vec{s}_{\vec{n}'\lambda'}(\vec{r})]\right), \tag{57}$$

$$Z_e(\imath\rho,\vec{r}) = \frac{1}{2\pi}\int_0^{\infty} d\omega\frac{\chi_{ei}(\omega,\vec{r})}{\imath\rho-\omega},$$

$$Z_m(\imath\rho,\vec{r}) = \frac{1}{2\pi}\int_0^{\infty} d\omega\frac{\chi_{mi}(\omega,\vec{r})}{\imath\rho-\omega}. \tag{58}$$

Here $\chi_{ei}(\omega,\vec{r})$ and $\chi_{mi}(\omega,\vec{r})$ are the imaginary parts of the electric and magnetic susceptibilities in the frequency domain, respectively. Equation (56) is a complicated algebraic equation for $\tilde{F}_{\vec{n}\lambda}(\rho)$ and may be solved by the iteration method. In the first step of the iteration method, $\tilde{F}_{\vec{n}\lambda}$ may be approximated by

$$\tilde{F}_{\vec{n}\lambda}(\rho) = \tilde{F}_{\vec{n}\lambda}^{(0)}(\rho) + \sum_{\vec{n}'}\sum_{\lambda'=1}^{2}\frac{\omega_{\vec{n}}W_{\vec{n}\lambda}^{\vec{n}'\lambda'}(\imath\rho)\tilde{F}_{\vec{n}'\lambda'}^{(0)}(\rho)}{\imath\rho-\omega_{\vec{n}}[1+W_{\vec{n}\lambda}^{\vec{n}\lambda}(\imath\rho)]}, \tag{59}$$

where

$$\tilde{F}_{\vec{n}\lambda}^{(0)}(\rho) = \frac{\imath e\sqrt{4\hbar\omega_{\vec{n}}/\varepsilon_0 V}\vec{r}_{12}\cdot\vec{u}_{\vec{n}\lambda}(\vec{R})}{\imath\hbar\rho-\hbar\omega_{\vec{n}}}\tilde{c}(\rho) \tag{60}$$

is the solution of (56) in the absence of the medium, that is, when $W_{\vec{n}\lambda}^{\vec{n}'\lambda'}=0$. Now the combination of Eq. (59) and the Laplace transformation of Eq. (54) yields

$$\imath\hbar[\tilde{c}(\rho)-c(0)] = \hbar\omega_0\tilde{c}(\rho) + \{\vec{r}_{12}^{\,*}\cdot[\tilde{G}^{(0)}(\vec{R},\vec{R},\imath\rho)+\tilde{G}(\vec{R},\vec{R},\imath\rho)]\cdot\vec{r}_{12}\}\tilde{c}(\rho), \tag{61}$$

where

$$\tilde{G}^{(0)}(\vec{R},\vec{R},\imath\rho) = \frac{4e^2}{\varepsilon_0 V}\sum_{\vec{n}}\sum_{\lambda=1}^{2}\frac{\omega_{\vec{n}}\vec{u}_{\vec{n}\lambda}(\vec{R})\vec{u}_{\vec{n}\lambda}(\vec{R})}{\imath\rho-\omega_{\vec{n}}}, \tag{62}$$

$$\tilde{G}(\vec{R},\vec{R},\imath\rho) = \frac{4e^2}{\varepsilon_0 V}\sum_{\vec{n},\vec{n}'}\sum_{\lambda,\lambda'=1}^{2}\frac{\sqrt{\omega_{\vec{n}}^3\omega_{\vec{n}'}}\vec{u}_{\vec{n}\lambda}(\vec{R})W_{\vec{n}\lambda}^{\vec{n}'\lambda'}(\imath\rho)\vec{u}_{\vec{n}'\lambda'}(\vec{R})}{\{\imath\rho-\omega_{\vec{n}}[1+W_{\vec{n}\lambda}^{\vec{n}\lambda}(\imath\rho)]\}(\imath\rho-\omega_{\vec{n}'})}. \tag{63}$$

The dyadic $\tilde{G}^{(0)}$ gives the contribution of the vacuum field for the spontaneous emission and level shift of the atom, that is, when there is no magnetodielectric medium, while the dyadic $\tilde{G}$ gives the spontaneous emission and frequency shift of the atom due to the presence of the medium. From definitions (57) and (58), it is obvious that $Z_e(\imath\rho,\vec{r})$, $Z_m(\imath\rho,\vec{r})$, and $W_{\vec{n}\lambda}^{\vec{n}'\lambda'}(\imath\rho)$ are analytic functions in the complex semiplane $\mathrm{Re}(\rho)>0$ and, since the denominator in dyadic $\tilde{G}$ has no zero in the semiplane $\mathrm{Re}(\rho)>0$, $\tilde{G}(\vec{R},\vec{R},\imath\rho)$ is analytic for $\mathrm{Re}(\rho)>0$. Taking the inverse Laplace transformation of (61), we find the following integro-differential equation:

$$\dot{c}(t) = -\imath\omega_0 c(t) - (\Gamma_0+\imath\Delta_0)c(t) + \int_0^t K(t-t')c(t')dt', \tag{64}$$

where

$$K(t-t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, e^{-\imath\omega(t-t')}\{\vec{r}_{12}\cdot[\tilde{G}(\vec{R},\vec{R},\omega+\imath\tau)]\cdot\vec{r}_{12}\}, \tag{65}$$

and Γ_0 and Δ_0 are the contribution of the vacuum field for the decay rate and the Lamb shift of the two-level atom [31]. Here we restrict our attention to the weak-coupling regime, where the Markov approximation [31,32] applies. That is to say, we may replace $c(t')$ in the integrand in (64) by

$$c(t') = c(t)e^{\imath\omega_0(t-t')}, \tag{66}$$

and approximate the time integral in (64) at very large times by

$$\int_0^t dt'\, e^{i(\omega_0-\omega)(t-t')} \approx \left(iP\frac{1}{\omega_0-\omega} + \pi\delta(\omega_0-\omega)\right), \tag{67}$$

where P denotes the principal Cauchy value. After some simple algebra and using the Kramers-Kronig relations for the dyadic $\tilde{G}$, we deduce

$$\dot{c}(t) = -\imath\omega_0 c(t) - (\Gamma_0+\Gamma+\imath\Delta_0+\imath\Delta)c(t), \tag{68}$$

where

$$\Gamma = -\frac{1}{\hbar}[\vec{r}_{12}\cdot \mathrm{Im}\ \tilde{G}(\vec{R},\vec{R},\omega_0+\imath 0^+)\cdot \vec{r}_{12}],$$

$$\Delta = \frac{1}{\hbar}[\vec{r}_{12}\cdot \mathrm{Re}\ \tilde{G}(\vec{R},\vec{R},\omega_0+\imath 0^+)\cdot \vec{r}_{12}] \qquad (69)$$

are the decay constant and the level shift due to the presence of the magnetodielectric medium, respectively.

There are many cases where the medium is homogeneous inside the region Ω and $W_{\vec{n}\lambda}^{\vec{n}\lambda}(\imath\rho)$ in the denominator of $\tilde{G}$ in (63) is independent of the polarization label λ and the vector label $\vec{n}$. For example, for the box $0<x<L_1$, $0<y<L_2$, $0<z<L_3/2$ filled uniformly with a homogeneous magnetodielectric medium, from the definition (57), we have

$$W_{\vec{n}\lambda}^{\vec{n}\lambda}(\imath\rho) \equiv W(\imath\rho) = \frac{1}{2}[Z_e(\imath\rho)+Z_m(\imath\rho)]. \qquad (70)$$

Generally speaking, for a piecewise homogeneous medium with N homogeneous pieces in volumes $\Omega_1,\Omega_2,\ldots,\Omega_N$, applying the definitions (2) and (17) in (57), we note that $W_{\vec{n}\lambda}^{\vec{n}\lambda}(\imath\rho)$ in the denominator of $\tilde{G}$ can be approximated by

$$W_{\vec{n}\lambda}^{\vec{n}\lambda}(\imath\rho) \equiv W(\imath\rho) \simeq \sum_{i=1}^{N}\frac{\Omega_i}{V}[Z_e^{(i)}(\imath\rho)+Z_m^{(i)}(\imath\rho)], \qquad (71)$$

where $Z_e^{(i)}$ and $Z_e^{(i)}$ denote the quantities Z_e and Z_m for the ith piece, respectively. Now we can substitute $W_{\vec{n}\lambda}^{\vec{n}'\lambda'}(\imath\rho)$ from (57) in the numerator of $\tilde{G}$ given by (63) and do the summations over λ,λ' by using the completeness relations

$$\sum_{\lambda=1}^{2} u_{\vec{n}\lambda}^{i}(\vec{R})u_{\vec{n}\lambda}^{j}(\vec{r}) = (\delta_{ij}-\hat{k}_{\vec{n}}^{i}\hat{k}_{\vec{n}}^{j})f_i(\vec{n},\vec{R})f_j(\vec{n},\vec{r}),$$

$$\sum_{\lambda=1}^{2} u_{\vec{n}\lambda}^{i}(\vec{R})s_{\vec{n}\lambda}^{j}(\vec{r}) = \sum_{\mu=1}^{3}\varepsilon_{j\mu i}\hat{k}_{\vec{n}\mu}f_i(\vec{n},\vec{R})g_j(\vec{n},\vec{r}). \qquad (72)$$

Now in the box frame we have

$$\tilde{G}_{\alpha\beta}(\vec{R},\vec{R},\imath\rho) = \frac{2e^2}{\varepsilon_0}\sum_{i=1}^{N}\int_{\Omega_i} d^3r \sum_{\gamma=1}^{3}[Z_e^{(i)}(\imath\rho)\,\eta_{\alpha\gamma}^{e}\zeta_{\beta\gamma}^{e} + Z_m^{(i)}(\imath\rho)\,\eta_{\alpha\gamma}^{m}\zeta_{\beta\gamma}^{m}], \qquad (73)$$

where

$$\eta_{\alpha\gamma}^{e}(\vec{R},\vec{r},\imath\rho) = \frac{4}{V}\sum_{\vec{n}}\frac{\omega_{\vec{n}}}{\imath\rho-\omega_{\vec{n}}[1+W(\imath\rho)]} \times(\delta_{\alpha\gamma}-\hat{k}_{\vec{n}}^{\alpha}\hat{k}_{\vec{n}}^{\gamma})f_\alpha(\vec{n},\vec{R})f_\gamma(\vec{n},\vec{r}),$$

$$\zeta_{\beta\gamma}^{e}(\vec{R},\vec{r},\imath\rho) = \frac{4}{V}\sum_{\vec{n}}\frac{\omega_{\vec{n}}}{\imath\rho-\omega_{\vec{n}}}(\delta_{\beta\gamma}-\hat{k}_{\vec{n}}^{\beta}\hat{k}_{\vec{n}}^{\gamma})f_\beta(\vec{n},\vec{R})f_\gamma(\vec{n},\vec{r}),$$

$$\eta_{\alpha\gamma}^{m}(\vec{R},\vec{r},\imath\rho) = \frac{4}{V}\sum_{\vec{n}}\sum_{\mu=1}^{3}\frac{\omega_{\vec{n}}}{\imath\rho-\omega_{\vec{n}}[1+W(\imath\rho)]} \times\varepsilon_{\gamma\mu\alpha}\hat{k}_{\vec{n}}^{\mu}f_\alpha(\vec{n},\vec{R})g_\gamma(\vec{n},\vec{r}),$$

$$\zeta_{\beta\gamma}^{m}(\vec{R},\vec{r},\imath\rho) = \frac{4}{V}\sum_{\vec{n}}\sum_{\mu=1}^{3}\frac{\omega_{\vec{n}}}{\imath\rho-\omega_{\vec{n}}} \times\varepsilon_{\gamma\mu\beta}\hat{k}_{\vec{n}}^{\mu}f_\beta(\vec{n},\vec{R})g_\gamma(\vec{n},\vec{r}). \qquad (74)$$

It is remarkable to note that, since we have assumed that the center of the atom is localized in a free region, then in the argument of the tensors η^e, ζ^e, η^m, and ζ^m in (74), we have $\vec{R}\neq\vec{r}$, and therefore the series above are all convergent. Inspection of the definitions (58) shows that the real and imaginary parts of $Z_e(\omega+\imath 0^+)$ and $Z_m(\omega+\imath 0^+)$ for real frequency ω are

$$\mathrm{Re}[Z_e(\omega+\imath 0^+)] = \frac{1}{2\pi}P\int_0^\infty d\omega'\frac{\chi_{ei}(\omega')}{\omega-\omega'} \equiv \alpha_e(\omega),$$

$$\mathrm{Im}[Z_e(\omega+\imath 0^+)] = -\frac{1}{2}\chi_{ei}(\omega)\vartheta(\omega) \equiv \gamma_e(\omega),$$

$$\mathrm{Re}[Z_m(\omega+\imath 0^+)] = \frac{1}{2\pi}P\int_0^\infty d\omega'\frac{\chi_{mi}(\omega')}{\omega-\omega'} \equiv \alpha_m(\omega),$$

$$\mathrm{Im}[Z_m(\omega+\imath 0^+)] = -\frac{1}{2}\chi_{mi}(\omega)\vartheta(\omega) \equiv \gamma_m(\omega), \qquad (75)$$

where $\vartheta(\omega)$ is the step function and P denotes the Cauchy principal value. Finally, using (69) and (73)–(75) and

$$\frac{1}{\omega+\imath 0^+-\omega'} = P\frac{1}{\omega-\omega'} - \imath\pi\delta(\omega-\omega'), \qquad (76)$$

we find the spontaneous emission and the shift frequency of a two-level atom due to the presence of a magnetodielectric medium. From (69) and (73) it is clear that the spontaneous emission and the Lamb shift depend on (i) the position of the atom, (ii) the orientation of the dipole of the atom, (iii) the size and type of the medium, (iv) the relative position of the medium with respect to the center of the atom, and (v) the volume and shape of the cavity. For a nonrectangular cavity, we must use a different set of eigenvectors and therefore the spontaneous emission also depends on the shape of the cavity.

IV. CONCLUSION

The scheme introduced by the present authors to quantize the electromagnetic field in a magnetodielectric medium is generalized to the case where some external charges are present in the medium. The spontaneous emission of a two-level atom embedded in a magnetodielectric medium is calculated as an application of the model.

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