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Renormalization of concurrence: The application of the quantum renormalization group to quantum-information systems

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We have combined the idea of renormalization group and quantum-information theory. We have shown how the entanglement or concurrence evolve as the size of the system becomes large, i.e., the finite size scaling is obtained. Moreover, we introduce how the renormalization-group approach can be implemented to obtain the quantum-information properties of a many-body system. We have obtained the concurrence as a measure of entanglement, its derivatives and their scaling behavior versus the size of system for the one-dimensional Ising model in transverse field. We have found that the derivative of concurrence between two blocks each containing half of the system size diverges at the critical point with the exponent, which is directly associated with the divergence of the correlation length.

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A fundamental difference between quantum and classical physics is the possible existence of nonclassical correlations in quantum systems called entanglement [1]. Recently, the study of strongly correlated systems in condensed matter physics from the perspective notions of quantum-information theory has received much attention. It seems that the main motivations for such treatment are two fold: (i) Over the last decade the entanglement has been realized to be a crucial resource to process and send information in ways such as quantum teleportation, supercoding, and algorithms for quantum computations [2]. (ii) The features of the ground state of many-body systems which consists of a superposition of a huge number of product states opens the question of how these states are interrelated.

The role of entanglement in quantum phase transition (QPT) [3] is of considerable interest [4]. Quantum phase transitions occur at absolute zero and are driven by quantum fluctuations. Entanglement as a direct measure of quantum correlations shows nonanalytic behavior such as discontinuity in the vicinity of the quantum phase transition point [5]. In the past few years the subject of several activities were to investigate the behavior of entanglement in the vicinity of quantum critical point for different spin models [4,6–10] as well as itinerant systems [11–13].

Our main purpose in this work is to combine the idea of quantum renormalization group (QRG) [14,15] and quantum-information properties (QIP). This will give two insights on (i) how a quantum information property evolves as the size of system becomes large and (ii) QRG connects the nonanalytic behavior of entanglement to the critical phenomenon properties of the model. To have a concrete discussion, the one-dimensional $S=\frac{1}{2}$ Ising model in a transverse field (ITF) has been considered by implementing the quantum renormalization group approach.

The main idea of the RG method is the mode elimination or thinning of the degrees of freedom followed by an iteration which reduces the number of variables step by step until reaching a fixed point. In Kadanoff's approach, the lattice is divided into blocks. Each block is treated independently to build the projection operator onto the lower energy subspace. The projection of the interblock interaction is mapped to an effective Hamiltonian (H^{eff}) which acts on the renormalized subspace [16,17].

We have considered the ITF model on a periodic chain of N sites with the Hamiltonian

$$H = -J\sum_{i=1}^{N} (\sigma_i^z\sigma_{i+1}^z + g\sigma_i^x). \qquad (1)$$

To implement QRG the Hamiltonian is divided to two-site blocks [18], $H^B=\Sigma_{I=1}^{N/2}h_I^B$ with $h_I^B=-J(\sigma_{1,I}^z\sigma_{2,I}^z+g\sigma_{1,I}^x)$. The remaining part of the Hamiltonian is included in the interblock part, $H^{BB}=-J\Sigma_{I=1}^{N/2}(\sigma_{2,I}^z\sigma_{1,I+1}^z+g\sigma_{2,I}^x)$, where $\sigma_{j,I}^\alpha$ refers to the α component of the Pauli matrix at site j of the block labeled by I. The Hamiltonian of each block (h_I^B) is diagonalized exactly and the projection operator (P_0) is constructed from the two lowest eigenstates, $P_0=|\psi_0\rangle\langle\psi_0|+|\psi_1\rangle\langle\psi_1|$. In this respect the effective Hamiltonian ($H^{eff}=P_0[H^B+H^{BB}]P_0$) is similar to the original one [Eq. (1)] replacing the couplings with the following renormalized coupling constants:

$$J' = J\frac{2(\sqrt{g^2+1}+g)}{1+(\sqrt{g^2+1}+g)^2}, \quad g' = g^2. \qquad (2)$$

Since the block Hamiltonian is treated exactly, the density matrix can be written in terms of the ground state of the two-site block, $\rho_{12}=|\psi_0\rangle\langle\psi_0|$. We confine our interest to the entanglement between two sites which is measured by concurrence. The concurrence in terms of the parameters defined for a two-site block is

$$C = \mathrm{Max}\{\lambda_1-\lambda_2-\lambda_3-\lambda_4, 0\}, \qquad (3)$$

where $\lambda_k(k=1,2,3,4)$ are the square roots of the eigenvalues in descending order of the operator R_{12},

$$R_{12}=\rho_{12}\tilde{\rho}_{12}, \quad \tilde{\rho}_{12}=(\sigma_1^y\otimes\sigma_2^y)\rho_{12}^*(\sigma_1^y\otimes\sigma_2^y).$$

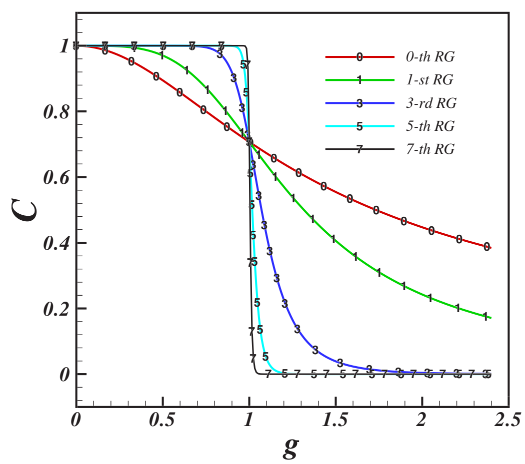

FIG. 1. (Color online) Concurrence of the ITF chain as a function of g (transverse field strength normalized to the exchange interaction) at different RG steps.

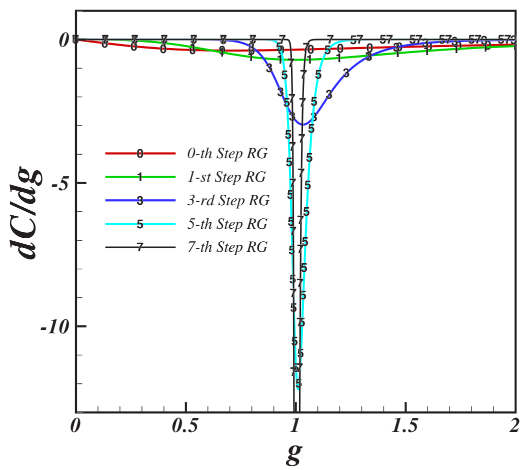

FIG. 2. (Color online) Evolution of the first derivative Concurrence under RG in the limit of large system (high RG step), the nonanalyticity behavior of the first derivative of concurrence is captured through the diverging.

The concurrence as a measure of entanglement is a local quantity which includes the global properties of a system. Generally, the global properties of a system enters into the entanglement effectively by summing over the whole degrees of freedom except the local one. In other words, a system can be supposed of a single site and a heat bath (the rest of system). It is supposed that the effect of a heat bath can be replaced by an effective single-site quantity, *the entanglement*. The effective single site represents the long-range properties of the model and not the microscopic ones. Having this in mind we can enter the global properties of the model to the entanglement (the local quantity) using the renormalization-group idea. In this respect, we always think of a two-site model which can be treated exactly as obtained above. However, the coupling constants of the two-site model are the effective ones which are given by the renormalization group procedure. This can be used as an approach to calculate the entanglement in a large system.

Before implementing the idea of renormalization group to calculate the entanglement, let us first briefly analyze the RG equations [Eq. (2)]. The RG equations give the flow of coupling constants in the phase diagram. Any point in the phase diagram runs to a stable fixed point (after enough iterations) which defines the stable phases. The unstable fixed points are important because they define the critical points which are at the border of different stable phases. There are two stable and one unstable fixed point for the RG equations of ITF model. The stable phases are represented by two fixed points $g=0$ (long-ranged ordered Ising phase) and $g=\infty$ (the paramagnetic phase). The critical point $g_c=1$ is the transition point between the Ising ($g<g_c$) and paramagnetic ($g>g_c$) phases. The obtained critical point is exactly the same as what can be obtained by transforming the ITF model to free fermions using the Jordan-Wigner transformation [19].

To implement the idea of RG approach in concurrence we have plotted in Fig. 1 the value of C versus g for different RG steps. In the n RG step the expression given in Eq. (3) is evaluated at the renormalized coupling given by the n iteration of g given in Eq. (2). The zero RG step means a bare two-site model, while in the first RG step the effective two-site model represents a four-site chain. Generally, in the n RG step, a chain of 2^{n+1} sites is represented effectively by the two sites with renormalized couplings. All plots in Fig. 1 cross each other at the critical point, $g_c=1$. In other words the critical point is a scale-free point where the quantum fluctuations extend over all length scales.

After few RG iterations (which represent a large enough system), the concurrence at the critical point (g_c) of the model discontinues, that is, a signature of phase transition. As shown in Fig. 1, the concurrence switches from 1 to 0. It means that the entanglement acts similar to an order parameter, i.e., for the paramagnetic phase the entanglement is zero and for the long-ranged ferromagnetic phase it is equal to one.

The nonanalytic behavior in some physical quantity is a feature of second-order quantum phase transition. It is also accompanied by a scaling behavior since the correlation length diverges and there is no characteristic length scale in the system at the critical point. Osterloh *et al.* [4] have verified that the entanglement in the vicinity of critical point of ITF model and XY model in transverse field shows a scaling behavior. They have concentrated on the concurrence between the two nearest-neighbor sites at various sizes of system. As we have stated in the RG approach for the ITF model, a large system, i.e., $N=2^{n+1}$, can be effectively described by two sites with the renormalized couplings of the nth RG step. Thus, the entanglement between the two renormalized sites represents the entanglement between two parts of the system each containing $N/2$ sites effectively. In this respect we can speak of *block entanglement*—the entanglement between a block and the rest of system—in a large system provided the size of the block and the rest of system is equal.

Having this in mind, the first derivative of concurrence is analyzed as a function of coupling g at different RG steps which manifest the size of the system. The derivative of concurrence with respect to the coupling constant $(\frac{dC}{dg})$ shows a singular behavior at the critical point (Fig. 2). The singular behavior is the result of discontinuous change of C at $g=g_c$.

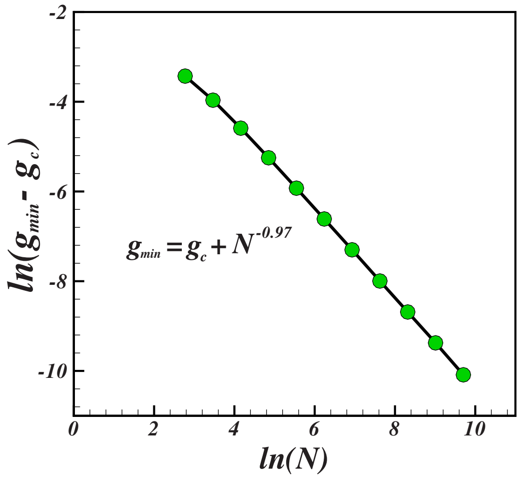

FIG. 3. (Color online) Scaling of the position (g_m) of the minimum of concurrence at different RG steps, it is seen that g_m goes to g_c as the size of the system becomes large as $g_m = g_c + N^{-0.97}$.

We have plotted $\frac{dC}{dg}$ versus g in Fig. 2 for different RG steps which shows the singular behavior as the size of the system becomes large (higher RG steps). A more detailed analysis shows that the position of the minimum (g_m) of $\frac{dC}{dg}$ tends towards the critical point similar to $g_m = g_c + N^{-\theta}$ with $\theta = 0.97$, which has been plotted in Fig. 3. A similar behavior has been reported in Osterloh's work [4], which shows that λ_m scales as $\lambda_m = 1 + N^{-1.87}$. It should be noticed that λ_m is the position of the minimum of the derivative of concurrence of two nearest-neighbor sites, which is different from our case. In our treatment the derivative of the concurrence of two blocks shows a singular behavior. Moreover, we have derived the scaling behavior of $y \equiv |\frac{dC}{dg}|_{g_m}|$ versus N. This has been plotted in Fig. 4, which shows a linear behavior of $\ln(y)$ versus $\ln(N)$. The scaling behavior is $|\frac{dC}{dg}|_{g_m}| \sim N^{\theta}$ with exponent $\theta = 1$. However, the minimum value of the derivative of concurrence for two nearest-neighbor sites diverges logarithmically [4], $\frac{dC}{d\lambda}|_{\lambda_m} = -0.2702 \ln N$.

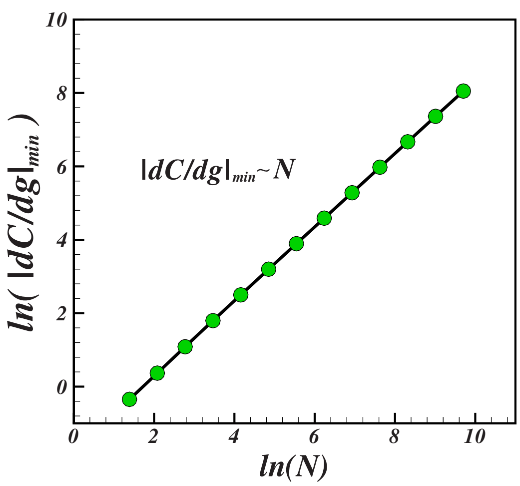

FIG. 4. (Color online) Scaling of the minimum of the first derivative of concurrence for various sizes of system. The RG procedure shows the minimum diverges as $|\frac{dC}{dg}|_{g_m}| \sim N$.

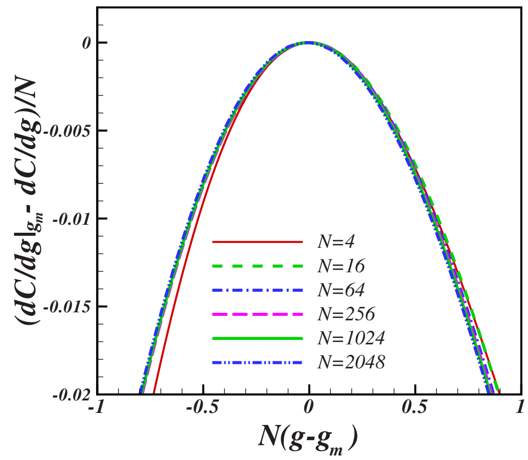

FIG. 5. (Color online) A manifestation of finite-size scaling through RG treatment. The curves corresponding to different lattice sizes collapse to a single graph.

We would like to show that the exponent θ is directly related to the correlation length exponent close to the critical point. The correlation length exponent, ν, gives the behavior of correlation length in the vicinity of g_c, i.e., $\xi \sim (g - g_c)^{-\nu}$. Under the RG transformation, Eq. (2), the correlation length scales in the nth RG step as $\xi^{(n)} \sim (g_n - g_c)^{-\nu} = \xi / n_B^n$, which immediately leads to an expression for $|\frac{dg_n}{dg}|_{g_c}$ in terms of ν and n_B (number of sites in each block). Dividing the last equation to $\xi \sim (g - g_c)^{-\nu}$ gives $|\frac{dg_n}{dg}|_{g_c} \sim N^{1/\nu}$, which implies $\theta = 1/\nu$, since $|\frac{dC}{dg}|_{g_m} \sim |\frac{dg_n}{dg}|_{g_c}$ at the critical point. It should also be noted that the scaling of the position of minimum, g_{min} (Fig. 3), also comes from the behavior of the correlation length near the critical point. As the critical point is approached and in the limit of large system size, the correlation length almost covers the size of the system, i.e., $\xi \sim N$, and a simple comparison with $\xi \sim (g - g_c)^{-\nu}$ results in the following scaling form $g_m = g_c + N^{-1/\nu}$.

To obtain the finite-size scaling behavior of concurrence we have followed a scaling trick in which all graphs collapse on each other. This is also a manifestation of the existence of finite-size scaling for the block entanglement. The scaling trick is based on the divergence of derivative of concurrence close to the critical point (Fig. 2) and the power-law scaling obtained in Figs. 3 and 4. We define the scaling variable $x = N^{1/\nu}(g - g_m)$ and obtain the universal function $F(x)$ such that $\frac{dC}{dg} \sim N^{1/\nu} F(x)$, where $F(x) = 1/(1 + x^2)$. In this way we have plotted $\left(\frac{dC}{dg}\Big|_{g_m} - \frac{dC}{dg}\right)/N$ versus $N(g - g_m)$ in Fig. 5. The curves which correspond to different system sizes clearly collapse on a single universal curve. This results justify that the RG implementation of entanglement truly capture the critical behavior of the ITF model.

To summarize, we have initiated the idea of renormalization group (RG) to study the quantum-information properties (QIP) of a system. In this respect some basic notions have been introduced: (i) The evolution of QIP, i.e., entanglement or concurrence, in terms of RG steps give how the properties develop from a finite-size system to its thermodynamic counterpart. In other words, there exist some finite-size scaling for the properties. (ii) The RG procedure can be implemented to obtain the QIP of a system in terms of the effective Hamiltonian which is described by the renormalized coupling constants. (iii) The RG procedure indicates that the exponents governing the nonanalytic behavior of the QIP in the vicinity of the critical point comes from the long-range properties of the model. This is the manifestation of the fact that some global properties of the system can be represented by local properties such as entanglement. These notions have been observed and approved in our study of the ITF model. Moreover, the RG approach shows that as the size of the system becomes large two effective sites are entangled for ferromagnetic, i.e., $g<g_c$ and disentangled for paramagnetic, i.e., $g>g_c$, and probably manifest the block entanglement is relevant for the existence of ferromagnetic correlations.

We have also implemented the idea of the QRG to the anisotropic Heisenberg model (*XXZ*) [20] to obtain the QIP. We have used the Von Neuman entropy for searching the entanglement between one site and the rest of a three-site block. We have been able to obtain the scaling behavior of the concurrence and entanglement of this model. This justifies our main idea.

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[1] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Communication* (Cambridge University Press, Cambridge, 2000).
[3] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 2000).
[4] A. Osterloh, Luigi Amico, G. Falci, and Rosario Fazio, Nature (London) **416**, 608 (2002).
[5] L. A. Wu, M. S. Sarandy, and D. A. Lidar, Phys. Rev. Lett. **93**, 250404 (2004).
[6] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).
[7] J. Vidal, G. Palacios, and R. Mosseri, Phys. Rev. A **69**, 022107 (2004).
[8] T. J. Osborne and M. A. Nielsen, Phys. Rev. A **66**, 032110 (2002).
[9] I. Bose and E. Chattopadhyay, Phys. Rev. A **66**, 062320 (2002).
[10] F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett. **92**, 027901 (2004).
[11] P. Zanardi and X. Wang, J. Phys. A **35**, 7947 (2002).
[12] Shi-Jian Gu, Shu-Sa Deng, You-Quan Li, and Hai-Qing Lin, Phys. Rev. Lett. **93**, 086402 (2004).
[13] Alberto Anfossi, Paolo Giorda, and Arianna Montorsi, Phys. Rev. B **75**, 165106 (2007).
[14] P. Pfeuty, R. Jullian, K. L. Penson, in *Real-Space Renormalization*, edited by T. W. Burkhardt and J. M. J. van Leeuwen (Springer, Berlin, 1982), Chap. 5.
[15] K. G. Wilson, Rev. Mod. Phys. **47**, 773 (1975).
[16] M. A. Martin-Delgado and G. Sierra, Int. J. Mod. Phys. A **11**, 3145 (1996).
[17] A. Langari, Phys. Rev. B **58**, 14467 (1998); **69**, 100402 (R) (2004); R. Jafari and A. Langari, *ibid.* **76**, 014412 (2007).
[18] M. A. Martin-Delgado and G. Sierra, Phys. Rev. Lett. **76**, 1146 (1996).
[19] J. I. Latorre, E. Rico, and G. Vidal, Quantum Inf. Comput. **4**, 48 (2004).
[20] M. Kargarian, R. Jafari, and A. Langari, e-print arXiv:0711.2358.