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Ancilla dimensions needed to carry out positive-operator-valued measurement

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To implement a positive-operator-valued measurement (POVM), which is defined on the d_S-dimensional Hilbert space of a physical system, one has to extend the Hilbert space to include d_A additional dimensions (called the ancilla). This is done via either the tensor product extension (TPE) or the direct sum extension (DSE). The implementation of a POVM utilizes the available resources more efficiently if it requires fewer additional dimensions. To determine how to implement a POVM with the least additional dimensions is, therefore, an important task in quantum information. We have determined the necessary and sufficient (hence minimal) number of the additional dimensions needed to implement the same POVM by the TPE and the DSE, respectively. If the POVM has n elements and r_i is the rank of the ith element, then the dimension of the minimal ancilla is $d_A=\Sigma_{i=1}^{n} r_i - d_S$ for the DSE implementation, and this represents a lower bound for the added dimensions in the TPE implementation. In the proof, we explicitly construct the DSE implementation of a general POVM with elements of arbitrary rank. As an example, we determine d_A for the unambiguous discrimination of N linearly independent states and provide the full DSE implementation of a state-discriminating POVM for $N=2$.

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Quantum measurements (QMs) are essential components of quantum-information processing. On the one hand, one can get a desired state by measurement as, e.g., in teleportation [1]. On the other, one has to measure the state of a quantum system to extract the information encoded in it as, e.g., in quantum cryptography and quantum algorithms [2,3]. The simplest QM is the projector-valued measurement (PVM), which can be described by a set of orthogonal projectors $\{P_i\}$, $P_iP_j=\delta_{ij}P_i$, acting on the Hilbert space of the system, with $\Sigma_i P_i = I$ [4].

An often sufficient generalization is the positive-operator-valued measure (POVM), which is a set of positive operators Π_i that satisfy the completeness relation, $\Sigma_{i=1}^{n}\Pi_i=I$. Each possible measurement outcome i is associated with a POVM element $\Pi_i \equiv M_i^\dagger M_i$, and the underlying M_i is called the detection operator. If outcome i occurs, the state of the system after the measurement is $\rho_i = M_i\rho M_i^\dagger$ if the initial state was ρ. The probability of obtaining outcome i is $p_i=\text{Tr}(M_i\rho M_i^\dagger) =\text{Tr}(M_i^\dagger M_i\rho)$. The detection operators are not necessarily orthogonal to each other, i.e., $M_i^\dagger M_j \neq \delta_{ij} M_i^\dagger M_i$ is allowed. Due to completeness, the generated probability distribution is normalized, $\Sigma_i p_i=1$.

A POVM is thus a decomposition of the identity in terms of positive but not necessarily orthogonal operators, and is defined on the Hilbert space of the system, H_S. The key question is how to realize such an operation. All implementations are based on Neumark's theorem [5], which states that there is a one-to-one correspondence between a POVM and the following constructive procedure (see also [4,6] for very readable accounts). First, we embed the system into a larger Hilbert space where the extra degrees of freedom are called the ancilla. Next, we introduce an interaction between the system and the ancilla which will unitarily entangle the system degrees of freedom with those of the ancilla. Finally, we perform standard PVMs on this extended Hilbert space. Looking at the system alone, the resulting operation is neither a unitary transform nor a projective measurement, but a POVM, since both operations were performed on a larger Hilbert space. Conversely, any POVM can be realized in this way by an appropriate choice of the unitary and the subsequent PVM.

Since there are two ways to extend the initial space H_S, there are two very different realizations of a POVM. The first is the tensor product extension (TPE); the extended space is the tensor product of the system space and the ancilla space, $H=H_S\otimes H_A$. The TPE implementation of the POVM can be schematically described as

$$U(H_S\otimes H_A) \rightarrow \text{followed by PVM on } H_A \rightarrow \{M_i^\dagger M_i, \quad i=1,\ldots,n\} \text{ on } H_S, \qquad (1)$$

where U is a unitary operator and projective measurements are performed on the ancilla system.

A second method is the direct sum extension (DSE); the extended space is the direct sum of the system space and ancilla space, $H=H_S\oplus H_A$. H_A may represent the so far unused extra dimensions of the original system. The DSE implementation of the POVM can be schematically described as

$$U(H_S\oplus H_A) \rightarrow \text{followed by PVM on } H_S\oplus H_A \rightarrow \{M_i^\dagger M_i, \quad i=1,\ldots,n\} \text{ on } H_S, \qquad (2)$$

where U is a unitary, and projective measurements can now be performed on the entire extended space H.

Both methods need additional dimensions to implement a POVM. It is more difficult to prepare and control a higher-dimensional system than a lower-dimensional one in experiments [7,8]. Also, an optimal use of available resources is often preferred. So it is important to find out how to implement a POVM with the *least* added dimensions. In this paper, we determine the least number of the added dimensions that is necessary to implement the same POVM by the methods of the TPE and DSE, respectively, and prove that the DSE needs fewer additional dimensions. The proof is constructive and, as a by-product, we obtain an explicit implementation of a general POVM by the DSE. It should be noted that in the standard references only the implementation of rank 1 elements is considered [4,6]. We construct the implementation of POVM elements of arbitrary rank explicitly. The method is illustrated on the example of unambiguous discrimination among N linearly independent states for which we derive the dimension of the minimal ancilla. Finally, we show that the DSE needs significantly less experimental cost than the TPE in many cases to realize the same POVM, and give a specific scheme to unambiguously discriminate two nonorthogonal states by DSE.

As seen above, a POVM acting on a Hilbert space H_S can be implemented by unitary operations and PVMs acting on a direct sum extension space $H=H_S\oplus H_A$. Let r_i denote the rank of the POVM element $M_i^\dagger M_i$ and d_S the dimension of the system Hilbert space H_S. Then the following theorem holds.

Theorem 1. An ancilla space H_A of dimension $d_A=r-d_S$, where $r=\Sigma_{i=1}^n r_i$, is necessary and sufficient to implement the POVM in the DSE method.

Proof. Let $\{|e_k\rangle,\ k=1,\ldots,d_S\}$ represent an orthonormal basis in H_S. M_i can be written in this basis as $M_i=\Sigma_{k,l=1}^{d_S} c_{kl}^{(i)}|e_l\rangle\langle e_k|$. Introducing $|\varphi_k^{(i)}\rangle\equiv\Sigma_{l=1}^{d_S} c_{kl}^{(i)}|e_l\rangle$, M_i can be expressed as

$$M_i=\sum_{k=1}^{d_S}|\varphi_k^{(i)}\rangle\langle e_k|, \tag{3}$$

for $i=1,\ldots,n$. $\{|\varphi_k^{(i)}\rangle\}$ is a set of unnormalized vectors in H_S. The rank r_i of $M_i^\dagger M_i$ is the number of the linearly independent vectors in this set for a fixed i. The effect of M_i is to transform the basis vector $|e_k\rangle$ into the postmeasurement state $|\varphi_k^{(i)}\rangle$.

To proceed, we first array the states in Eq. (3) as

$$\begin{matrix} |\varphi_1^{(1)}\rangle & \cdots & |\varphi_1^{(n)}\rangle \\ \vdots & & \vdots \\ |\varphi_{d_S}^{(1)}\rangle & \cdots & |\varphi_{d_S}^{(n)}\rangle, \end{matrix} \tag{4}$$

where the states of the ith column in (4) are those in M_i. For column i we can find r_i normalized and orthogonal vectors $\{|f_1^{(i)}\rangle,\ldots,|f_{r_i}^{(i)}\rangle\}$ such that $|\varphi_1^{(i)}\rangle,\ldots,|\varphi_{d_S}^{(i)}\rangle$ are linear superpositions of these vectors,

$$|\varphi_k^{(i)}\rangle=\sum_{l=1}^{r_i}u_{k,l}^{(i)}|f_l^{(i)}\rangle, \tag{5}$$

where $u_{k,l}^{(i)}$ are expansion coefficents for state k in column i $(1\leq k\leq d_S,\ 1\leq i\leq n)$. The vectors in $\{|f_1^{(i)}\rangle,\ldots,|f_{r_i}^{(i)}\rangle\}$ are not necessarily orthogonal to those in $\{|f_1^{(j)}\rangle,\ldots,|f_{r_j}^{(j)}\rangle\}$ for $i\neq j$. From the completeness of the POVM $\Sigma_i^n\Sigma_l^{r_i}u_{kl}^{(i)}u_{k'l}^{(i)*}=\delta_{kk'}$ follows. Thus the coefficients in the rows of (4) form the first d_S rows of an $r\times r$ unitary matrix U. The matrix elements in row k $(1\leq k\leq d_S)$, $u_{k,l}$, come from the first column in (4) for $l=1,\ldots,r_1$, from the second column for $l=r_1+1,\ldots,r_1+r_2$, etc. The remaining $r-d_S$ rows in U are undetermined within the constraint that the full U is unitary. This constraint is easy to satisfy if we notice that the first d_S rows represent d_S mutually orthogonal and normalized vectors in a d_S-dimensional subspace of the full r-dimensional space of U. Choosing the remaining rows to be basis vectors in the complementary $r-d_S$ dimensional subspace meets the requirements.

Then, let us extend H_S into $H_S\oplus H_A$, and unitarily rotate the bases $|e_1\rangle,\ldots,|e_{d_S}\rangle$ of H_S in the whole extended space H such that

$$U|e_j\rangle=\sum_{k=1}^{r}u_{j,k}|k\rangle,\quad j=1,\ldots,d_S, \tag{6}$$

where $u_{j,k}$ are elements of the $r\times r$ unitary matrix U and $\{|k\rangle;\ k=1,\ldots,r\}$ is a set of orthogonal basis vectors of the extended space H. Then we perform a set of PVMs $\{P_i,\ i=1,\ldots,n\}$ on the extended space such that

$$P_i=\sum_{m=1}^{r_i}\left|\sum_{l=0}^{i-1}r_l+m\right\rangle\left\langle\sum_{l=0}^{i-1}r_l+m\right|, \tag{7}$$

where $r_0=0$. The combined effect of U in (6) and the PVM in (7) can be described by the operators,

$$P_i'=\sum_{j=1}^{d_S}\left(\sum_{m=1}^{r_i}u_{j,\Sigma_{l=0}^{i-1}r_l+m}\left|\sum_{l=0}^{i-1}r_l+m\right\rangle\right)\langle e_j|, \tag{8}$$

where, again, $r_0=0$ and $i=1,\ldots,n$; namely, P_i' transforms the bases $\{|e_j\rangle,\ j=1,\ldots,d_S\}$ of the space H_S into the states $\{P_iU|e_j\rangle\equiv P_i'|e_j\rangle,\ j=1,\ldots,d_S\}$, corresponding to the outcome P_i in the extended space H.

Finally, if the outcome was i, we perform a unitary transformation which transforms the states $\{|k\rangle,\ k=\Sigma_{l=1}^{i-1}r_l+1,\ldots,\Sigma_{l=1}^{i-1}r_l+r_i\}$ into the r_i orthogonal vectors $\{|f_1^{(i)}\rangle,\ldots,|f_{r_i}^{(i)}\rangle\}$ of Eq. (5), and thus implement the POVM element $M_i^\dagger M_i$. We proceed similarly for the other outcomes and thus implement the full POVM (for all $i=1,\ldots,n$). Therefore, the extension is sufficient.

Since the implementation of the POVM is achieved by PVMs on the extended space Eq. (7), and the rank of the projector P_i is not less than the rank r_i of M_i [9], the dimensionality of the extended space is at least $r=\Sigma_{i=1}^n r_i$. So one needs at least a $(d_A=r-d_S)$-dimensional ancilla space H_A to implement the POVM by the DSE. Therefore, the extension is also necessary. ■

Corollary. If we use the TPE to implement the POVM, we need an n-dimensional ancilla, at the least. The total dimension of the tensor product space is then at least $n\times d_S$, of which at least $(n-1)d_S$ are the added dimensions. Obviously, $(n-1)d_S\geq r-d_S=d_A$, since the rank of each POVM element is less than d_S. Thus the number of added dimensions for the DSE, d_A, represents a lower bound for the additonal dimensions necessary for the TPE. The DSE, in general, requires fewer added dimensions and utilizes resources more economically than the TPE.

As an application of Theorem 1, we consider unambiguous discrimination (UD) among N states $\{|\psi_i\rangle,\ i=1,\ldots,N\}$. The necessary and sufficient condition for UD is that the states be linearly independent [10]. UD can be accomplished by a POVM such that $\Pi_i|\psi_j\rangle=0$ if $i\neq j$ [11]. Then the element Π_i unambigously identifies the state as $|\psi_i\rangle$. This mandates the choice $\Pi_i=a_i|\psi_i'\rangle\langle\psi_i'|$, $0<a_i<1$ (for all $i=1,\ldots,N$), where $|\psi_i'\rangle$ is orthogonal to all states of the set $\{|\psi_j\rangle\}$ with $|\psi_i\rangle$ omitted. It was shown that these POVM elements cannot span the identity and we have to allow for one more outcome [11] Π_{N+1}, such that $\sum_{i=1}^{N+1}\Pi_i=I_S$, where I_S is the identity in H_S. Further, $\Pi_{N+1}|\psi_i\rangle\neq 0$ for all i and, thus, $N+1$ corresponds to an inconclusive outcome. If outcome $i<N+1$ occurs, we learn that the given state was $|\psi_i\rangle$. If outcome $N+1$ occurs, corresponding to failure, we learn nothing of the state given.

In the DSE implemention we choose a unitary operator U which acts on the extended Hilbert space as

$$U|\psi_i\rangle=\sqrt{p_i}|e_i\rangle+\sqrt{q_i}|\phi_i\rangle, \tag{9}$$

where all $|e_i\rangle$ are orthogonal to each other and to all $|\phi_i\rangle$ for $i=1,\ldots,N$, so they form an orthonormal basis for an N-dimensional subspace of the extended Hilbert space H. For simplicity, they can be chosen to be the basis vectors of H_S. In what follows, we will make this choice, which is completely general but by no means mandatory. We can then perform a projective measurement using the set of complete orthogonal projectors $\{P_i=|e_i\rangle\langle e_i|,\ i=1,\ldots,N\}$ on H_S and $P_{N+1}=I-\sum_{i=1}^{N}P_i$ projecting on the ancilla subspace. If the outcome is i, $1\leq i\leq N$, we obtain the state $|\psi_i\rangle$, and if the outcome is $N+1$, we fail.

Theorem 2. An $(N-1)$-dimensional ancilla space is sufficient to unambiguously discriminate among the N given linearly independent states.

Proof. The proof is based on the method of demonstrating that the opposite assumption leads to contradiction. N linearly independent states can be distinguished unambiguously, i.e., they can be expressed in the form shown in Eq. (9). The failure states $\{|\phi_i\rangle,\ i=1,\ldots,N\}$ must be linearly dependent, otherwise they can be unambiguously distinguished, giving further information about the initial state, contrary to our assumption that the subspace $I-\sum_{i=1}^{N}P_i$ is associated with failure. So the failure states $\{|\phi_i\rangle,\ i=1,\ldots,N\}$ span a subspace of H that has at most $N-1$ dimensions. This means that the rank of P_{N+1} or Π_{N+1} is bounded, $r_{N+1}\leq N-1$. Since the rank of Π_i is 1 for $i=1,\ldots,N$, and H_S has N dimensions, from Theorem 1 it immediately follows that the number of added dimensions is at most $N-1$. ■

Corollary. In the TPE implementation we need an ancilla of at least $N+1$ dimensions, so the dimensionality of the enlarged Hilbert space is $N(N+1)$, of which $N(N+1)-N=N^2$ are the added dimensions. Since $N^2>N-1$, the DSE requires fewer resources than the TPE.

Theorems 1 and 2 determine the minimum dimensionality of the ancilla. A higher-dimensional ancilla can always be used but it does not improve the performance of the POVM. In the last example, we can use an ancilla of more than $N-1$ dimensions but this will only increase the rank of the unambiguously discriminating elements. Typically, the TPE is employed more often in experiments, but we will show that the DSE, if available, can be more economical in some cases. To this end, we analyze the detailed implementation of a POVM with the DSE and TPE, respectively. As shown in Eq. (1), the TPE needs an ancilla system and unitary operations acting jointly on the system and the ancilla. Although the ancilla may be very cheap, the experimental implementation of the unitary operations is difficult since it requires controlled interactions between the two systems. By contrast, the DSE needs to enlarge the dimensions of the system, which can be accomplished by using extra dimensions of the system itself if they are available, and controlled interactions are not involved. Often, additional unused degrees of freedom are not available, but they can be found in some cases. For example, in cavity QED [12,13], two levels of a multilevel atom can be regarded as a qubit for quantum-information processing. If we choose an atom with three or more levels, the extra levels can be employed to implement a POVM on the two-dimensional Hilbert space spanned by two levels of the atom. To choose a three- or four-level atom is relatively straightforward [13,15], so this scheme is experimentally feasible if the rank r of the POVM elements is not too large. Another example is the two orthogonal polarizations of a photon forming a qubit. If we exploit the path degrees of freedom of the photon (such as two different paths), then the paths and the polarizations form a four-dimensional Hilbert space. By Theorem 1 we can implement any POVM, provided the sum of ranks of all POVM elements is at most 4, on the polarization space via the help of the paths under current technologies [16]. Furthermore, in trapped ion systems [17], either the internal levels or the vibration levels can be regarded as a qubit. It is possible that a POVM on the space spanned by internal (or vibration) levels of an ion can be implemented with the help of the vibrational (or internal) degrees of freedom of the ion.

To show the advantage of the DSE in some cases more clearly, we give a specific scheme to unambiguously discriminate two nonorthogonal but linearly independent states by the DSE. Consider a three-level atom in a two-mode cavity. The atom states are denoted by $|g\rangle$, $|e\rangle$, and $|l\rangle$. We wish to unambiguously discriminate two nonorthogonal states $|\Psi_1\rangle=|g\rangle$ and $|\Psi_2\rangle=(1/\sqrt{2})(|g\rangle+|e\rangle)$. By Theorem 2, we want three possible outcomes from our POVM: first state, second state, and inconclusive [11]. By Theorem 1, we need only a three-dimensional space (or a one-dimensional ancilla space) to implement the UD. Using $|l\rangle$ as ancilla space, we can implement the UD as follows. One first generates one of the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$, and then performs unitary operation U on the three-dimensional Hilbert space so that $U|g\rangle=(1/\sqrt{3})|g\rangle+\sqrt{2/3}|l\rangle$; $U|e\rangle=-(1/\sqrt{3})|g\rangle+\sqrt{1/2}|e\rangle+\sqrt{1/6}|l\rangle$; $U|l\rangle=-(1/\sqrt{3})|g\rangle-\sqrt{1/2}|e\rangle+\sqrt{1/6}|l\rangle$. Thus $U|\Psi_1\rangle=(1/\sqrt{3})|g\rangle+\sqrt{2/3}|l\rangle$ and $U|\Psi_2\rangle=(1/2)|e\rangle+(\sqrt{3}/2)|l\rangle$. Finally, one measures the states with the operators $P_1=|g\rangle\langle g|$, $P_2=|e\rangle\langle e|$, $P_3=|l\rangle\langle l|$. If one gets the outcome P_1 (P_2), the generated state is unambiguously $|\Psi_1\rangle$ ($|\Psi_2\rangle$),and the outcome P_3 means failure. This scheme can be realized with current technologies [12–15] and, obviously, needs less experimental cost than the TPE. Unambiguous discrimination among three nonorthogonal states has been performed experimentally based on the DSE implementation of the optimal POVM, following these lines [19]. A TPE implementation would not have been feasible with available experimental technologies. Recently, more experimental [16] and theoretical [18] works have been carried out to further explore this promising direction.

In summary, we determined the minimum dimensionality of the ancilla required to implement the same POVM by the methods of the tensor product extension and the direct sum extension of the system Hilbert space, respectively. The method is constructive and applicable to POVM elements of arbitrary rank. It is shown that the DSE is more economical than the TPE and, in some cases, easier to implement experimentally. The results are of importance for the design of efficient programmable quantum processors [20–22] and other quantum-information processing schemes related to quantum measurement, such as entanglement purification and entanglement distillation [23]. A possible direction for further research is the design of experimental schemes based on the DSE for the implementation of optimal POVMs employed in quantum-information-processing protocols.

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