

Dimensionality-induced entanglement in macroscopic dimer systems

Dagomir Kaszlikowski,¹ Wonmin Son,² and Vlatko Vedral^{1,2}

¹*Department of Physics, National University of Singapore, 117542 Singapore, Singapore*

²*The School of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, United Kingdom*

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We investigate entanglement properties of mixtures of short-range spin- s dimer coverings in lattices of arbitrary topology and dimension. We show that in one spatial dimension nearest neighbor entanglement exists for any spin s . Surprisingly, in higher spatial dimensions there is a threshold value of spin s below which the nearest neighbor entanglement disappears. The traditional “classical” limit of large spin value corresponds to the highest nearest neighbor entanglement that we quantify using the negativity.

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Recently, there has been a considerable interest in a possible link between entanglement and properties of many-body systems [1]. Although a role of entanglement in the description of such large macroscopic systems is not entirely clear there are some promising results showing the connection between the critical phenomena and entanglement [2]. Our present paper is a small contribution to this field.

Consider a macroscopic system of N subsystems with an arbitrary spin s . Maximal entanglement of such a system corresponds to the configuration where pairs of subsystems exist in maximally entangled states (dimers). The amount of entanglement quantified by the relative entropy in this case is $(N/2)\log(2s+1)$. The dimer configuration is also the most robust to noise in the sense that one has to destroy entanglement between each pair of dimers in order to destroy all entanglement. However, there are many dimer configurations leading to the same maximal amount of entanglement. Frequently in nature there is no reason why some of them should be more likely than others, which is why they occur as superpositions or mixtures of all possible dimer coverings.

Here we analyze how superposing and mixing of dimers affects the robustness of their entanglement. We limit ourselves to short-range dimers because in practice forces generating entanglement between subsystems are themselves especially short range (in momentum space this, of course, need not be the case, an issue that is briefly addressed at the end of the paper).

Let us first consider a one-dimensional lattice that is a union of two sublattices L_A and L_B . The sites belonging to the lattice L_A (L_B) are enumerated by odd (even) numbers. We assume that the lattice is described by the so-called spin liquid, i.e., a state without magnetic order,

$$\begin{aligned} |\tilde{\psi}\rangle &= |(ab)_{12}\rangle|(ab)_{34}\rangle\cdots|(ab)_{(2N-1)(2N)}\rangle \\ &+ |(ab)_{(2N)1}\rangle|(ab)_{23}\rangle\cdots|(ab)_{(2N-2)(2N-1)}\rangle = |c_1\rangle + |c_2\rangle, \end{aligned} \quad (1)$$

with $|(ab)_{n(n+1)}\rangle = (1/\sqrt{S})\sum_{k=0}^{S-1}\gamma_S^{kb}|k\rangle_n|k+a\rangle_{n+1}$, $S=2s+1$, $a, b=0, 1, \dots, S-1$, $\gamma = \exp(2i\pi/S)$, and the index n refers to the n th site on the ring. The states $|(ab)_{n(n+1)}\rangle$ form a generalized Bell basis for two particles with spin s , i.e., they are complete and maximally entangled. In analogy with the spin- $\frac{1}{2}$ case a maximally entangled state between two sites is called

a *dimer* and the states $|c_1\rangle, |c_2\rangle$ are called *dimer coverings*. We are interested in entanglement properties of the state $|\tilde{\psi}\rangle$ in the thermodynamical limit, i.e., for $N \rightarrow \infty$. Note that it makes no difference in this limit whether we are superposing or mixing two dimer coverings. As will be proven below all the phase information is absent from the local entanglement properties, which we are interested in.

We can prove the following facts for a one-dimensional case.

(1) There is always nearest neighbor entanglement for an arbitrary value of s . Furthermore, we present the negativity as the function of s and show that it increases with s .

(2) The subset of even (or odd) sites does not contain entanglement.

(3) The subset of even (or odd) sites is maximally entangled to the rest.

Proof of (1). Let us first derive a density matrix for the nearest neighbors. Due to the translational invariance of the state it is enough to consider the density matrix ρ_{12} of the first two spins. We have

$$\begin{aligned} |c_1\rangle &= \sum_{k_1, k_2, \dots, k_N} \frac{\gamma^{(k_1+k_2+\dots+k_N)b}}{\sqrt{S^N}} |k, k_1+a\rangle_{12} \\ &\times |k_2, k_3, \dots, k_N\rangle_{357\dots} |k_2+a, k_3+a, \dots, k_N+a\rangle_{468\dots}, \\ |c_2\rangle &= \sum_{k_1, k_2, \dots, k_N} \frac{\gamma^{(k_1+k_2+\dots+k_N)b}}{\sqrt{S^N}} |k_1+a, k_2\rangle_{12} \\ &\times |k_2+a, k_3+a, \dots, k_N+a\rangle_{357\dots} \\ &\times |k_3, k_4, \dots, k_{N-1}, k_1\rangle_{468\dots} \end{aligned} \quad (2)$$

After some tedious but straightforward algebra we get

$$\text{Tr}_{345\dots(2N)}(|c_1\rangle\langle c_1|) = |(ab)\rangle\langle(ab)|,$$

$$\text{Tr}_{345\dots(2N)}(|c_2\rangle\langle c_2|) = \frac{1}{S^2},$$

$$\text{Tr}_{345\dots(2N)}(|c_1\rangle\langle c_2|) = \frac{\gamma^{abN}}{S^N} |(ab)\rangle\langle[(a+2Na)b]|, \quad (3)$$

where $\mathbb{1}$ is the identity matrix. Thus the unnormalized density matrix $\tilde{\rho}_{12}$ reads

$$\tilde{\rho}_{12} = \frac{1}{S^2} + |(ab)\rangle\langle(ab)| + \frac{\gamma^{abN}}{S^N} |(ab)\rangle\langle[(a+2Na)b]| + \text{H.c.} \quad (4)$$

The trace of the matrix $\tilde{\rho}_{12}$ equals $M=2+2S^{1-N} \times \cos(2\pi abN/S)$ if the number of the sites is a multiplicity of S , i.e., $2N=mS$ (m is an integer) and 2 otherwise.

In the thermodynamic limit the normalized state ρ_{12} becomes an equal mixture of the maximally entangled state $|(ab)\rangle$ and the white noise

$$\rho_{12} \underset{N \rightarrow \infty}{\approx} \frac{1}{2S^2} + \frac{1}{2} |(ab)\rangle\langle(ab)|. \quad (5)$$

In a similar way we can compute the density matrix between next nearest neighbors, for instance, between the first and the third spin

$$\tilde{\rho}_{13} = \frac{2}{S^2} \mathbb{1} + \frac{1}{S^N} \sum_{k_1, k_2} \gamma^{-abN} |k_1, k_2\rangle\langle k_2 + 2(N-1)a, k_1 + 2a| + \text{H.c.}, \quad (6)$$

which becomes the white noise in the thermodynamical limit because the off diagonal elements rapidly vanish with N . It can be checked that for larger separation one always gets the white noise for the same reason, i.e., the off-diagonal elements vanish with N .

We finally calculate the negativity between nearest neighbor spins. Since their state is a mixture of a maximally entangled state with identity matrix as is expressed in Eq. (5), the overall eigenvalues will be the same mixtures of the eigenvalues of the identity matrix and the eigenvalues of the maximally entangled state. This is also true for the partially transposed state since a density matrix is diagonalized simply by unitary operation as

$$\rho_{12}^{T_1} \rightarrow U \rho_{12}^{T_1} U^\dagger = \frac{1}{2S^2} + \frac{1}{2} U [|(ab)\rangle\langle(ab)|]^{T_1} U^\dagger. \quad (7)$$

T_1 means partial transposition of the density matrix on the particle 1. It is therefore easy to see that the negative eigenvalues of the partially transposed state all have the absolute value of $(S-1)/2S^2$. At the same time, the number of negative eigenvalues of the partially transposed total state is calculated to be $(S^2-S)/2$. The total negativity, which is the summation of all negative eigenvalues, measures the amount of entanglement in the state [3] and the total negativity is hence $S(S-1)^2/4S^2$ for the state. We see that, for large spin value, the total negativity grows linearly with the size of spin as claimed earlier.

Proof of (2) and (3). The density matrix ρ_o (ρ_e) describing the subset of all the odd (even) sites has the following form:

$$\tilde{\rho}_o = 2 + \frac{\gamma^{-Nab}}{S^N} \sum_{k_1, k_2, \dots, k_N} |k_1, k_2, \dots, k_N\rangle \times \langle k_N + 2a, k_1 + 2a, \dots, k_{N-1} + 2a| + \text{H.c.},$$

$$\tilde{\rho}_e = 2 + \frac{1}{S^N} \sum_{k_1, k_2, \dots, k_N} |k_1 + a, k_2 + a, \dots, k_N + a\rangle \times \langle k_2, k_3, \dots, k_N, k_1| + \text{H.c.} \quad (8)$$

Both of them become the white noise in the thermodynamic limit, which implies that there is no entanglement between any subset of odd (even) sites. However, the set of all the odd sites is maximally entangled with the set of all the even sites.

The entanglement between the subsets of odd and even sites can be seen already at the level of four sites. For instance, the state of the first four sites is given by

$$\begin{aligned} \tilde{\rho}_{1234} &= |(ab)_{12}, (ab)_{34}\rangle\langle(ab)_{12}, (ab)_{34}| \\ &+ \frac{1}{S} \otimes |(ab)_{23}\rangle\langle(ab)_{23}| \otimes \frac{1}{S} \\ &+ O(S^{2-N}) |(ab)_{13}, (ab)_{24}\rangle\langle(ab)_{14}, (ab)_{23}| + \text{H.c.} \end{aligned} \quad (9)$$

It is clear that the sites 1 and 3 treated as one subsystem are entangled to the subsystem consisting of the sites 2 and 4 but there is no genuine multipartite entanglement. The above formula can be easily generalized for an arbitrary subset containing even and odd sites and it can be seen that as long as the size of the subsets is fixed the subset is not genuinely multipartite entangled in the thermodynamical limit.

There are situations in which different dimer coverings have the same energy. This happens, for example, in the Majumdar-Ghosh Hamiltonian [4], where we have nearest and next nearest neighbor interactions. Unless there is some broken symmetry mechanism each different covering will contribute to the overall state with equal weight in the thermodynamic equilibrium. This means that it is more appropriate to consider a mixture of dimer coverings rather than their superposition. In any spatial dimension such a mixture, σ , has the following form

$$\sigma = \frac{1}{M} \sum_{k=1}^M |c_k\rangle\langle c_k|, \quad (10)$$

where $|c_k\rangle$ is the k th dimer covering and M is the number of all possible dimer coverings. For simplicity we present our result in two dimensions. The generalization to higher dimensions is straightforward. (See Refs. [5,6].)

We consider an infinite two-dimensional square lattice that is a union of two sublattices L_A and L_B . A site belonging to the sublattices L_A has neighbors belonging to the sublattices L_B (coordination number is 4). In this case, the state σ is defined as the incoherent mixture of all possible dimer coverings between the sublattices L_A and L_B . The following question arises, given a link of a pair of spins: What is the ratio between coverings that admit maximal entangled pair on the spins and all the other states?

To answer this question let us imagine a situation when two arbitrary neighbors, say A and B , are in a maximally entangled state, i.e., they form a dimer. The rest of the sites can be covered by dimers and there are Σ of such coverings. For an infinite lattice Σ is, of course, infinite but we can assume that the lattice is very large in which case Σ is also

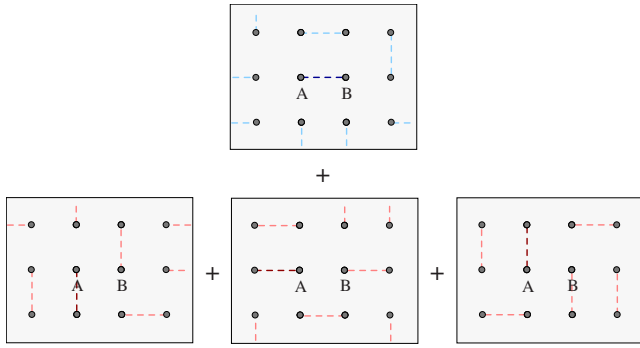


FIG. 1. (Color online) Mixture of four possible dimer coverings, $\{|c_k\rangle, k=1, \dots, 4\}$, for an infinite lattice site. The top box depicts one of the dimer coverings when the neighbors A and B are maximally entangled. The other three show some of the possible coverings when the state of A and B is maximally mixed (therefore disentangled). We argue that the ratio of the number of the coverings when A and B are maximally entangled to the number of coverings when A and B are maximally mixed is 1:3 in the thermodynamic limit (just like in the figure).

large but finite. Suppose now that the same neighbors are not in a maximally entangled state, i.e., they form dimers with their other neighbors. As is illustrated in Fig. 1, it is easy to see that in each such case (there are three of them) the number of the remaining dimer coverings Σ' is equal and approximately the same as in the previous case where the two neighbors A and B were in a maximally entangled state, i.e., $\Sigma \approx \Sigma'$. Therefore, for an infinite lattice the ratio of the dimer coverings including the sites A and B to the dimer coverings not including the sites A and B equals $\frac{1}{3}$. The consequence of this is that after tracing out all the other sites the density matrix describing the neighboring sites A and B (A and B can be, in fact, any neighbors) is given by a generalized Werner state containing $\frac{3}{4}$ of the white noise. Similar to the 1D case examined in the proof of (1), one can find that all the negative eigenvalues of the partially transposed reduced density matrix of two given neighboring spins A and B are $(3-S)/4S^2$. Thus, for a system consisting of spin $\frac{1}{2}$, there is no entanglement between the neighbors and the situation changes for larger spins $S > 3$ (recall that $S=2s+1$, where s is the spin). It agrees with the previous result that the higher amount of the white noise $S/(S+1)$ can be admixed to a maximally entangled state without destroying entanglement [7].

Similar reasoning can be applied to a “honeycomb” two-dimensional lattice (coordination number 3). In this case each spin has only three neighbors which gives us the amount of the white noise in the Werner state between the nearest neighbors equal to $\frac{2}{3}$. For spin- $\frac{1}{2}$ we do not have entanglement (the state is on the verge of being entangled), whereas entanglement exists for larger spins, i.e., for $S > 2$.

In higher than two dimensions we can apply exactly the same logic. All we need to do is to calculate the ratio of the number of coverings containing a dimer between two neighboring sites and those that do not. This ratio is always a fraction $1/R$, where R is a finite number that is the function of the coordination number (for a simple regular lattice it is

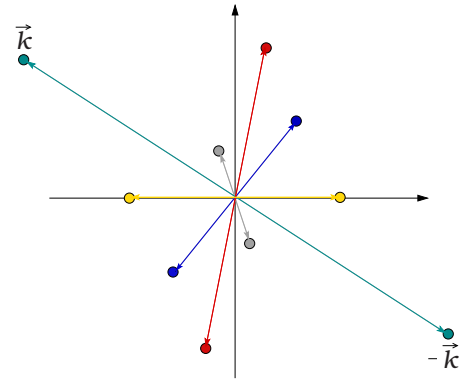


FIG. 2. (Color online) Arrangement of dimers in the BCS ground state in the momentum space. The pairs of points represent pairs of spin entangled electrons with momenta of the same magnitude but of opposite directions. As we see this arrangement is not isotropic, which is why our considerations no longer apply. In this state we have fermionic pair condensation and yet each of the fermions is spin- $\frac{1}{2}$. This is in contrast to the isotropic case when the minimal spin required for entanglement in three dimensions is $\frac{5}{2}$ (see text for detailed explanation).

equal to twice the spatial dimension). Therefore, in any dimension and any lattice structure, entanglement will always exist for spins with magnitude higher than R ; likewise it vanishes below this value.

A possible test of our prediction that there is a critical value of the spin below which entanglement does not exist could go as follows. Recently, condensates of fermionic atoms were observed in the laboratory [8]. In order to achieve condensation fermions have to form entangled pairs (so-called Cooper pairs), which then behave like quasibosons. Entangled fermionic pairs are formed through a scattering process the strength of which can be controlled experimentally by Feshbach resonance. Our calculation implies that only atoms with sufficiently high spins can form entangled pairs. In three dimensions we require atoms to have spin higher than 2 (i.e., $s \geq \frac{5}{2}$). The experiment by Regal *et al.* [8] uses potassium atoms, ^{40}K , whose spin is $\frac{9}{2}$. Were they to use atoms with spin of $\frac{3}{2}$ or lower, we conjecture that no fermionic condensation would be possible. Note that allowing long-range dimers can only increase the spin value necessary for entanglement between any two sites on the lattice. This is because the ratio of the dimer coverings contributing to the maximally entangled state between any two points on the lattice to the rest of the coverings decreases.

It is important to emphasize that our assumptions do not need to hold in practice. For example, the BCS model of fermionic condensation [9] has a ground state where momenta of spins are coupled in opposite directions as is schematically shown in Fig. 2. Namely, spin dimers now exist only between specific points on the lattice. This affects the ratio of mixing and allows for entanglement with low spin in higher dimensions. This model is highly nonisotropic in the sense of lattice points not having the neighborhood.

As a conclusion, we investigate entanglement properties of mixtures of short-range spin- s dimer coverings in lattices of arbitrary topology and dimension. We show that, in one

spatial dimension, nearest neighbor entanglement exists for any spin s . In higher spatial dimensions, there is a threshold value of spin s below which the nearest neighbor entanglement disappears and the threshold is found as $s=(2D-1)/2$ when D is the spatial dimensionality. Moreover, the traditional “classical” limit of large spin, $s \rightarrow \infty$, corresponds to the highest nearest neighbor entanglement that we quantify using the negativity.

Dimer coverings present very simple entangled states. Yet they are physically relevant as they frequently occur in nature. Their entanglement structure is simple. Here we show that overall entanglement is fully characterized only in terms of nearest neighbor entanglements. In spite of its simplicity,

we encountered one surprising fact. Namely, in higher than two dimensions the existence of entanglement is dependent on the size of the spin in each lattice site. Many questions remain open. For example, what happens when a degree of anisotropy is introduced into dimer coverings? Also, when does long range entanglement arise if we allow long-range dimers? We hope that our work stimulates a number of other interesting directions of research.

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- [1] M. Williamson and V. Vedral, *Phys. Rev. A* **76**, 032115 (2007).
[2] A. Osterloch, L. Amico, G. Falci, and R. Fazio, *Nature (London)* **416**, 608 (2002).
[3] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002).
[4] C. K. Majumdar and D. K. Ghosh, *J. Math. Phys.* **10**, 1388 (1969).
[5] Long range spin-1/2 dimer coverings, called spin gas, in two dimensions and higher were considered in [6].
[6] A. Chandran, D. Kaszlikowski, A. Sen De, U. Sen, and V. Vedral, *Phys. Rev. Lett.* **99**, 170502 (2007).
[7] R. F. Werner, *Phys. Rev. A* **40**, 4277 (1989); M. Horodecki, P. Horodecki, and R. Horodecki, *ibid.* **60**, 1888 (1999); M. Horodecki and P. Horodecki, *ibid.* **59**, 4206 (1999).
[8] C. A. Regal, M. Greiner, and D. S. Jin, *Phys. Rev. Lett.* **92**, 040403 (2004).
[9] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957).