Bethe's aperture theory for arrays

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Bethe's theory is applied to the optical transmission through an array of apertures in an infinitely thin perfect electric conductor. This method shows that, in the small-aperture limit, the lowest-order TM evanescent mode governs the transmission process, allowing for both 100% transmission at the passband wavelength and zero transmission at the Wood's anomaly. The applicability of this theory to total transmission in other systems is discussed, and a specific example of a single aperture in a transverse screen of a rectangular waveguide is given.

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I. INTRODUCTION

The study of arrays of small apertures in an infinitely thin perfect electric conductor (PEC) sheet has received considerable attention since the report of extraordinary optical transmission through apertures in a metal film [1]. That work claimed to have extraordinary optical transmission (EOT) with respect to Bethe's theory; however, Bethe's theory was formulated only for a single aperture and not an array [2]. Compared with past results on the transmission through aperture arrays, the extraordinary optical transmission is actually a *reduction* in transmission—total transmission is expected at the passband wavelength of the array structure [3–5]. For large apertures, analytical approximations to the transmission were presented, yet for the small-aperture limit it has been claimed that "all simple formulas are no longer reliable" [5]. Therefore Bethe's theory, which replaces a subwavelength aperture with an infinitesimal magnetic polarization current for normal incidence, is of interest for solving for the transmission through aperture arrays.

A solution for the transmission through aperture arrays was presented recently [6]. In that work, Babinet's principle was used to replace the aperture array with a disk array and then the overall polarization of the disk array was derived by an infinite summation over the mutual dipole coupling between the disks. The final analytic result was derived "from the divergent terms of the sum over the reciprocal lattice vectors" [6]. While it is clear from that work that the total transmission comes from multiple scattering, the role of the bound modes was not clear. Here, it will be shown that the physical origin of this divergent component is simply from the contribution of a single bound mode which governs the transmission in the small-aperture limit.

In this work, a different approach is taken to derive an analytic expression for transmission. The self-consistent-field distribution is solved in the presence of the magnetic polarization current excitation from the apertures. The electric field at the surface is represented by a Fourier expansion. This allows for a physical description of the total transmission phenomenon—the magnetic field at the surface is dominated by the lowest-order bound mode and then rescattered by the aperture array to allow for total transmission. As a result, the previously derived transmission dependence for the small-aperture limit is reproduced, showing clearly that the divergent contribution is from the lowest-order bound mode.

The main contribution of this work is to provide a physical mechanism for total transmission through an array of small holes in a thin metal screen: it arises from only a single TM bound mode that is near the cutoff. The TM bound mode has a divergent magnetic field component at the cutoff, so that all of the incident energy is coupled into that mode through the magnetic polarization current of the aperture. The energy is then reradiated on the other side of the screen. With this understanding of the physics of total transmission, it is possible to extend the study of total transmission to other geometries. In this work, total transmission through a single aperture in a transverse screen of a rectangular waveguide is discussed.

II. BETHE'S APERTURE THEORY FOR AN ARRAY

Figure 1(a) shows the geometry of the aperture array. A square unit cell of the array of apertures is considered with side length a. The aperture is centered at the origin, and it lies in a PEC in the x-y plane. A plane wave is normally incident from the negative z direction, with unity electric field linearly polarized in the x direction. The x component of the electric field at the surface may be represented by a Fourier expansion on the side of incidence,

$$E_x(x,y,z=0^-) = 1 + \sum_{m,n} r_{mn} \cos\left(\frac{2m\pi x}{a}\right)\cos\left(\frac{2n\pi y}{a}\right), \tag{1}$$

and on the side of transmission,

$$E_x(x,y,z=0^+) = \sum_{m,n} t_{mn} \cos\left(\frac{2m\pi x}{a}\right)\cos\left(\frac{2n\pi y}{a}\right). \tag{2}$$

These fields must equal one another. Furthermore, they should be equal to zero on the surface of the PEC. Figure 1(b) shows a schematic of the field distribution at the surface. By considering these boundary conditions in the context of the transmitted plane wave, it is clear that the contri-

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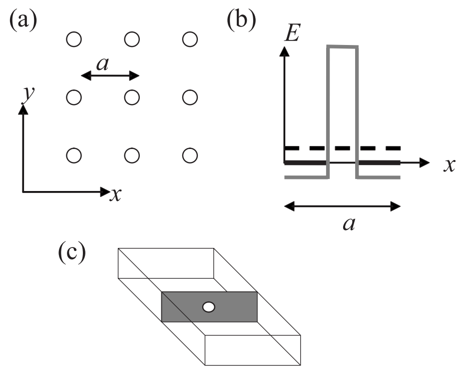

FIG. 1. (a) Geometry of aperture array for the case of circular holes. (b) Schematic of transverse electric field distribution for $y=z=0$. The PEC is shown with a thick black line, the constant electric field of the transmitted plane wave is shown with a dashed line, and the nonconstant field is shown with a gray line. The constant and nonconstant field components cancel each other out over the PEC. (c) Schematic of aperture in transverse PEC screen in a rectangular waveguide.

bution for the nonconstant Fourier terms with m,n not both equal to zero must give

$$\int\int_h E_x(x,y,z=0)dx\,dy = a^2 t_{00}, \tag{3}$$

where h represents the region of the aperture. In the small-aperture limit and for m,n small enough that the cosine dependence may be considered constant over the aperture, the Fourier transform implies

$$t_{mn} \simeq 4t_{00} \tag{4}$$

for $m,n \neq 0$ and

$$t_{mn} \simeq 2t_{00} \tag{5}$$

for $m,n=0,\ m\neq n$. This formulation shows that whatever light makes it through the aperture is bunched up at the aperture. Furthermore, the electric field in the aperture contributes to the evanescent modes equally, as long as their wave vector in the x-y plane is small.

Since we are considering normal incidence, Bethe's theory requires knowledge of the magnetic field at the aperture. For $\lambda \to a^+$, a single component dominates the magnetic field—that is, for $m=1,\ n=0$. For all other values of m,n, the field remains finite as $\lambda \to a^+$ and the series converges for finite apertures. Therefore, for $\lambda \gtrsim a$, only the lowest-order bound mode at the surface need be considered in addition to the incident, reflected, and transmitted plane waves. Based on the Fourier description of the electric field at the surface, it may be shown that the y component of the magnetic field on the transmission side at the surface is given by

$$H_y(0,0,0^+) \simeq \frac{t_{00}}{Z_0} - \frac{i2t_{00}}{Z_0\sqrt{\frac{\lambda^2}{a^2}-1}}, \tag{6}$$

where Z_0 is the characteristic impedance of free space. Here the plane-wave contribution is retained as an orthogonal component to the evanescent waves; it is $\pi/2$ out of phase with the rest of the field. Similarly, the magnetic field on the side of incidence may be expressed as

$$H_y(0,0,0^-) \simeq \frac{1-r_{00}}{Z_0} + \frac{i2t_{00}}{Z_0\sqrt{\frac{\lambda^2}{a^2}-1}}. \tag{7}$$

In the derivation of Eqs. (6) and (7) Maxwell's equations have been used to provide a relation between the electric and magnetic fields for the lowest-order TM bound mode, as outlined in the Appendix. It is clear that the divergent term $1/\sqrt{\frac{\lambda^2}{a^2}-1}$ arises from the ratio between the electric and magnetic fields of the lowest-order TM bound mode. In particular, close to the cutoff of that mode, the magnetic field diverges for a finite electric field. As a result, that single mode dominates the transmission process in the regime of applicability of Bethe's theory.

It is now possible to apply the aperture theory and replace the aperture with an equivalent magnetic polarization current. The equivalent magnetic polarization current is proportional to the magnetic field:

$$p_m(x,y,z) = \alpha_m[H_y(0,0,0^-) - H_y(0,0,0^+)]\delta(x)\delta(y)\delta(z), \tag{8}$$

where α_m is the magnetic polarizability and $\delta(x)$ is the Dirac-delta function.

Using the boundary condition $1+r_{00}=t_{00}$ gives

$$p_m(x,y,z) = \frac{2\alpha_m}{Z_0}\left(1 - t_{00} + \frac{i2t_{00}}{\sqrt{\frac{\lambda^2}{a^2}-1}}\right)\delta(x)\delta(y)\delta(z). \tag{9}$$

The transmitted field t_{00} is now solved self-consistently as coming from the excitation by an array of magnetic polarization currents, so that

$$t_{00} = \frac{-i2\pi Z_o}{a^2\lambda}\int\int\int_h p_m(x,y,z)dx\,dy\,dz. \tag{10}$$

Using this result, we can find the total transmittance $T=|t_{00}|^2$:

$$T = \frac{1}{1+\left(\frac{a^2}{2\pi k}\left[\frac{\pi}{2\alpha_m} - \frac{4\pi k}{a^2\sqrt{\frac{\lambda^2}{a^2}-1}}\right]\right)^2}, \tag{11}$$

where $k=2\pi/\lambda$. Here $\alpha_m=D^3/6$ for a circular hole with diameter D and $\alpha_m=\pi l d^2/16$ for a rectangular hole where l and d are the sides of the rectangle.

This result is accurate for small apertures and wavelengths $\lambda > a$. While the truncation of the magnetic field ex-

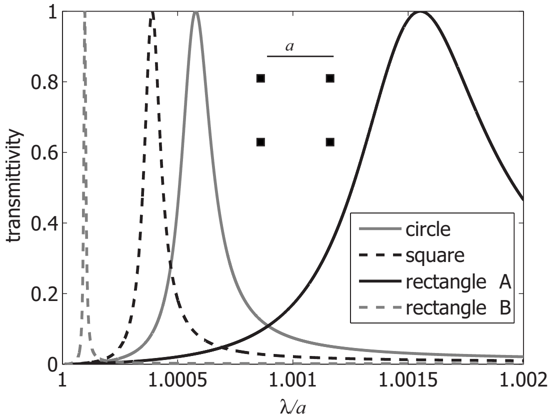

FIG. 2. Transmittivity for arrays of circles, holes, and rectangles, each with the apertures covering 2% of the total area. Rectangle *A* is half as wide in the direction of the electric polarization, and rectangle *B* is twice as wide in that direction.

pansion of Eqs. (6) and (7) was only valid for wavelengths close to the array periodicity, here we see that the passband occurs in that regime. For other wavelengths, the dominant term involves the magnetic polarizability, so that Eq. (11) is valid for all $\lambda > a$.

As $\lambda \to a^+$, $T \to 0$, which is expected since this is the wavelength of the Wood's anomaly. The transmission goes to zero due to the divergent large magnetic field buildup in the lowest-order TM bound mode. In the small-aperture limit, it is possible to obtain complete transmission for the wavelength

$$\lambda_t = \frac{128\pi^2\alpha_m^2}{a^5} + a. \tag{12}$$

III. DISCUSSION

A. Shape effect

Several experimental and theoretical works have considered the influence of aperture shape on optical transmission through metals for single holes [7,8] and for arrays [9–11]. For finite-thickness metal films, the Fabry-Perot (FP) resonances play an important role in the transmission [8]. These FP resonances are strongly influenced by the modes within the aperture and thereby the aperture shape. The separation of FP-related phenomena from phenomena that come from coupling to the bound surface modes has been attempted by considering double-hole and elliptical hole structures [11]. For the theory presented here, however, the FP resonances are naturally excluded since the metal is infinitely thin, and therefore, it is interesting to see the origin of the shape effect in this case. Furthermore, the metal is a PEC so that surface-plasmon-related phenomena are excluded.

Figure 2 shows the transmittivity for various aperture shapes of the same surface area within the small-hole limit of Bethe's theory. It is clear that a strong shape dependence exists here that is the result of the magnetic polarizability of the aperture, but is not the result of propagation within the aperture since the screen is infinitely thin. Nevertheless, some of the features of the shape effect in finite-thickness apertures are reproduced, such as the redshift in wavelength of the transmission peak for narrower rectangular apertures [9]. In this work, however, the shape effect arises purely from the polarizability of the aperture. The details of the shape effect can be found in how the polarizability allows for excitation of the lowest-order TM bound mode. More specifically, the shape of the aperture plays a role in scaling the coupling between the lowest-order bound mode and the incident plane wave, to cause the second term in the denominator of Eq. (11) to go to zero.

B. Comparison with past EOT theories

Most of the discussion in the literature is centered around how much of a role the surface-plasmon character of the metals plays in the EOT (e.g., [12–15]). In some cases, it is argued that the surface plasmon polariton plays the most important role in EOT [13], sometimes even a negative role [14], whereas in other works, diffraction from all the modes is thought to be necessary [12,15]. Several works have also discussed the importance of FP resonances to EOT [16–18].

In this paper, it is shown that a single bound mode provides the dominant contribution, allowing for total transmission, even for an infinitely thin PEC screen. It is important that the material be a PEC, because this means that no surface plasmon phenomena are present. It is also important that the PEC screen be infinitely thin because this means that no FP resonances can be supported within the apertures, so clearly these have no role in the present theory. The most important contribution of this paper is to show that a single TM bound mode near the cutoff dominates the transmission process and allows for total transmission in the limit of Bethe's theory (i.e., small apertures).

It is interesting to note that recent works have shown closer to 100% transmission in experiments on subwavelength hole arrays [19–21], whereas past works only showed 1%–10% transmission [1]. These newer experimental studies have been in longer-wavelength systems that are more closely related to the conditions of this work. In particular, at longer wavelengths the metal more closely resembles a PEC. Furthermore, the films are thinner with respect to the wavelength, and so the infinitely thin-film limit is better approximated by those experimental works (e.g., [20]).

C. Comparison with the dipole-scattering method

In the small-aperture limit of Bethe's theory, Eq. (11) is the same as was derived previously [6]. In that work, the dipole-dipole interaction dyadics gave an infinite series with a $1/\sqrt{\frac{\lambda}{a}-1}$ divergence. Here, the lowest-order evanescent TM mode gives the dependence of $1/\sqrt{\frac{\lambda^2}{a^2}-1}$, which approaches $1/\sqrt{2\left(\frac{\lambda}{a}-1\right)}$ for $\lambda \to a^+$; therefore, it shows the same divergence in the limit of Bethe's theory.

It is clear that this common divergence is the result of the lowest-order bound mode, which governs the transmission in the small-hole limit. For the lowest-order TM bound mode, the magnetic field diverges for a finite electric field when it is near the cutoff wavelength, and therefore, that mode dominates the transmission process in the small-hole limit near the cutoff. The demonstration of the important role of this single bound TM mode for the total transmission phenomenon is the main contribution of the theory presented here. In the next section, the applicability of this theory to other systems is discussed.

D. Extension to other systems: Waveguides

The theory presented here provides a general mechanism for total transmission through small apertures in a PEC screen. The ingredients for total transmission are the existence of only a single propagating mode and a bound mode at the surface. A further requirement is that the bound mode have a divergent transverse magnetic field component near the cutoff wavelength of that mode. It is interesting to consider if other systems will allow for these conditions to be met.

Figure 1(c) shows a schematic of an example system for total transmission: a rectangular waveguide in a PEC with an infinitely thin transverse PEC screen (perpendicular to the walls of the waveguide) and a small aperture in the center of the screen. The waveguide should be designed to allow propagating modes TE_{10}, TE_{20}, TE_{01}, TE_{11}, TE_{02}, and TM_{11} and to be just narrower than the cutoff of the TM_{12} mode. If the TE_{10} field is incident on the screen, none of the other propagating modes are excited by the aperture[1] and they may be neglected entirely. Thereby, the condition of having a single propagating mode is met.

In this configuration, the TM_{12} mode will have a divergent magnetic field component in the plane of the screen as the wavelength decreases to the cutoff value. This TM_{12} mode is excited by the aperture, and so it will dominate the transmission process within the Bethe limit. From these considerations, it is expected that this system will show total transmission too, albeit with a slightly different formulation than the one given by Eq. (11) due to the differences between waveguide modes and free-space propagation.

IV. CONCLUSION

In conclusion, Bethe's aperture theory was applied to a square array of apertures. This theory provides a simple physical mechanism for the phenomenon of EOT through an array of small apertures: the lowest-order TM bound mode dominates the transmission process near its cutoff wavelength. The formulation shows that an array of apertures has a passband wavelength with 100% transmission, as well as the Wood's anomaly with zero transmission. Furthermore, it agrees quantitatively with a previously derived result using an infinite summation over dipole-dipole interactions. The physical understanding of the total transmission phenomenon provided by this theory allows for the application to other small-aperture systems. One such system, proposed in this work, is the rectangular waveguide with a transverse PEC screen with an aperture in the center of the screen.

APPENDIX: MAGNETIC FIELD OF THE LOWEST-ORDER TM BOUND MODE

The electric field of the lowest-order TM bound mode above the surface of the screen (for $z>0$) may be expressed as

$$\vec{E}(x,y,z) = \left[\hat{x}\cos\left(\frac{2\pi x}{a}\right) - \hat{z}\frac{\lambda}{a\alpha}\sin\left(\frac{2\pi x}{a}\right)\right]\exp(-k\alpha z), \tag{A1}$$

where $\hat{x}$ and $\hat{z}$ are the unit vectors in the x and z directions, $k=\frac{2\pi}{\lambda}$, with the free-space wavelength of λ, and $\alpha=\sqrt{\frac{\lambda^2}{a^2}-1}$.

Using Faraday's law of induction in differential form gives the corresponding magnetic field above the screen:

$$\vec{H}(x,y,z) = \frac{i\hat{y}}{\mu_0\omega}\left[k\alpha - \frac{2\pi\lambda}{\alpha a^2}\right]\cos\left(\frac{2\pi x}{a}\right)\exp(-k\alpha z), \tag{A2}$$

where $\omega=ck$ and c is the speed of light in free space.

Considering the situation of interest for Bethe's aperture theory for arrays, where $\lambda \to a^+$, allows for the magnetic field at the location of the hole, $x=y=z=0$, to be expressed as

$$H(0,0,0) \simeq \frac{-i}{Z_0\sqrt{\frac{\lambda^2}{a^2}-1}}, \tag{A3}$$

where $Z_0=c\mu_0$. Since $E(0,0,0)=1$, the above equation is also the ratio between the magnetic and electric fields, as used in Eqs. (6) and (7). Below the surface, the ratio changes sign.

[1] It is important to note that electric field components perpendicular to the aperture may also excite waveguide modes at the aperture, so that the geometry described here has been specifically designed so that all of these components are zero at the center of the aperture. The waveguide system is different from the main topic of this paper; therefore, detailed analysis of the transmission phenomenon is not provided.

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