Excitation spectrum and phase separation of double Bose-Einstein condensates in optical lattices

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The excitation spectrum for symmetric two-component Bose-Einstein condensate (BECs) in optical lattices is obtained, with emphasis on its quantitative dependence on intercomponent interaction parameter. The resulting two branches of the excitation spectrum have a linear spectrum asymptotically in the low momentum limit, which is expected as a behavior of superfluid. The critical superfluid velocity, depending on the intercomponent interaction parameter, is then obtained, which is quite different from the result obtained in other reference. A requirement of real excitation spectrum also results in a condition on the interaction parameters, which coincides with the phase separation condition. Hence in real spectrum excitation, it is proposed that there should be two branches of superfluid with different critical superfluid velocities, while in the complex spectrum it is predicted that there should be phase separation in the two-branch superfluid system. The excitation spectrum for an asymmetric two-component BEC system is also presented and analyzed.

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I. INTRODUCTION

Alongside development of laser cooling techniques and setups, the experimental research on the Bose-Einstein condensate (BECs) flourishes, and numerous phenomena have been found. Principles of laser cooling and ideas about relevant experimental setups for different cooling traps can be found in Refs. [1,2]. Different phenomena of BECs are demonstrated, such as superfluidity and quantized vortices [3], quantum phase transition [4], dynamics of collapse and explosion [5], and dynamics of revival [6].

Experimentally, the interaction between two bosons of the same kind, as measured by the scattering length a_s, can be tuned in a large range (from attractive to repulsive) by using the Feshbach resonance [7], and different traps [1] and different optical lattices [8] can be formed and applied to cool atoms. For two-component BECs systems, spin-dependent optical lattices can also be used to control nonlinear interactions between the BECs components [9]. All of these experimental accessible controls of BECs systems make it convenient to conduct various experimental research on BECs systems.

For one-component BECs in optical lattices, the gapless superfluid phonon spectrum is verified in experiment [4]. In this paper, we concentrate on the superfluid excitation spectrum for two-component BECs in optical lattice. The phonon excitation spectrum, which is expected as a behavior of superfluid is obtained. Analysis on the excitation spectrum indicates that there should be phase separation happening in the two-component superfluid state.

II. THE MODEL

The general second-quantized model Hamiltonian for a two-component boson gas in a three-dimensional (3D) optical lattice has the following form [10,11]:

$$\begin{aligned}H=\sum_i\Bigg[&\int d\mathbf{y}\Psi_i^\dagger(\mathbf{y})\left(-\frac{\hbar^2}{2m_i}+V_i(\mathbf{y})\right)\Psi_i(\mathbf{y})\\&+\int d\mathbf{y}\frac{g_i}{2}\Psi_i^\dagger(\mathbf{y})\Psi_i^\dagger(\mathbf{y})\Psi_i(\mathbf{y})\Psi_i(\mathbf{y})\Bigg]\\&+\int d\mathbf{y}\frac{g_{AB}}{2}\Psi_A^\dagger(\mathbf{y})\Psi_B^\dagger(\mathbf{y})\Psi_B(\mathbf{y})\Psi_A(\mathbf{y})\\&+\int d\mathbf{y}\frac{g_{BA}}{2}\Psi_B^\dagger(\mathbf{y})\Psi_A^\dagger(\mathbf{y})\Psi_A(\mathbf{y})\Psi_B(\mathbf{y}),\end{aligned}\tag{1}$$

where the index i is summation in the range of A and B, which labels the components of the bosons. $\Psi_i^\dagger(\mathbf{y})$ and its conjugate denotes the boson field operators of component i, and the s-wave scattering strength of intracomponent and intercomponent bosons are, respectively,

$$g_i=4\pi a_i\hbar^2/m,\quad i=A,B,$$

$$g_{AB}=g_{BA}=2\pi a_{AB}\hbar^2/m_{AB},\tag{2}$$

with m_{AB} being the reduced mass of the two boson species.

Introducing the Wannier basis, after discretizing and transforming the Hamiltonian into the quasimomentum space, we get the Hamiltonian of the two-component BECs,

$$\begin{aligned}H=\sum_{i,\mathbf{k}}&[\epsilon_i-J_iz(\cos k_1a_l+\cos k_2a_l+\cos k_3a_l)]b_{\mathbf{k}i}^\dagger b_{\mathbf{k}i}\\&+\frac{q_A}{2N_s}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}b_{\mathbf{k}_1A}^\dagger b_{\mathbf{k}_2A}^\dagger b_{\mathbf{k}_3A}b_{\mathbf{k}_4A}\delta(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4)\\&+\frac{q_B}{2N_s}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}b_{\mathbf{k}_1B}^\dagger b_{\mathbf{k}_2B}^\dagger b_{\mathbf{k}_3B}b_{\mathbf{k}_4B}\delta(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4)\\&+\frac{q_{AB}}{N_s}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}b_{\mathbf{k}_1A}^\dagger b_{\mathbf{k}_2A}^\dagger b_{\mathbf{k}_3B}b_{\mathbf{k}_4B}\delta(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4),\end{aligned}\tag{3}$$

where $z=2$ and k_i are components of wave vector $\mathbf{k}$ and the parameters for each component (indicating on-site energy ϵ_i, tunneling coefficients J_i, and intracomponent interaction parameters q_i) are similarly defined as those of one-component BECs in an optical lattice [12,13], while the intercomponent interaction parameter q_{AB} is [14]

$$q_{AB}=\frac{2\pi a_{AB}\hbar^2}{m_{AB}}\int|w_A(\mathbf{y})|^2|w_B(\mathbf{y})|^2d\mathbf{y}. \tag{4}$$

By using the harmonic potential approximation to the optical lattice potential similar to that used in the one-component case [12,13],

$$V_i(\mathbf{y})=\frac{1}{2}m_i\omega_i^2(y_1^2+y_2^2+y_3^2), \tag{5}$$

$$\omega_i^2=\frac{2k_l^2V_0^i}{m_i}, \tag{6}$$

$$\alpha_i=\sqrt{\frac{m_i\omega_i}{\hbar}}, \tag{7}$$

$$w_i(\mathbf{y})=\sqrt{\left(\frac{\alpha_i}{\sqrt{\pi}}\right)^3}\exp\left(-\frac{1}{2}\alpha_i^2(y_1^2+y_2^2+y_3^2)\right), \tag{8}$$

with $i=A,B$, q_{AB} can be evaluated in an explicit form

$$q_{AB}=\frac{2\pi a_{AB}\hbar^2}{m_{AB}}\left(\frac{\alpha_A\alpha_B}{\alpha_{AB}}\right)^3\left(\frac{1}{\sqrt{\pi}}\right)^3, \tag{9}$$

where

$$\alpha_{AB}^2=\alpha_A^2+\alpha_B^2. \tag{10}$$

Terms in the Hamiltonian are grouped into H_0, H_{intra}, and H_{inter}, namely,

$$H_0=\sum_{i,\mathbf{k}}[\epsilon_i-J_iz(\cos k_1a+\cos k_2a+\cos k_3a)]b_{\mathbf{k}i}^\dagger b_{\mathbf{k}i},$$

$$H_{\text{intra}}=\frac{q_A}{2N_s}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}b_{\mathbf{k}_1A}^\dagger b_{\mathbf{k}_2A}^\dagger b_{\mathbf{k}_3A}b_{\mathbf{k}_4A}\delta(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4)+\frac{q_B}{2N_s}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}b_{\mathbf{k}_1B}^\dagger b_{\mathbf{k}_2B}^\dagger b_{\mathbf{k}_3B}b_{\mathbf{k}_4B}\delta(\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4),$$

$$H_{\text{inter}}=\frac{q_{AB}}{N_s}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}b_{\mathbf{k}_1A}^\dagger b_{\mathbf{k}_2A}b_{\mathbf{k}_3B}^\dagger b_{\mathbf{k}_4B}\delta(\mathbf{k}_1+\mathbf{k}_3,\mathbf{k}_2+\mathbf{k}_4). \tag{11}$$

For the two-body interaction terms, by retaining the interaction between the zero quasimomentum mode and other nonzero quasimomentum modes, H_0 and H_{intra} can be similarly processed as in the situation of one-component BECs in an optical lattice, while H_{inter} can be simplified as follows:

$$\mathbf{k}_1=\mathbf{k}_3=\mathbf{k}_2=\mathbf{k}_4=0\rightarrow b_{0A}^\dagger b_{0A}b_{0B}^\dagger b_{0B}=N_{0A}N_{0B},$$

$$\mathbf{k}_3=\mathbf{k}_4\neq 0,\quad \mathbf{k}_1=\mathbf{k}_2=0\rightarrow b_{0A}^\dagger b_{0A}{\sum_{\mathbf{k}}}' b_{\mathbf{k}B}^\dagger b_{\mathbf{k}B}=N_{0A}{\sum_{\mathbf{k}}}' b_{\mathbf{k}B}^\dagger b_{\mathbf{k}B},$$

$$\mathbf{k}_1=\mathbf{k}_2\neq 0,\quad \mathbf{k}_3=\mathbf{k}_4=0\rightarrow b_{0B}^\dagger b_{0B}{\sum_{\mathbf{k}}}' b_{\mathbf{k}A}^\dagger b_{\mathbf{k}A}=N_{0B}{\sum_{\mathbf{k}}}' b_{\mathbf{k}A}^\dagger b_{\mathbf{k}A},$$

$$\mathbf{k}_1=\mathbf{k}_4\neq 0,\quad \mathbf{k}_2=\mathbf{k}_3=0\rightarrow b_{0A}b_{0B}^\dagger{\sum_{\mathbf{k}}}' b_{\mathbf{k}A}^\dagger b_{\mathbf{k}B},$$

$$\mathbf{k}_2=\mathbf{k}_3\neq 0,\quad \mathbf{k}_1=\mathbf{k}_4=0\rightarrow b_{0A}^\dagger b_{0B}{\sum_{\mathbf{k}}}' b_{\mathbf{k}A}b_{\mathbf{k}B}^\dagger,$$

$$\mathbf{k}_1=-\mathbf{k}_3\neq 0,\quad \mathbf{k}_2=\mathbf{k}_4=0\rightarrow b_{0A}b_{0B}{\sum_{\mathbf{k}}}' b_{\mathbf{k}A}^\dagger b_{-\mathbf{k}B}^\dagger,$$

$$\mathbf{k}_2=-\mathbf{k}_4\neq 0,\quad \mathbf{k}_1=\mathbf{k}_3=0\rightarrow b_{0A}^\dagger b_{0B}^\dagger{\sum_{\mathbf{k}}}' b_{\mathbf{k}A}b_{-\mathbf{k}B}. \tag{12}$$

Collecting all of these interaction terms and assuming that the interspecies and intraspecies interactions between particles are weak so that most bosons are not excited from the zero momentum mode, we can replace the zero quasimomentum field operators with a c-number $\sqrt{N_{0A}}$ or $\sqrt{N_{0B}}$ (and then use $N_{0i}=N_i-\Sigma_{\mathbf{k}}' b_{\mathbf{k}i}^\dagger b_{\mathbf{k}i}$ for both components and neglect all of the terms of the four operators product of nonzero $\mathbf{k}$ modes), and get the whole Hamiltonian as

$$H=\sum_{i=A,B}\left(N_i(\epsilon_i-J_iz)+\frac{q_i}{2N_s}(N_i^2-N_i)\right)+\frac{q_{AB}}{N_s}N_AN_B+{\sum_{\mathbf{k}}}'[J_Az(1-\cos k_1a_l)+q_An_A]b_{\mathbf{k}A}^\dagger b_{\mathbf{k}A}+\frac{q_An_A}{2}{\sum_{\mathbf{k}}}'(b_{\mathbf{k}A}^\dagger b_{-\mathbf{k}A}^\dagger+b_{\mathbf{k}A}b_{-\mathbf{k}A}){\sum_{\mathbf{k}}}'[J_Bz(1-\cos k_1a_l)+q_Bn_B]b_{\mathbf{k}B}^\dagger b_{\mathbf{k}B}+\frac{q_Bn_B}{2}{\sum_{\mathbf{k}}}'(b_{\mathbf{k}B}^\dagger b_{-\mathbf{k}B}^\dagger+b_{\mathbf{k}B}b_{-\mathbf{k}B})+q_{AB}\sqrt{n_An_B}{\sum_{\mathbf{k}}}'(b_{\mathbf{k}A}^\dagger b_{\mathbf{k}B}+b_{\mathbf{k}A}b_{\mathbf{k}B}^\dagger+b_{\mathbf{k}A}^\dagger b_{-\mathbf{k}B}^\dagger+b_{\mathbf{k}A}b_{-\mathbf{k}B}), \tag{13}$$

where $n_A=N_A/N_s$ and $n_B=N_B/N_s$ are defined. For convenience, here we use the band spectrum of a one-dimensional optical lattice. For a three-dimensional optical lattice, we can simply replace the one-dimension band spectrum $\epsilon_k=Jz(1-\cos k_1a_l)$ by $\epsilon_{\mathbf{k}}=Jz(3-\cos k_1a_l-\cos k_2a_l-\cos k_3a_l)$ and the constant Jz term by $3Jz$.

Before going into discussion about the excitation spectrum, we define a system of two-component BECs to be a symmetric one when it preserves condition $J_A=J_B$.

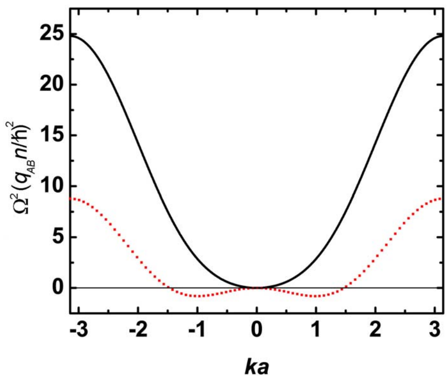

FIG. 1. (Color online) The two-branch excitation spectra at $Jz/q_{AB}n=2$.

III. THE EXCITATION SPECTRUM AND SUPERFLUID VELOCITY

The Hamiltonian in (13) can be diagonalized and the excitation spectrum can be obtained by following the diagonalization method developed in [17]. First we would like to define the following vectors of operators:

$$\langle b| \equiv (b_{\mathbf{k}A}^{\dagger}, b_{-\mathbf{k}A}^{\dagger}, b_{\mathbf{k}B}^{\dagger}, b_{-\mathbf{k}B}^{\dagger}, b_{\mathbf{k}A}, b_{-\mathbf{k}A}, b_{\mathbf{k}B}, b_{-\mathbf{k}B}),$$

$$|b\rangle \equiv (b_{\mathbf{k}A}, b_{-\mathbf{k}A}, b_{\mathbf{k}B}, b_{-\mathbf{k}B}, b_{\mathbf{k}A}^{\dagger}, b_{-\mathbf{k}A}^{\dagger}, b_{\mathbf{k}B}^{\dagger}, b_{-\mathbf{k}B}^{\dagger})^T, \quad (14)$$

where T means transpose into column. The boson commutation of the complex conjugate operator vectors, as a generalization of that of boson operators, can be expressed in terms of a matrix J_P,

$$|b\rangle\langle b| - (|b^{\dagger}\rangle\langle b^{\dagger}|)^T = J_P \equiv \begin{pmatrix} I_{4\times 4} & 0 \\ 0 & -I_{4\times 4} \end{pmatrix}, \quad (15)$$

where I is a unit matrix. The Hamiltonian of given $\pm\mathbf{k}$ modes of both components in (13) can be expressed as follows (unimportant constants are neglected):

$$H = \frac{1}{2}\langle b|H_P|b\rangle,$$

$$H_P = \begin{pmatrix} w & f \\ f & w \end{pmatrix}, \quad (16)$$

where w and f matrices are

$$w = \begin{pmatrix} E_A & 0 & C & 0 \\ 0 & E_A & 0 & C \\ C & 0 & E_B & 0 \\ 0 & C & 0 & E_B \end{pmatrix},$$

$$f = \begin{pmatrix} 0 & F_A & 0 & C \\ F_A & 0 & C & 0 \\ 0 & C & 0 & F_B \\ C & 0 & F_B & 0 \end{pmatrix}, \quad (17)$$

and the parameters in the matrices are defined by

$$F_A = q_A n_A, \quad F_B = q_B n_B,$$

$$E_A = J_A z(1 - \cos k_1 a_l) + q_A n_A = \epsilon_k^A + F_A,$$

$$E_B = J_B z(1 - \cos k_1 a_l) + q_B n_B = \epsilon_k^B + F_B,$$

$$C = q_{AB}\sqrt{n_A n_B}. \quad (18)$$

To obtain the excitation spectrum, we should find out a symplectic transformation (or a generalized Bogoliubov transformation) to diagonalize the Hamiltonian. The transformation can be represented by a matrix T as follows:

$$\langle B| = \langle b|T, \quad |B\rangle = T^{\dagger}|b\rangle. \quad (19)$$

That the transformation T is a symplectic transformation or a generalized Bogoliubov transformation means that the new set of operator vectors $\langle B|$ and $|B\rangle$ should preserve the boson commutation relation as indicated in (15). This condition becomes a requirement on the matrix T [17],

$$T^{\dagger}J_P T = J_P. \quad (20)$$

When a transformation matrix T is found to make the Hamiltonian diagonal

$$H = \frac{1}{2}\langle B|H_D|B\rangle, \quad (21)$$

namely, H_D is a diagonal matrix, the excitation spectrum Ω, expected to be indicated in the diagonal matrix H_D (also in the $H_D J_P$), can be obtained. The mathematical discussion of existence and uniqueness of a matrix T that satisfies the requirement (20) and diagonalizes the Hamiltonian can be found in [15,16].

A natural question arises about how H_D is related to the original Hamiltonian matrix H_P through the transformation matrix T. This can be found directly from above

$$TH_D T^{\dagger} = H_P, \quad (22)$$

and it can be further transformed into

$$TH_D T^{\dagger}J_P = H_P J_P. \quad (23)$$

It is emphasized that the transformation matrix T should preserve the above condition (20). By applying this condition in an alternative way $T^{\dagger}J_P = J_P T^{-1}$ on (23), we can find a useful relation

$$TH_D J_P T^{-1} = H_P J_P, \quad (24)$$

which means that $H_D J_P$ is the eigenvalue matrix of $H_P J_P$ as they are connected by the similar transformation matrix (T and T^{-1}).

For our purpose, we are only concerned with the excitation spectrum, but not the exact form of T. This excitation spectrum Ω can be determined through evaluation of the eigenvalues $|\lambda| = \Omega$ of matrix $H_P J_P$ or equivalently the roots of the following secular equation [17]:

$$\mathrm{Det}(H_P J_P - \lambda I_{8\times 8}) = 0. \quad (25)$$

The reason is already shown above, as the two matrices $H_P J_P$ and $H_D J_P$ are related by a similar matrix.

For the specific case $J_A = J_B$ or equivalently

$$\epsilon_k^A = \epsilon_k^B = \epsilon_k, \tag{26}$$

which is fulfilled by a system of symmetric two-component BECs, the excitation spectra or the eigenvalues of the matrix $H_P J_P$ can be obtained in a concise explicit form

$$\Omega_\pm = \sqrt{\epsilon_k^2 + [F_A + F_B \pm \sqrt{(F_A - F_B)^2 + 4C^2}]\epsilon_k}. \tag{27}$$

For the case when $F_A = F_B = qn$ and $C=0$, which in fact corresponds to a one-component BEC system, (27) does reduce to the spectrum for the single-component case,

$$\Omega = \sqrt{\epsilon_k^2 + 2qn\epsilon_k}. \tag{28}$$

The requirement of a real spectrum imposes a condition that

$$F_A + F_B - \sqrt{(F_A - F_B)^2 + 4C^2} > 0, \tag{29}$$

implying that $q_A q_B > q_{AB}^2$, which is proved to be the opposite of the *phase separation condition* $U_A U_B \leq U_{AB}^2$ introduced in [11,18–21]. In the real spectrum case, the critical superfluid velocity is obtained,

$$v_s^\pm = \left.\frac{\partial \Omega_\pm}{\partial k}\right|_{k\to 0} = \sqrt{[q_A n_A + q_B n_B \pm \sqrt{(q_A n_A - q_B n_B)^2 + 4q_{AB}^2 n_A n_B}]Jza_l^2/2}. \tag{30}$$

The velocity depends not only on the intracomponent interaction parameter $(q_A,\ q_B)$ but also *on the intercomponent interaction parameter* q_{AB}. This result is different from the result obtained in [10]. The two components are coupled to each other to exhibit a superfluid state and there are two branches of superfluid with different critical velocities in the whole BECs. Each branch is a combination of the two BECs components. For the case when $q_A n_A = q_B n_B = qn$, the critical superfluid velocity in the absence of intercomponent interaction is given by

$$v_s^0 = \sqrt{qnJza_l^2}. \tag{31}$$

When the intercomponent interaction is switched on, the critical velocity is equal to

$$v_s^\pm / v_s^0 = \sqrt{1 \pm \kappa}, \tag{32}$$

where $\kappa = q/q_{AB}$. By tuning physical parameters characterizing an optical lattice, the transition to superfluid state from Mott insulator of one-component BECs in optical lattices has already been experimentally accomplished. Therefore, we expect that the superfluid state of two-component BECs and the linear excitation spectra listed in (27) can be verified experimentally in the near future.

Correspondingly, the critical superfluid velocity for 3D optical lattices is obtained similarly,

$$v_s^\pm = \left.\frac{\partial \Omega_\pm}{\partial \mathbf{k}}\right|_{\mathbf{k}\to 0} = \sqrt{[q_A n_A + q_B n_B \pm \sqrt{(q_A n_A - q_B n_B)^2 + 4q_{AB}^2 n_A n_B}]Jza^2/2} \tag{33}$$

which, as expected, not only depends on the intracomponent interaction parameters $(q_A,\ q_B)$ but also on the intercomponent interaction parameter q_{AB}. We emphasize that the superfluid state should be understood in a whole, which means that the two components are correlated with the exhibit of a superfluid state, due to the intercomponent interaction, and such interaction refutes their individual division into two components when they are in a stable or nonseparating superfluid state.

On the other hand, when the complex excitation spectrum is satisfied, which also corresponds to the phase separation condition, it is then expected that there should be phase separation occurring in the two-branch superfluid system. In order to characterize such phase-separated two-branch excitation spectra, we choose a ratio of interaction parameters ($n_A = n_B$ is assumed)

$$\chi = \frac{Jz}{q_{AB} n} = 2 \tag{34}$$

and that of interaction parameters

$$q_A : q_{AB} : q_B = 0.1:1:0.1 \tag{35}$$

from which it can be directly verified that the phase separation condition is satisfied. The excitation spectra are then obtained,

$$\Omega_\pm = \sqrt{[Jz(1 - \cos ka)]^2 + [q_A n + q_B n \pm \sqrt{(q_A n - q_B n)^2 + 4(q_{AB} n)^2}]Jz(1 - \cos ka)}. \tag{36}$$

It can be found that one branch (the negative sign branch) of the excitation spectra falls in the complex domain for a certain region of k, which corresponds to the dashed branch in Fig. 1. We shall call such a branch as the phase separation branch, while the other as the phase amalgamation branch. The instability in the phase separation branch, which arises from the complex spectrum, can cause a redistribution of bosons from $k=0$ mode together with the phase separating process. Such a redistribution process is recommended to be understood in an analogy to the context of the attractive interaction model of ^{4}He [23]. Experimentally, there has been a phase separating process reported in other kinds of trap potential for the double-condensate system of ^{87}Rb [22].

To examine the excitation spectrum in experiment in the case of phase separation, it comes to the question of whether the whole excitation spectrum is still reasonable in physics as it contains a branch of unstable excitation spectrum. We tend to believe that the phase separation branch should be replaced by the stable single component BECs excitation spectrum in optical lattices, while the phase amalgamation branch can still be kept.

IV. EXCITATION SPECTRUM FOR ASYMMETRIC TWO-COMPONENT BECs

The above analysis is in the case of $J_A = J_B$ or for a symmetric two-component BEC system. For an asymmetric two-component BEC system, $J_A \neq J_B$, by following the same diagonalization procedure, the excitation spectra can also be obtained:

$$\bar{\Omega}_{\pm}^2 = \frac{\epsilon_{\mathbf{k}}^A(\epsilon_{\mathbf{k}}^A + 2F_A) + \epsilon_{\mathbf{k}}^B(\epsilon_{\mathbf{k}}^B + 2F_B)}{2} \pm \frac{\sqrt{[\epsilon_{\mathbf{k}}^A(\epsilon_{\mathbf{k}}^A + 2F_A) - \epsilon_{\mathbf{k}}^B(\epsilon_{\mathbf{k}}^B + 2F_B)]^2 + 4\epsilon_{\mathbf{k}}^A\epsilon_{\mathbf{k}}^B 4C^2}}{2}, \quad (37)$$

which can be easily checked to reduce to (27) when $\epsilon_{\mathbf{k}}^A = \epsilon_{\mathbf{k}}^B$. From the requirement for the reality of the excitation spectrum, we can find a condition

$$(2F_A + \epsilon_{\mathbf{k}}^A)(2F_B + \epsilon_{\mathbf{k}}^B) > 4C^2, \quad (38)$$

which is a generalization of (29) and, however, depends on the band spectrum $\epsilon_{\mathbf{k}}^i$ $(i = A, B)$. As $\epsilon_{\mathbf{k}}^i \geq 0$, by setting $\epsilon_{\mathbf{k}}^i = 0$, a stricter condition on real excitation spectrum gives

$$F_A F_B > C^2, \quad (39)$$

or equivalently

$$q_A q_B > q_{AB}^2, \quad (40)$$

which is the same as that of a symmetric two-component BEC system. This leads to a conclusion that somehow the asymmetry relaxes the condition of the real excitation spectrum compared with that of the symmetric systems. The critical superfluid velocity is also found,

$$\bar{v}_s^{\pm} = \left.\frac{\partial \bar{\Omega}_{\pm}}{\partial \mathbf{k}}\right|_{\mathbf{k}\to 0} = \sqrt{[F_A J_A z + F_B J_B z \pm \sqrt{(F_A J_A z - F_B J_B z)^2 + 4 J_A z J_B z C^2}] a_l^2 / 2}. \quad (41)$$

As pointed out previously, there should be two branches of superfluid with different critical velocities in the whole BECs, and each branch should be a combination of two BECs components.

Finally, it is worthy to point out that the two-component BECs systems considered so far are characterized by repulsive interaction in intracomponent, and hence the s-scattering length between bosons in each component is positive,

$$a_A, a_B > 0. \quad (42)$$

However, the general excitation spectrum in (37) is an even function of the intercomponent interaction (denoted by C), and hence it should remain unchanged for the case where the intercomponent interaction is attractive. Furthermore, in the starting Hamiltonian, only interactions between the zero mode and the nonzero $\mathbf{k}$ modes are kept, which is still in the framework of the Bogoliubov approximation. Therefore, the result obtained here should be valid only in the weak interaction regime (both the intracomponent and intercomponent scattering lengths are small). We point out that such an excitation spectrum can be generalized to the model of two-component BECs in the free space, in which we just replace ϵ_k in (27) with the following free space kinetic spectrum:

$$\epsilon_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m}, \quad (43)$$

while the two-branch structure of the spectrum remains unchanged. Such a spectrum can be treated as a generalization of the original Bogoliubov spectrum for single component boson gas in free space [24].

V. CONCLUSION

In conclusion, the excitation spectrum for two-component BECs in the optical lattice is studied. A two-branch linear spectrum is obtained, hence the critical velocity, which is dependent on the intercomponent interaction. It is predicted that there should be a two-branch superfluid composing of the two correlated components when the system is stable in the sense that the excitation spectrum is real. Also, it is predicted that there should be phase separation in the two-component superfluid systems, when the excitation spectrum becomes complex.

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