Measuring the phase of a Bose-Einstein condensate

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The phase of the mean-field wave function of a Bose-Einstein condensate can be recovered from a time series of images. We adapt an iterative retrieval method that has been successful in linear electron and optical imaging systems to solve the nonlinear Gross-Pitaevskii equation. We address a number of issues related to the successful application of this method to the nonlinear system, including the retrieval of wave functions with nonzero net topological charge (i.e., containing vortices) and the effects of repulsive and attractive interactions in the condensate on the convergence properties of the method. Recovering the phase from the continuity-of-density equation is also investigated. An understanding of these issues is of importance for the practical implementation of phase retrieval to Bose-Einstein condensates.

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I. INTRODUCTION

In several experiments on Bose-Einstein condensates (BECs), knowledge of the phase of the mean-field wave function is important in understanding the dynamics of the system. Examples include the first observation of a vortex in a BEC [1], understanding the stable dynamics of bright solitons in an attractive BEC [2], and investigations of rotational and irrotational condensates [3]. If both the density and the phase are measured in an experiment, the mean-field wave function for the condensate can be constructed. Using the mean-field wave function and the Gross-Pitaevskii equation, the dynamics of the condensate can be explored. The problem of recovering the phase of a wave function has been well studied in the optics, x-ray, and electron microscopy communities using both interferometric and noninterferometric techniques [4–13]. Research has recently explored how these techniques might be applied to BECs [14–16]. In particular, the prospect of noninterferometric phase recovery from a time series of density measurements appears promising [14]. In the approach of Ref. [14], an iterative method is used to solve for the phase of a wave function obeying the Gross-Pitaevskii equation. We investigate how the iterative method performs on wave functions with nonzero net topological charge and how its performance is affected by repulsive and attractive interactions.

In optical and electron imaging the phase of a wave function has also been obtained from the transport-of-intensity equation [9,10]. It has been proposed that this equation can be solved to obtain the phase of a wave function in the presence of nonlinear interactions [17,18]. The equivalent equation for a condensate expresses the continuity of density. We demonstrate a successful phase retrieval using this equation and show that errors in the solution appear when the time step between density measurements is greater than the correlation time.

A solution to the phase problem using a series of intensity measurements was first proposed by Gerchberg and Saxton [4]. Their approach was iterative and variations of this method have been successfully applied in both electron, optical, and x-ray systems [7,8]. Other methods have also been developed for these systems, including a direct solution based on the transport-of-intensity equation [9,10] and least squares minimization methods [12,13]. These methods find the phase of a wave function satisfying a linear first-order differential wave equation,

$$\frac{\partial}{\partial z}\psi(\mathbf{r}_\perp,z)=\frac{i}{4\pi k}\nabla_\perp^2\psi(\mathbf{r}_\perp,z). \tag{1}$$

This equation describes the propagation in the z direction of the wave function of a photon or fast electron, $\psi(\mathbf{r}_\perp,z)$, with wave number k, in the paraxial approximation. The wave function can be written in the form $\psi(\mathbf{r}_\perp,z)=\sqrt{\rho(\mathbf{r}_\perp,z)}e^{i\phi(\mathbf{r}_\perp,z)}$, where $\rho(\mathbf{r}_\perp,z)$ is the intensity of the wave and $\phi(\mathbf{r}_\perp,z)$ is the phase. The wave function is recovered in the two dimensional $\mathbf{r}_\perp$ plane from intensity measurements taken at different values of the defocus parameter z.

In the mean-field approximation a BEC can be described by a complex scalar function and its time evolution is governed by the nonlinear Gross-Pitaevskii equation,

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}_\perp,t)=\left(-\frac{\hbar^2}{2m}\nabla_\perp^2+V_{\mathrm{trap}}(\mathbf{r}_\perp)+g|\psi(\mathbf{r}_\perp,t)|^2\right)\psi(\mathbf{r}_\perp,t), \tag{2}$$

where $g=4\pi\hbar^2a/m$, with a the s-wave scattering length and m the mass. The term $V_{\mathrm{trap}}(\mathbf{r}_\perp)$ is an external trapping potential which in this study is assumed to be parabolic: $V_{\mathrm{trap}}(\mathbf{r}_\perp)=\frac{1}{2}m\omega^2\mathbf{r}_\perp^2$, where ω is the trap frequency. Notice that if the strength of the nonlinear term is zero and the trapping potential is removed, the nonlinear equation reduces to the form of Eq. (1), with the identification $z=vt$ and $k=mv/(2\pi\hbar)$. The quantity $\rho(\mathbf{r}_\perp,t)$ is now interpreted as the density of the condensate. The phase problem for a BEC is to recover the phase from density measurements recorded at different times. Although a condensate is three dimensional, we will only consider condensates that have a density which can be effectively projected onto a two-dimensional (2D) plane. For example, under tight harmonic trapping in one dimension the dynamics along this dimension can be suppressed, producing a quasi-2D condensate [19]. It is also possible to remove a two-dimensional slice from a three-

dimensional condensate and image it separately [20]. When the condensate is effectively 2D, the density of the condensate can be inferred from an image of the condensate. The most common method of imaging condensates is absorption imaging, but this is destructive [21,22]. To use this technique for phase retrieval would require images of multiple condensates prepared under the same conditions and imaged at different times. Two condensates prepared under identical conditions will share a common phase, $\phi(\mathbf{r}_\perp,z)$, up to an overall constant. Since this overall constant does not affect the dynamics of the condensate, it is possible to retrieve a phase from images of separately prepared condensates. There are several sources of noise for a time series of images. For instance there are technical difficulties associated with reproducing condensates under identical conditions, such as variations in the number of trapped atoms. Another source of noise is fluctuations in probe light intensity. Even in an experimentally ideal situation, quantum fluctuations of the condensate density during time evolution present a noise level on these images of 5–10 % [23] which is uncorrelated between images. It has been shown previously that the iterative method is robust in the presence of noise levels up to 10% on the brightest pixel [7]. Hence if the noise present in the images from all sources is less than this level, phase retrieval should be possible. As an alternative to absorption imaging, the nondestructive technique of phase-contrast imaging can take multiple images of a single condensate [22,24,25]. This technique avoids the issues associated with reproducing identical condensates. Currently the main limitation of applying phase contrast images to phase retrieval is the signal-to-noise ratio that is currently achievable.

Phase vortices are a topological feature found commonly in condensates and optical waves. Single charged vortices can be produced in BECs by mechanical rotation of an asymmetric trap [26] and by direct optical phase imprinting [27]. This technique has also been used to produce a vortex lattice [28]. Vortices with nonzero topological charge can be created in laser fields [29] and in x-ray fields using phase plates [30]. Iterative methods have successfully retrieved the phase of wave functions containing vortex-antivortex pairs which have a net topological charge of zero [14,31,32]. When the net topological charge of the wave function is not zero a further issue arises. If an iterative method cannot alter the net topological charge of the trial wave function, then the correct charge must be known *a priori* for a successful retrieval. We use an iterative method of phase retrieval, similar to that in Refs. [7,14]. We investigate whether it is affected by the issue of net topological charge, and, if so, how.

Using a Feshbach resonance, the sign of the interaction can be changed to produce a condensate with attractive interactions [33]. Tan *et al.* [14] have suggested that the presence of repulsive nonlinear interactions improves the convergence of their iterative method of phase retrieval. This is explored here systematically. The analysis is then extended to show that convergence is slower with attractive interactions and that when the interaction is strongly attractive the method can fail.

As an alternative to iterative methods, the phase of a wave function can also be solved directly from the continuity-of-density equation. The continuity-of-density equation is

$$\frac{m}{\hbar}\frac{\partial}{\partial t}\rho(\mathbf{r}_\perp,t) = \nabla_\perp \cdot [\rho(\mathbf{r}_\perp,t)\nabla_\perp \phi(\mathbf{r}_\perp,t)]. \tag{3}$$

This equation applies equally to wave functions that obey either Eq. (1) or Eq. (2), up to an appropriate choice of constants. For electron or optical wave functions, it is more commonly known as the transport-of-intensity equation [9,10]. The continuity-of-density equation can be inverted to find the phase if the density and the time derivative of the density are known. These techniques utilize as few as two density measurements to estimate the derivative of the density with respect to time. The variations in these densities caused by the nonlinear interaction introduce an error in the solution. In this paper it is demonstrated that the phase of a BEC can nevertheless be obtained from the continuity-of-density equation provided that the time step between density measurements is less than the correlation time for the condensate.

II. ITERATIVE LOOP METHOD OF PHASE RETRIEVAL

Iterative methods retrieve the phase from a number of density measurements: $\rho'(\mathbf{r}_\perp,t_1)$, $\rho'(\mathbf{r}_\perp,t_2),\ldots$ $\rho'(\mathbf{r}_\perp,t_N)$. In this paper an iterative loop method is used. To begin a trial wave function is formed using the first measured density and a guessed phase. This trial wave function is evolved to the time of the next density measurement, where the density of the time-evolved wave function is replaced with the measured density. This new wave function is then evolved to the next time where the density is replaced again and so on. Once this is performed for the final measurement, the process is repeated in reverse. This constitutes one full iteration. Before the density is replaced, the density of the trial wave function, $\rho(\mathbf{r}_\perp,t)$, is compared to the measured density, $\rho'(\mathbf{r}_\perp,t)$, using the sum-squared error (SSE),

$$\mathrm{SSE} = \frac{\sum [\sqrt{\rho'(\mathbf{r}_\perp,t)} - \sqrt{\rho(\mathbf{r}_\perp,t)}]^2}{\sum \rho'(\mathbf{r}_\perp,t)}. \tag{4}$$

At the end of each iteration, the SSE at each time is averaged to estimate the overall convergence of the algorithm.

It will be helpful for the discussion below to also describe the iterative loop method mathematically. The evolution of a wave function according to the Gross-Pitaevskii equation, Eq. (2), can be written as

$$\psi(t+t_0) = U(t,t_0)\psi(t_0) \tag{5}$$

and the replacement of density operation, in which the density of a trial wave function is replaced with the measured density, can be written as

$$\begin{aligned}\psi'(\mathbf{r}_\perp,t_n) &= T_n[\psi(\mathbf{r}_\perp,t_n)] = T_n\left[\sqrt{\rho(\mathbf{r}_\perp,t_n)}e^{i\phi(\mathbf{r}_\perp,t_n)}\right] \\ &\equiv \sqrt{\rho'(\mathbf{r}_\perp,t_n)}e^{i\phi(\mathbf{r}_\perp,t_n)},\end{aligned} \tag{6}$$

where $\psi(\mathbf{r}_\perp,t_n)$ has been evolved from the previous time and $\rho'(\mathbf{r}_\perp,t_n)$ is the measured density at time t_n. In this notation the wave function at the $(i+1)$ iteration of the method can be written as

$$\psi_{i+1}(\mathbf{r}_\perp,t_1)=\prod_{n'=2}^{N}T_{n'-1}U_{t_{n'-1},t_{n'}}\prod_{n=N}^{2}T_nU_{t_n,t_{n-1}}\psi_i(\mathbf{r}_\perp,t_1). \tag{7}$$

The replacement of density operation, T_n, can be interpreted as applying a constraint at time n. The N density measurements provide N constraints which we require our wave function to satisfy. The iterative loop method applies these constraints consecutively and repeatedly. When applied to electron or optical systems, the unitary evolution operation, U, is interpreted as free space propagation and the time parameter is replaced by a spatial propagation parameter.

There are also iterative methods that are more sophisticated than the loop method, such as the iterative wave function reconstruction (IWFR) method [8], which can also be applied to BECs. The loop method has been used here as it is computationally faster.

III. CONSERVATION OF TOPOLOGICAL CHARGE IN THE ITERATIVE LOOP METHOD

In two dimensions a phase vortex is a topological feature that can occur wherever the density of the wave function is zero at a point. At this point the phase is undefined. For a continuous and differentiable wave function, the integral of the phase over a closed loop Σ encompassing this point can be any nonzero integer multiple of 2π:

$$\oint_\Sigma \nabla_\perp\phi(\mathbf{r}_\perp)\cdot d\hat{\mathbf{n}}=2\pi l, \tag{8}$$

where $\hat{\mathbf{n}}$ is a unit vector tangential to the loop Σ. The integer l is the called the topological charge and the sign of l indicates the orientation of the vortex. The net topological charge of a wave function is defined by the integral of the phase over a closed loop that encompasses all the singularities.

The iterative loop method has successfully retrieved the phase of wave functions with vortices when the net topological charge of the wave function is zero [14]. When the net topological charge of the wave function is not zero, it is important to know whether the method is capable of changing the net topological charge of a trial wave function. If the method conserves the charge of the trial wave function, then the correct net charge must be known *a priori* for a successful retrieval. Recall that the iterative loop method consists of two kinds of operation, unitary evolution U and replacement of density T_n. Do these operations conserve the net topological charge of a wave function?

Since unitary evolution constitutes a continuous deformation of the wave function, the topological charge must remain fixed within a closed loop unless a singularity crosses the loop [34]. As a consequence unitary evolution can introduce vortex-antivortex pairs or a single charged vortex can come "from infinity," entering via the edge of the condensate. This is how the rotation of an asymmetric trap can introduce vortices into a condensate [26]. In our simulations, time evolution will correspond to the free expansion of a condensate after the trapping potential has been turned off. Since there is no external source of angular momentum, only pairs of vortices will enter or leave the condensate during time evolution, but not single vortices. Thus time evolution will not change the net topological charge of a wave function.

The replacement of density operation, T_n, only modifies the density and leaves the phase of the wave unaltered. Hence this operation should leave the topological charge of the wave function unchanged. However, there is an exception when the measured density contains an extended region of zero density. If after time evolution a vortex is located in this region, the density around the vortex core will be set to zero when the density is replaced, as discussed in Ref. [35]. Hence, the vortex is removed from the trial wave function and the net topological charge is changed. Since in an experiment a BEC is situated within a trap, an image of a BEC will include an extended region of zero density. However, this mechanism of vortex removal depends on the dynamics of the vortices during time evolution. There is no guarantee that a particular vortex which we would like removed from a trial wave function will move into the region of zero density, and thus we do not expect this to happen routinely.

To illustrate this, we investigate the performance of the iterative method for two different starting guesses for the phase. In particular, we compare guesses that do and do not have the correct net topological charge. We also wish to know whether the nonlinear interaction potential affects convergence in these cases. To test the iterative method a steady state solution to Eq. (2) was found for a trapped condensate with a first-order vortex. This was achieved with the imaginary time method [36]. The density and phase of this solution in the absence of interactions are shown in Figs. 1(a) and 1(b), while the density and phase shown in Figs. 1(c) and 1(d) are for a ^{23}Na condensate with a scattering length of $a=2.9$ nm. A trap frequency of $2\pi\times 3.3$ rad s^{-1} was used. This value of the trap frequency represents a typical value used in current experiments [37]. Note that with $a=0$ the condensate, in a steady state in the parabolic trap, is smaller in size. The condensate was normalized to a value of 50 atoms nm^{-1}, which represents a density in the third spatial dimension. To form the time series the trap was turned off and the condensate allowed to expand. The time series consists of five densities at 5 ms intervals. When the phase guess is zero, shown in Fig. 1(e), the retrieved phases after 100 iterations do not show a vortex, Figs. 1(g) and 1(i). The iterative method failed to change the topological charge of the wave function irrespective of the interaction strength. Inspection of the convergence curves shown in Fig. 2 indicates that the method stagnated at the outset.

Next we see that the method is successful when a guess for the phase with the correct net topological charge is used. If the iterative loop method is robust, it should be capable of converging to the correct solution from any guess for the phase with the correct net topological charge. To randomly select a function with a net topological charge of one, we can combine an ideal first-order vortex with a smoothly varying random function. The requirement that the function is smoothly varying (i.e., contains no vortices) is satisfied if the smallest features of this random function are greater than the width of a pixel. Such a function was obtained by generating

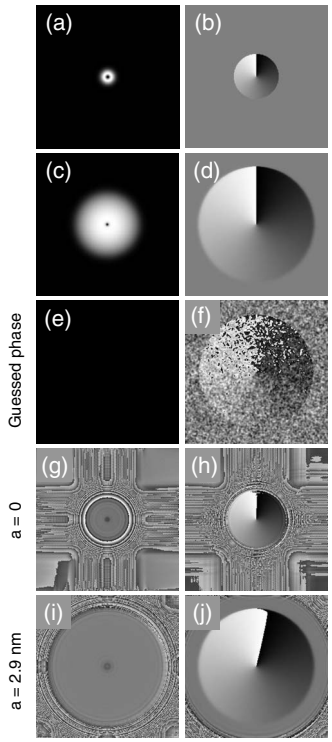

FIG. 1. Density and phase of the first-order vortex in a parabolic trap is shown for the case without interactions, (a) and (b), and for the case with interactions, (c) and (d). In the case of nonlinear time evolution, a ^{23}Na condensate was assumed with a scattering length of 2.9 nm. The size of the image is $(3\times10^2\ \mu\text{m})^2$ and the trap frequency is $2\pi\times3.3$ rad s^{-1}. The phase retrievals were performed with two different starting guesses (e) and (f), as discussed in the text. The retrieved phases using linear time evolution are shown in (g) and (h) and those for nonlinear evolution in (i) and (j). Each phase was retrieved from five images starting at $t=0$ with time intervals of 5 ms and is shown after 100 iterations.

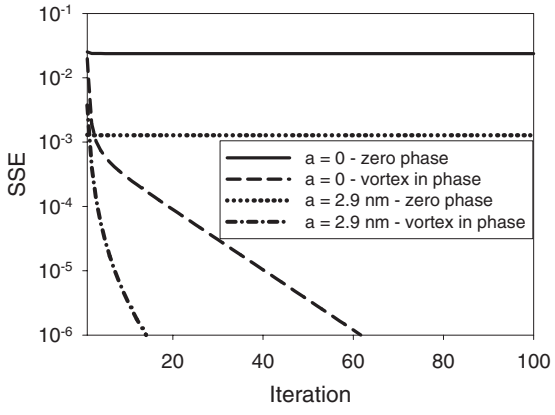

FIG. 2. The comparison of the convergence profiles for the phase retrievals starting with different initial guesses shown in Fig. 1. The curves labeled "zero phase" refer to the starting phase shown in Fig. 1(e) and those labeled "vortex in phase" refer to the starting phase in Fig. 1(f). The values of a distinguish the linear and nonlinear cases.

a random number for each pixel between zero and 2π and then smoothing this random distribution with a Gaussian with a $1/e$ width of 5520 nm. This width was chosen to produce a random distribution whose smallest features are 5–8 times wider than the width of a pixel. This function was added to the ideal phase vortex in Fig. 1(d) to produce the phase guess shown in Fig. 1(f). The correct phase has been retrieved by 100 iterations in both the case where the interaction term was omitted, Fig. 1(h), and where it was included, Fig. 1(j). Figure 2 shows that the convergence of these phase retrievals is steady and rapid. The test was repeated starting from several other randomly selected guesses for the phase with the correct charge, which also varied the location of the underlying vortex, and each time the retrieval was successful. Hence if the topological charge is correctly specified in the phase guess then the iterative method is successful.

Tests were conducted on wave functions containing a single vortex with a charge greater than one. The results were consistent with the first-order case. A phase guess with the correct topological charge converged but a phase guess of zero stagnated without introducing vortices into the wave function.

We also considered the case where there were several vortices of charge 1, so that the net topological charge was greater than 1. The greatest success was achieved when the phase guess contained vortices located at the density zeros. When the vortices in the guess for the phase were located elsewhere, interaction between the vortices prevented the method from relocating all the vortices correctly and convergence was not achieved. While it is simple enough to place vortices at the density zeros in practice, the sign of each vortex (and thus the net topological charge) must still be guessed. The signs of the vortices can be identified by the retrieval that achieves the greatest level of convergence.

Variation of the parameters such as the density, scattering length, and the trap frequency will effectively alter the strength of the interaction potential. Further tests over a range of interaction strengths were consistent with the results shown in Fig. 1. Hence, we regard our conclusions concerning the retrieval of wave functions with nonzero net topological charge by the iterative method to be applicable to condensates generally, irrespective of trap frequency, scattering length, or density. We do expect that the interaction strength will have a significant effect on the convergence of the iterative method, and this will systematically be studied in Sec. IV.

It is also possible to use a guess for the phase which has a random value on each pixel. This guess for the phase may lead, after iteration, to a trial wave function with the correct net topological charge. However, this guess may also lead to a trial wave function with the incorrect net topological charge. Hence we do not consider the use of a random start to be a reliable approach to the retrieval of wave functions with a nonzero net topological charge. The most reliable implementation of the iterative method is when the guess for the phase has the correct net topological charge.

These results are also important for the application of phase retrieval to singular optics [35]. The case where $a=0$ is analogous to the propagation of an optical field. As a conse-

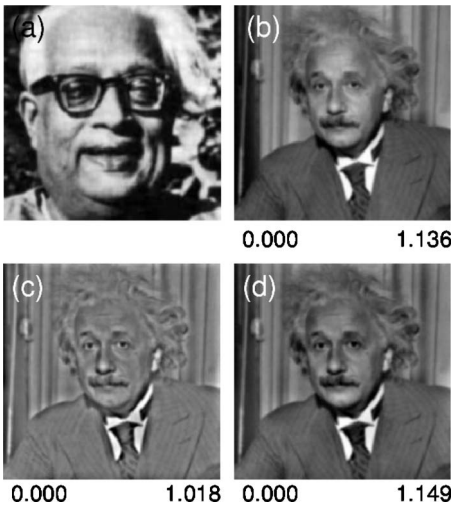

FIG. 3. Density (a) and the phase (b) of the test object. The phase retrievals after 50 iterations are shown for (c) $g|\psi|^2_{\max}=9\times10^{-14}$ eV ($\omega=2\pi\times21.7$ rad s^{-1}) and (d) $g|\psi|^2_{\max}=9\times10^{-13}$ eV ($\omega=2\pi\times217$ rad s^{-1}). The object size is $(8\times10^2\ \mu\text{m})^2$. The density was normalized to a value of 50 atoms nm^{-1}. The minimum and maximum values of the phase are indicated beneath each phase map in radians.

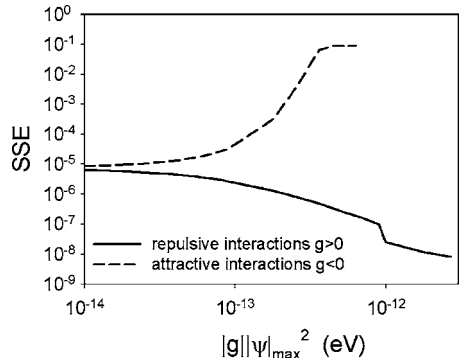

FIG. 4. Plot of the convergence for a retrieval of the test object shown in Fig. 3 as the strength of the nonlinear potential is varied. The convergence is measured after 50 iterations. The retrievals are performed on five densities at 5 ms intervals.

quence of the results shown in Figs. 1(g) and 1(h), we conclude that the phase of wave field containing optical vortices can be successfully retrieved with this method if the net topological charge is known.

IV. CONVERGENCE PROPERTIES OF THE ITERATIVE LOOP METHOD FOR ATTRACTIVE AND REPULSIVE NONLINEAR POTENTIALS

Leaving the issue of nonzero net topological charge, we now consider how the strength of the interactions affects the performance of the iterative loop method for wave functions with a net topological charge of zero. Previous research has found that increasing the strength of the nonlinear potential improves convergence [14]. This research considered repulsive interactions (a positive nonlinear potential). The aim in this section is to systematically explore this convergence behavior for wave functions with a net topological charge of zero and extend the analysis to attractive interactions (a negative nonlinear potential).

The test object for this section was constructed using a portrait of Bose for the density and a portrait of Einstein for the phase, as shown in Figs. 3(a) and 3(b), and represents a general wave function of zero topological charge without any symmetries. The time series contained five densities at 5 ms intervals. In this section the variation of the nonlinear potential, $g|\psi|^2_{\max}$, has been used to quantify the interaction and to emphasize that this potential can be changed by varying the interaction strength, g, or by changing the density of the condensate. Figure 4 shows the convergence of the iterative method after 50 iterations as the strength of the nonlinear potential is varied. It confirms that for repulsive interactions convergence improves as the strength of the nonlinear potential increases. This is because a stronger nonlinear potential produces greater variations between the densities in the time series which assists the convergence of the method. This idea has been previously invoked in the choice of appropriate time intervals between successive measurements [38]. If the interval is too small, the lower spatial frequencies of the wave function will be minimally changed by the time evolution and hence unable to be recovered by the algorithm. The same effect is evident in Figs. 3(c) and 3(d) which compare the retrievals after 50 iterations for two different strengths of the nonlinear potential. In the case of stronger interactions ($g|\psi|^2_{\max}=9\times10^{-13}$ eV) the retrieval is qualitatively very accurate after 50 iterations and has the correct phase range to an accuracy of 1%. But in the case of weaker interactions ($g|\psi|^2_{\max}=9\times10^{-14}$ eV) there is less variation between the densities in the time series. As a consequence, convergence is slower and there is still a significant disagreement with the correct solution after 50 iterations. The use of a larger time step between densities in the time series and further iterations improves this result.

As Fig. 4 indicates, a significant improvement in convergence can be obtained if the strength of the nonlinear potential of a repulsive condensate is increased by two orders of magnitude. As is often done in phase retrieval studies, we have obtained Fig. 4 with a wave function that was chosen to rigorously test the capabilities of the method. We expect this result to be applicable to wave functions in general, and hence to trapped condensates. In that case, there are a number of experimental parameters which ultimately determine the strength of the nonlinear potential. For example, the strength of nonlinear potential depends on g, which is proportional to s-wave scattering length and inversely proportional to the mass of the atoms. The strength of the nonlinear potential is also dependent on the density, which is determined by the trap frequency, the mass, and the number of atoms in the condensate. In the Thomas-Fermi limit [39], one can show that the peak density of a 2D condensate in a harmonic trap is

$$|\psi|^2_{\max} = \sqrt{\frac{mn_z\omega_x\omega_y}{\pi g}}, \tag{9}$$

where n_z is the density of atoms in the third dimension and ω_x and ω_y are trap frequencies along the x and y directions, respectively. Hence we can infer the experimental conditions which will optimize the performance of the method.

Attractive interactions are modeled by setting the sign of g to be negative. With attractive interactions the convergence properties change remarkably. As shown in Fig. 4, for interaction strengths $|g||\psi|^2_{\max}$ up to 1×10^{-13} eV the convergence of the method is slower and for strengths of $|g||\psi|^2_{\max}$ approaching 1×10^{-12} eV the method fails completely. This slow convergence is due to the way in which the attractive interaction changes the density through the time series. The attractive interaction leads to an increase in the density in certain regions of the image and a decrease everywhere else. Phase retrieval relies on interference to encode the phase of the wave function at one point in time in the density at a later time. If the density decreases in a certain region due to a noninterferometric process, such as an attractive interaction, then the method becomes less sensitive to interference in that region. The reduced sensitivity to regions of the measured density is detrimental to the performance of the method. If the density becomes very high on very few pixels, then the Gross-Pitaevskii equation must be solved very accurately, by sampling time finely, which requires increased computing power. For the case of very strong attractive interactions, $|g||\psi|^2_{\max}$ approaching 1×10^{-12} eV, the failure could be because the Gross-Pitaevskii equation has not been solved with sufficient accuracy. A fine enough sampling was not found due to constraints on computing time.

In practice the measured densities contain noise due to the statistics of the detector and due to quantum fluctuations. Noise due to the statistics of the detector was added to the simulated densities by specifying a percentage error on the brightest pixel and determining the noise levels on other pixels in accordance with Poisson statistics. The tolerance of the iterative method to Poisson noise has been investigated previously [7], and it has been shown that a successful retrieval can be achieved with a level of 10% noise on the brightest pixel. By repeating these tests it was found that the inclusion of the nonlinear potential does not affect the iterative method's tolerance to noise.

V. SOLVING THE CONTINUITY-OF-DENSITY EQUATION IN THE PRESENCE OF A NONLINEAR POTENTIAL

The phase of a wave function can be recovered by solving the continuity-of-density equation, Eq. (3), so long as there are no points of zero density (i.e., no vortices). This method requires knowledge of the derivative of the density with respect to time. This can be estimated from a time series with a minimum of two density measurements. Care must be taken to interpret the continuity-of-density equation correctly. The nonlinear potential does not appear in this equation and hence the first-order time derivative of the density is independent of the nonlinear interaction. The error associated with the estimate of the first-order time derivative from a number of density measurements is dependent on higher-order time derivatives [38]. Higher-order time derivatives of the density are dependent on the nonlinear interaction. Hence, the nonlinear potential contributes to the error associated with the estimate of the first-order time derivative.

The test object from Sec. IV has been used to demonstrate that the continuity-of-density equation can be solved for the phase. The continuity-of-density equation has been solved here from knowledge of the density at three different times. The estimate of the time derivative of the density was made from two densities symmetric around the time of interest. For instance, to retrieve the phase at $t=t_0$ with a 1 ms time step, the densities at $t=t_0-1$ ms and t_0+1 ms are used to estimate the derivative. To obtain a quantitative measure of the method's accuracy, the retrieved phase and the phase of the test object have been compared using a sum-squared error, with the form of Eq. (4). Since the phase is only determined up to an additive constant, this constant was varied to minimize the sum-squared error. Although this measure will not be available in experiment it is useful for understanding the properties of the method. The sum-squared error calculated using the retrieved phase and the phase of the test object will be abbreviated SSE(Ph) to distinguish it from the SSE calculated using densities.

Figure 5 shows the results of phase retrieval using the continuity-of-density equation. In the absence of interactions, a qualitative inspection indicates that the phase has been retrieved successfully for a step size as large as 20 ms. The SSE(Ph) is not zero and increases as the step size increases because there is an error in the estimate of the time derivative of the density. For convenience, the inclusion of interactions is specified by the value of the scattering length a. The mass number is still assumed to be 23. When the scattering length is set to $a=29$ nm, the retrieval is still qualitatively successful for step sizes up to 10 ms. The retrieval using a 20 ms time step shows a loss of resolution and the finer details have been lost. The SSE(Ph) is higher when interactions are included and significantly higher when the time step is 20 ms. This increase in the SSE(Ph) is due to an additional error in the estimate of the time derivative of the density due to the nonlinear potential.

The correlation time is the characteristic time scale over which the nonlinear interactions cause significant variations in the density. By dividing the healing length by the speed of sound in the condensate, the correlation time is estimated to be [40]

$$t_{\text{corr}} = \frac{m}{2\sqrt{2}hna}, \tag{10}$$

where m is the mass of the atoms, n is the mean density of the condensate, a is the scattering length, and h is Planck's constant. Using the parameters for the simulations in Fig. 5 with $a=29$ nm, the correlation time is found to be 9 ms. We can use the correlation time to estimate the step size for which the nonlinear interactions begin to introduce a significant error in the retrieved phase. Figure 6 shows the SSE(Ph) as the step size was increased. The SSE(Ph) is only significantly above that of the $a=0$ case when the step size is above

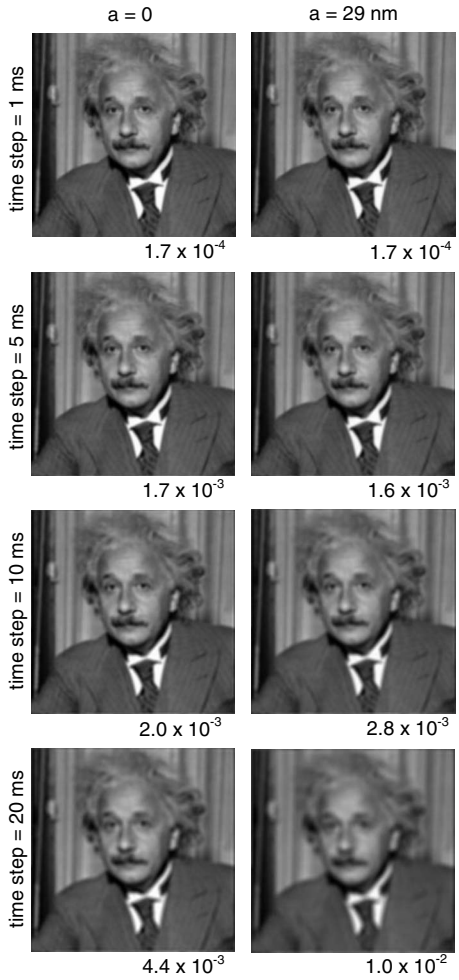

FIG. 5. The phase maps retrieved from the continuity-of-density equation for different time steps for the test object shown in Figs. 3(a) and 3(b). The test object and parameters are the same as Sec. IV. The SSE(Ph) shown below each phase was calculated using the retrieved phase and the original phase.

the correlation time. This is consistent with Fig. 5 which shows that when the step size was 20 ms, in excess of the correlation time, the nonlinear interactions lead to a significant error in the retrieved phase. When $a=2.9$ nm, which corresponds to the strength of sodium interactions, the correlation time is 89 ms. Hence Fig. 6 does not show any significant error above that of the $a=0$ case. The correlation time given by Eq. (10) is not precise for a general condensate since it does not take into account the condensate's specific geometry. However, based on these simulations we expect Eq. (10) to be a good guide for choosing an appropriate time step.

The capacity of the continuity-of-density equation method to tolerate noise has been investigated previously [7]. In particular the solution for the phase is susceptible to low frequency noise. When applied to measurements of a BEC, the solution will be susceptible to noise due to quantum fluctuations irrespective of the imaging technique used. The treatment of this noise, either through low-frequency filtering or otherwise, will be important for the successful application of the continuity-of-density solution to BECs. Since noise has been successfully treated in both applications to x rays and fast electrons [9,41], the presence of noise should not discount the use of the continuity-of-density equation.

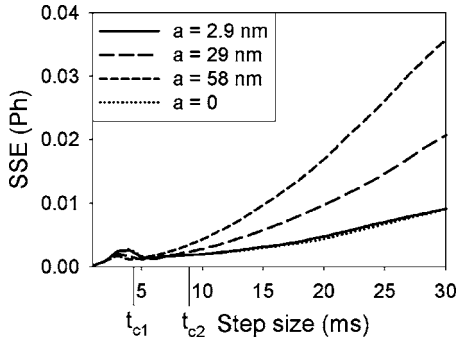

FIG. 6. Comparing the retrieved phase to the phase of the test object for the different interaction strengths and step sizes. The time t_{c1} is the correlation time of the wave function with an interaction strength of $a=58$ nm. The time t_{c2} is the correlation time when the strength of the interaction is $a=29$ nm.

VI. CONCLUSION

An iterative method has been successfully applied to retrieve the phase of a BEC for an experimentally realizable set of parameters. It was shown that the method can successfully retrieve a single charge vortex if the correct net topological charge is specified in the initial phase guess. Increasing the strength of a repulsive interaction improved convergence, but increasing the strength of an attractive interaction slowed convergence. The method was unsuccessful in the presence of strong attractive interactions.

The continuity-of-density equation has been successfully solved to retrieve the phase of a wave function, without zeros in the density, in the presence of a nonlinear potential. The correlation time of the condensate indicates the point where the nonlinear interaction introduces a significant error into the solution. Hence the correlation time is a guide for selecting an appropriate time step for the estimate of the density derivative.

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