Optimal bang-bang control for SU(1,1) coherent states

Jian-Wu Wu,[1,*] Chun-Wen Li,[1] Tzyh-Jong Tarn,[2] and Jing Zhang[1]
[1]*Department of Automation, Tsinghua University, Beijing, 100084, People's Republic of China*
[2]*Department of Systems Science and Mathematics, Washington University, St. Louis, Missouri 63130, USA*
(Received 19 August 2007; published 6 November 2007)

In this paper, the problem of achieving an arbitrary SU(1,1) coherent state is considered via switching the control field back and forth between admissible values with minimal number of switching times. When the controlled system Hamiltonian is hyperbolical or parabolical, the results show that the minimal switching number is one or two, which lies on whether the argument of the involved control is adjustable or not, and is independent of the target SU(1,1) coherent state. While for the elliptical case, the results indicate that the minimal number of switches needed depends on the target SU(1,1) coherent state and is provided as a function of it. In this case, one switch can also be saved if the argument of the involved control is adjustable. The theory developed here can also be extended to solve the optimal bang-bang control problem for a general SU(1,1) time evolution.

DOI: 10.1103/PhysRevA.76.053403 PACS number(s): 32.80.Qk, 02.20.−a, 42.50.Dv, 02.30.Yy

I. INTRODUCTION

In order to realize the desired states transition, strategy based on Lie group decomposition has been presented for both compact [1] and noncompact [2] quantum systems in recent years. In comparison with other existing quantum control techniques, the merit of this method is that it does not need any approximations or iterative calculations. The main idea of this approach is to decompose the desired system evolution operator U_f as the following product (in the system of units such that $\hbar=1$):

$$U_f=\prod_{k=1}^{Q} e^{-it_k(H_0+u_kH_I)}, \quad (1)$$

where H_0 and H_I are the drift Hamiltonian and the control Hamiltonian, respectively. Accordingly, one can obtain the piecewise constant control field $u(t)$, which takes value u_k in the time period $\Sigma_{l=1}^{k-1}t_l \le t \le \Sigma_{l=1}^{k}t_l$ and switches $Q-1$ times.

In practice, however, there always exists a rise or decay time between two adjacent control pulses. In addition, it is difficult to implement the required switch exactly at the ideal switch point. To reduce the error introduced by the switches, it is natural to consider the problem of controlling the systems with minimal switches, i.e., achieving the decomposition (1) for the desired U_f with minimal number of factors. This minimization problem is mathematically related to the uniform finite generation problem on Lie groups [3–8], which has been extensively explored since the 1970s. If the Lie algebra of a connected Lie group G is generated by $X_1,X_2,\ldots,X_n$, the problem on the uniform generation is to find the minimum integer number k such that every element of G can be decomposed as the product of $\exp(t_iX_i)$ with k factors. The integer k is called the order of the generation of G with respect to the generators $X_1,X_2,\ldots,X_n$. For a system to evolve on the Lie group G, one can obtain the switching control law with its switches less than the generation order and realize any desired evolution. The generation order is always finite when G is compact [8], however, it may be infinite when G is noncompact [5]. To further reduce the noise that might be introduced to the system by the switches, we shall look for the exact minimal number of switches needed to realize a desired system evolution. In [9], with the control field switching back and forth between two different values, D'Alessandro obtained the optimal switching controls for the systems evolving on the rotation group.

In this paper, we consider the quantum system whose Hamiltonian preserves, during the evolution process, the SU(1,1) coherent states (CS's). The aim is to realize an arbitrary SU(1,1) CS from the vacuum state by piecewise constant control fields with minimal number of switching times. The properties of quantum systems with SU(1,1) symmetry have been extensively studied in the literature [10–20], and such dynamical models can be successfully used to describe various physical processes. Notice that any SU(1,1) CS can be obtained from the vacuum state by the action of the squeezing operator, the optimal switching control problem can be solved by considering the time evolution of the corresponding propagator. By introducing the unimodular Möbius transformation, we obtain the minimal switches for all the possible situations.

The paper is organized as follows. In Sec. II, we first introduce the controlled system model for the SU(1,1) CS's. Then, by the unimodular Möbius transformation, the desired squeezing operators are one-to-one mapped to the points on the open unit disk, which provides the main approach in this paper. In Sec. III, the main results are obtained on the optimal switching control problem for all the possible situations, which include the hyperbolical, parabolical, and elliptical cases. The conclusion is drawn in Sec. IV.

II. FORMULATION

A. Model

In this paper, we consider the following Hamiltonian that preserves SU(1,1) CS's [11,12]

*wujw03@mails.tsinghua.edu.cn

$$H(t) = 2\omega_0 K_0 + u(t)K_+ + \bar{u}(t)K_-, \tag{2}$$

where $u(t)$ is an arbitrary complex function of time which is referred to as the external control field. This time-dependent Hamiltonian can be used to describe many physical processes such as degenerate parametric amplifier [12]. The operators K_0, $K_\pm = K_1 \pm iK_2$ are the generators of the Lie algebra of SU(1,1), which satisfy the following commutation relations:

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0. \tag{3}$$

Accordingly, the invariant Casimir operator is given by $C = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+)$, which has eigenvalue $k(k-1)$ under the positive discrete series unitary irreducible representation denoted by $\mathcal{D}^+(k)$, where the non-negative integer k is the so-called Bargmann index. Denote the basis states of $\mathcal{D}^+(k)$ as $|m,k\rangle$, where $m=0,1,\ldots$, then K_0 is diagonalized as

$$K_0|m,k\rangle = (m+k)|m,k\rangle. \tag{4}$$

Accordingly, $K_\pm$ act as raising and lowering operators as

$$K_+|m,k\rangle = [(m+1)(m+2k)]^{1/2}|m+1,k\rangle,$$

$$K_-|m,k\rangle = [m(m+2k-1)]^{1/2}|m-1,k\rangle. \tag{5}$$

Following Perelomov, the SU(1,1) CS's are then given by [11,21]

$$\begin{aligned}|\xi,k\rangle &= D(\alpha)|0,k\rangle \\ &= \exp(\alpha K_+ - \bar{\alpha}K_-)|0,k\rangle \\ &= (1-|\xi|^2)^k \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+2k)}{m!\,\Gamma(2k)}\right)^{1/2} \xi^m |m,k\rangle,\end{aligned} \tag{6}$$

where $\alpha = -(\theta/2)e^{-i\varphi}$, $\xi = -\tanh(\theta/2)e^{-i\varphi}$, with the parameters φ and θ obeying $0 \le \theta < \infty$, $0 \le \varphi < 2\pi$. Assume that the involved quantum state is initially prepared as the vacuum state $|0,k\rangle$ at time $t=0$. Then, the desired SU(1,1) CS $|\xi,k\rangle$ can be achieved by steering the system

$$i\dot{U}(t) = [2\omega_0 K_0 + u(t)K_+ + \bar{u}(t)K_-]U(t) \tag{7}$$

from its initial $U(0)=I$ to the terminal $U(T)=D(\alpha)$.

In practice, the amplitude of the control $u(t)$ in (2) is usually restricted by an upper bound c, i.e., the admissible control should satisfy $|u(t)| < c$. Here, we consider the case when the control field can switch back and forth between two extremal values u_1 and u_2, where $u_1 = -u_2 = ce^{i\phi}$ $(0 \le \phi < 2\pi)$. In the control theory, this control strategy is well known as the bang-bang control. Accordingly, we obtain two Hamiltonians

$$H_1 = 2\omega_0 K_0 + ce^{i\phi}K_+ + ce^{-i\phi}K_- \tag{8}$$

and

$$H_2 = 2\omega_0 K_0 - ce^{i\phi}K_+ - ce^{-i\phi}K_-. \tag{9}$$

By switching back and forth between H_1 and H_2, the final evolution operator that can be generated is of the following form:

$$U_f = e^{-iH_2 t_n} e^{-iH_1 t_{n-1}} \cdots e^{-iH_2 t_2} e^{-iH_1 t_1}, \tag{10}$$

where t_1, $t_n \ge 0$, and $t_i > 0$ $(i=2,3,\ldots,n-1)$. The goal of this paper is to determine the minimal number of factors needed, for any desired squeezing operator $D(\alpha)$, such that $D(\alpha) = e^{-iH_0 t_0''} U_f e^{-iH_0 t_0'}$, where the drift term $H_0 = 2\omega_0 K_0$ of the system Hamiltonian is corresponding to the control $u=0$.

B. Main Idea

We shall be interested in the 2×2 non-Hermitian realization of the Lie algebra SU(1,1), in which the generators are correspondingly identified as

$$K_0 = \frac{1}{2}\sigma_3, \quad K_1 = \frac{i}{2}\sigma_2, \quad K_2 = -\frac{i}{2}\sigma_1, \tag{11}$$

where σ_i are the Pauli matrices. Accordingly, any given squeezing operator $D(\alpha)$ is then an element of the Lie group SU(1,1), and can be written as

$$\begin{aligned}D(\alpha) &= \begin{pmatrix} \cosh|\alpha| & \frac{\bar{\alpha}}{|\alpha|}\sinh|\alpha| \\ \frac{\bar{\alpha}}{|\alpha|}\sinh|\alpha| & \cosh|\alpha| \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\theta/2) & e^{i(\pi-\varphi)}\sinh(\theta/2) \\ e^{-i(\pi-\varphi)}\sinh(\theta/2) & \cosh(\theta/2) \end{pmatrix},\end{aligned} \tag{12}$$

where $\alpha = -(\theta/2)e^{-i\varphi}$, $0 \le \theta < \infty$, and $0 \le \varphi < 2\pi$. Similarly, the time evolution propagators corresponding to the Hamiltonian H_l $(l=1,2)$ can be written as

$$U(t) = \exp(-itH_l) = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \tag{13}$$

where

$$a = \cos(t\sqrt{\omega_0^2 - c^2}) - \frac{i\omega_0}{\sqrt{\omega_0^2 - c^2}}\sin(t\sqrt{\omega_0^2 - c^2}),$$

$$b = -\frac{ice^{i\phi}e^{i(l-1)\pi}}{\sqrt{\omega_0^2 - c^2}}\sin(t\sqrt{\omega_0^2 - c^2}), \tag{14}$$

when $\omega_0 > c$, and

$$a = 1 - i\omega_0 t,$$

$$b = -ice^{i\phi}e^{i(l-1)\pi}t, \tag{15}$$

when $\omega_0 = c$, and

$$a = \cosh(t\sqrt{c^2 - \omega_0^2}) - \frac{i\omega_0}{\sqrt{c^2 - \omega_0^2}}\sinh(t\sqrt{c^2 - \omega_0^2}),$$

$$b = -\frac{ice^{i\phi}e^{i(l-1)\pi}}{\sqrt{c^2 - \omega_0^2}}\sinh(t\sqrt{c^2 - \omega_0^2}), \tag{16}$$

when $\omega_0 < c$. We call the Hamiltonian H_l $(l=1,2)$ elliptical (parabolical or hyperbolical) when $\omega_0 > c$ $(\omega_0 = c$ or $\omega_0 < c)$.

For any given SU(1,1) element

$$X=\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2-|b|^2=1, \tag{17}$$

one can introduce the unimodular Möbius transformation R defined by

$$R(z;X):=\frac{az+b}{\bar{b}z+\bar{a}}. \tag{18}$$

For any given $X_1,X_2\in \mathrm{SU}(1,1)$, it can be verified that $R(z;X_2X_1)=R(R(z;X_1);X_2)$. Let

$$M(X):=R(0;X), \tag{19}$$

where $X\in \mathrm{SU}(1,1)$. Since any given point in the open unit disk $\mathbb{D}=\{z\in\mathbb{C} \| z|<1\}$ can be written as $e^{i(\pi-\varphi)}\tanh(\theta/2)$, the mapping M maps the squeezing operators one to one and onto $\mathbb{D}$. Notice that $R(0;X)=0$ holds if, and only if, $X=\exp(-iH_0t_0)$ for some $t_0\in[0,2\pi/\omega_0)$, the subgroup $G=\{X\in \mathrm{SU}(1,1)|X=\exp(-iH_0t_0),t_0\in[0,2\pi/\omega_0)\}$ is the maximum isotropy group, with respect to the mapping $M(\cdot)$, of the point $z=0$. Thus, $M(\cdot)$ is an isomorphism from the coset space $\mathrm{SU}(1,1)/G$ to $\mathbb{D}$. This indicates that for arbitrary $X_1,X_2\in \mathrm{SU}(1,1)$, $R(0;X_1)=R(0;X_2)$ holds if, and only if, $X_1=X_2\exp(-iH_0t_0)$ for some $t_0\in[0,2\pi/\omega_0)$.

Based on the above discussion, one can obtain the optimal switching control fields steering the quantum system from the initial vacuum state to a desired SU(1,1) CS state, $|\xi,k\rangle$, as follows.

Step 1. Calculate the squeezing operator $D(\alpha)$ such that $|\xi,k\rangle=D(\alpha)|0,k\rangle$.

Step 2. Obtain the point z_f in the open unit disk that the squeezing operator $D(\alpha)$ is mapped to by the mapping $M(\cdot)$.

Step 3. Realize the evolution operator U_f with minimal number of factors such that $|R(0,U_f)|=|z_f|$ is in the form $U_f=e^{-iH_2t_n}e^{-iH_1t_{n-1}}\cdots e^{-iH_2t_2}e^{-iH_1t_1}$.

Step 4. Determine the parameter t_0'' such that $R(0,\exp(-iH_0t_0'')U_f)=z_f$.

Step 5. Determine the parameter t_0' such that $D(\alpha)=\exp(-iH_0t_0'')U_f\exp(-iH_0t_0')$.

III. MAIN RESULTS

In this section, we evaluate the minimal switches needed to realize a desired squeezing operator $D(\alpha)$ by looking for the path that connects the origin $z_0=0$ and the target point $z_f=M(D(\alpha))=e^{i(\pi-\varphi)}\tanh(\theta/2)$, which follows the trajectories generated by $R(z,e^{-iH_1t})$, $R(z,e^{-iH_2t})$, and $R(z,e^{-iH_0t})$. Three different cases with respect to the controlled Hamiltonians H_1 and H_2 will be taken into account.

A. Hyperbolical case

When $c>\omega_0$, H_1 and H_2 are hyperbolical corresponding to the control $u_1=ce^{i\phi}$ and $u_2=-ce^{i\phi}$, respectively, where $\phi\in[0,2\pi)$ is a fixed real number. Consider the trajectories given by

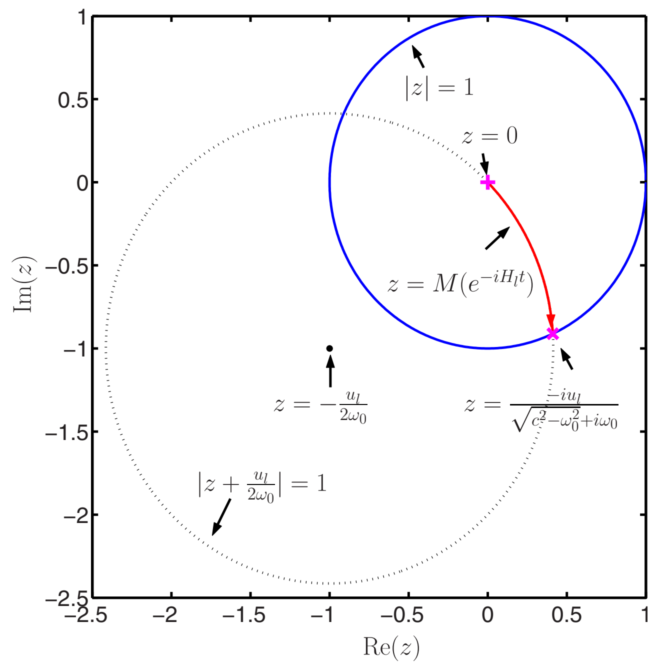

FIG. 1. (Color online) The trajectory of $M(\exp(-itH_l))$ ($l=1$ or 2) on the complex plane with respect to $c>\omega_0$, where $u_l/\omega_0=2+2i$.

$$\begin{aligned} M(\exp(-iH_lt)) &= \frac{-\dfrac{ice^{i\phi}e^{i(l-1)\pi}}{\sqrt{c^2-\omega_0^2}}\sinh(t\sqrt{c^2-\omega_0^2})}{\cosh(t\sqrt{c^2-\omega_0^2})+\dfrac{i\omega_0}{\sqrt{c^2-\omega_0^2}}\sinh(t\sqrt{c^2-\omega_0^2})} \\ &= \frac{-ice^{i\phi}e^{i(l-1)\pi}\tanh(t\sqrt{c^2-\omega_0^2})}{\sqrt{c^2-\omega_0^2}+i\omega_0\tanh(t\sqrt{c^2-\omega_0^2})}, \end{aligned} \tag{20}$$

where $l=1,2$. For any time t, we have

$$\begin{aligned} \left| M(\exp(-iH_lt))+\frac{u_l}{2\omega_0}\right| &= \frac{c}{2\omega_0}\left|1-\frac{i2\omega_0\tanh(t\sqrt{c^2-\omega_0^2})}{\sqrt{c^2-\omega_0^2}+i\omega_0\tanh(t\sqrt{c^2-\omega_0^2})}\right| \\ &= \frac{c}{2\omega_0}>\frac{1}{2}. \end{aligned} \tag{21}$$

This equation shows that the time evolution of $M(\exp(-iH_lt))$ follows a circular trajectory centered at $z=-\frac{u_l}{2\omega_0}$ with radius $r=\frac{c}{2\omega_0}$ (see Fig. 1).

It is easy to show that $|M(\exp(-iH_lt))|$ ($l=1,2$) increases monotonously as the time t increases. Moreover, as t goes to infinite, we have

$$\begin{aligned} \lim_{t\to+\infty}|M(\exp(-iH_lt))| &= \lim_{t\to+\infty}\left|\frac{-ice^{i\phi}e^{i(l-1)\pi}\tanh(t\sqrt{c^2-\omega_0^2})}{\sqrt{c^2-\omega_0^2}+i\omega_0\tanh(t\sqrt{c^2-\omega_0^2})}\right| \\ &= \left|\frac{-ice^{i\phi}e^{i(l-1)\pi}}{\sqrt{c^2-\omega_0^2}+i\omega_0}\right|=1. \end{aligned} \tag{22}$$

Therefore, to reach the target point z_f from the origin $z_0=0$, one can first reach a point z_1 such that $|z_1|=\tanh(\theta/2)$ by following the trajectory $M(\exp(-iH_1t))$ [or $M(\exp(-iH_2t))$]. If $z_1 \neq z_f$, one can reach the point z_f from z_1 by following the trajectory $R(z_1, \exp(-iH_0t))$, because it follows a circle centered at $z=0$ with radius $r=\tanh(\theta/2)$. This indicates that one can construct an evolution operator of the form

$$U_f = \exp(-iH_0t_0'')\exp(-iH_lt_1), \tag{23}$$

where $l=1$ or 2, such that $M(U_f)=z_f$, which enables one to realize the desired squeezing operator $D(\alpha)$ as

$$D(\alpha) = \exp(-iH_0t_0'')\exp(-iH_lt_1)\exp(-iH_0t_0'). \tag{24}$$

If the parameter ϕ can be adjusted to be any value in $[0,2\pi)$, then, for any fixed time t, the argument of $M(\exp(-iH_lt))$ $(l=1,2)$ can take any value in $[0,2\pi)$ by appropriately tuning ϕ. Combining with (22), one can immediately draw the conclusion that $\{z=M(\exp(-iH_lt))|t \geq 0, \phi \in [0,2\pi)\}=\mathbb{D}(l=1,2)$, which implies that any target point z_f in $\mathbb{D}$ can be reached from the origin $z_0=0$ without any switch by following the trajectory of $M(\exp(-iH_1t))$ or $M(\exp(-iH_2t))$. This enables one to realize the desired squeezing operator $D(\alpha)$ by only one switch, i.e.,

$$D(\alpha) = \exp(-iH_lt_1)\exp(-iH_0t_0), \tag{25}$$

where $l=1$ or 2.

In conclusion, we have the following proposition.

Proposition III.1. Consider the case that the magnitude c of the controls u_1 and u_2 is greater than ω_0. If the control u_1 has a fixed argument, then at least two switches are needed to realize a desired SU(1,1) CS. If the argument of the control u_1 is adjustable in $[0,2\pi)$, then only one switch is required.

B. Parabolical case

In this case, H_1 and H_2 are parabolical Hamiltonians corresponding to the controls $u_1=\omega_0 e^{i\phi}$ and $u_2=-\omega_0 e^{i\phi}$, where $\phi \in [0,2\pi)$. The corresponding trajectories are then given by

$$M(\exp(-iH_lt)) = \frac{-i\omega_0 e^{i\phi}e^{i(l-1)\pi}t}{1+i\omega_0 t}, \tag{26}$$

where $l=1,2$. It also can be verified that, for any time t,

$$\left| M(\exp(-iH_lt)) + \frac{u_l}{2\omega_0} \right| = \frac{1}{2}\left| 1 - \frac{i2\omega_0 t}{1+i\omega_0 t} \right| = \frac{1}{2}. \tag{27}$$

This implies that $M(\exp(-iH_lt))$ $(l=1,2)$ evolves on the circle $|z+\frac{u_l}{2\omega_0}|=\frac{1}{2}$, which is tangent to the unit circle $|z|=1$ at the point $z=-e^{i\phi}e^{i(l-1)\pi}$ (see Fig. 2).

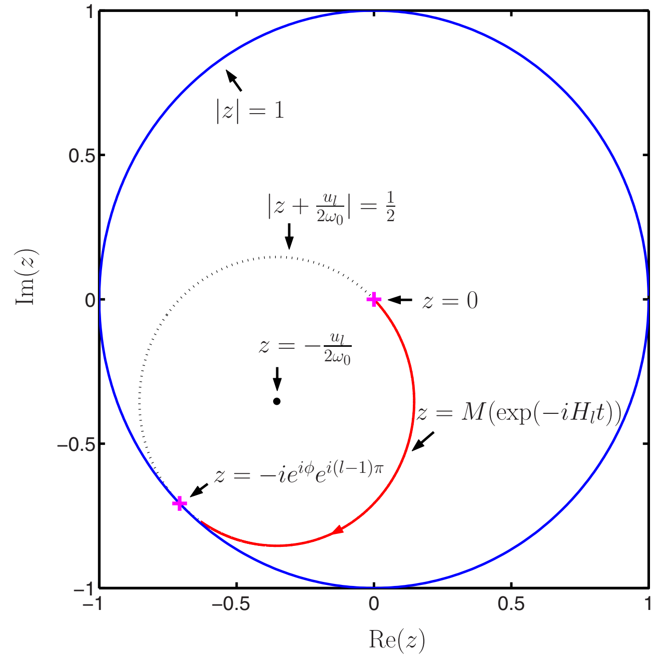

FIG. 2. (Color online) The trajectory of $M(\exp(-itH_l))$ on the complex plane with $u_l=\frac{\sqrt{2}}{2}(1+i)\omega_0$.

In this case, $|M(\exp(-iH_lt))|$ $(l=1,2)$ also increases monotonously with time t, and

$$\lim_{t\to+\infty} |M(\exp(-iH_lt))| = \lim_{t\to+\infty} \left| \frac{-i\omega_0 e^{i\phi}e^{i(l-1)\pi}t}{1+i\omega_0 t} \right| = 1. \tag{28}$$

Similar to the hyperbolical case, we have the following proposition.

Proposition III.2. For the case that the magnitude c of the controls u_1 and u_2 is exactly ω_0, at least two switches are required to realize an arbitrary desired SU(1,1) CS with the argument, ϕ, of u_1 fixed. If the argument of the control u_1 is adjustable in $[0,2\pi)$, then only one switch is needed.

C. Elliptical case

When $c<\omega_0$, H_1 and H_2 are elliptical Hamiltonians corresponding to the control $u_1=ce^{i\phi}$ and $u_2=-ce^{i\phi}$, where $\phi \in [0,2\pi)$. In this case, the trajectory driven by H_l $(l=1,2)$ starting from the point $z=ku_l$ with $0 \leq k < 1/c$ can be determined by

$$R(ku_l;\exp(-iH_lt)) = \frac{\left[\cos(t\sqrt{\omega_0^2-c^2}) - \frac{i\omega_0}{\sqrt{\omega_0^2-c^2}}\sin(t\sqrt{\omega_0^2-c^2})\right]ku_l - \frac{ice^{i\phi}e^{i(l-1)\pi}}{\sqrt{\omega_0^2-c^2}}\sin(t\sqrt{\omega_0^2-c^2})}{\frac{ice^{-i\phi}e^{-i(l-1)\pi}}{\sqrt{\omega_0^2-c^2}}\sin(t\sqrt{\omega_0^2-c^2})ku_l + \cos(t\sqrt{\omega_0^2-c^2}) + \frac{i\omega_0}{\sqrt{\omega_0^2-c^2}}\sin(t\sqrt{\omega_0^2-c^2})}$$

$$= \frac{k\sqrt{\omega_0^2 - c^2} - i(k\omega_0 + 1)\tan(t\sqrt{\omega_0^2 - c^2})}{\sqrt{\omega_0^2 - c^2} + i(kc^2 + \omega_0)\tan(t\sqrt{\omega_0^2 - c^2})} u_l. \tag{29}$$

One can verify that for every $t > 0$,

$$\left| R(ku_l;\exp(-iH_l t)) - \frac{k^2c^2 - 1}{2(kc^2 + \omega_0)} u_l \right|$$
$$= \left| \frac{k\sqrt{\omega_0^2 - c^2} - i(k\omega_0 + 1)\tan(t\sqrt{\omega_0^2 - c^2})}{\sqrt{\omega_0^2 - c^2} + i(kc^2 + \omega_0)\tan(t\sqrt{\omega_0^2 - c^2})} - \frac{k^2c^2 - 1}{2(kc^2 + \omega_0)} \right| c$$
$$= \frac{k^2c^2 + 2k\omega_0 + 1}{2(k^2c^2 + \omega_0)} c. \tag{30}$$

Thus, the trajectory $R(ku_l;\exp(-iH_l t))$ $(l=1,2)$ is restricted on a circle centered at $z=\frac{k^2c^2-1}{2(kc^2+\omega_0)}u_l$ with radius $r=\frac{k^2c^2+2k\omega_0+1}{2(k^2c^2+\omega_0)}c$ in the open unit disk $\mathbb{D}$ (see Fig. 3).

Analysis of (29) shows that $|R(ku_l;\exp(-iH_l t))|$ $(l=1,2)$ increases monotonously with t when $t \in \left[\frac{n\pi}{\sqrt{\omega_0^2-c^2}}, \frac{(2n+1)\pi}{2\sqrt{\omega_0^2-c^2}}\right]$ $(n=0,1,2,\ldots)$, while it is monotonously decreasing on the interval $\left[\frac{(2n+1)\pi}{2\sqrt{\omega_0^2-c^2}}, \frac{(n+1)\pi}{\sqrt{\omega_0^2-c^2}}\right]$ $(n=0,1,2,\ldots)$. The values of $R(\cdot;\cdot)$ at $t'=\frac{n\pi}{\sqrt{\omega_0^2-c^2}}$ and $t''=\frac{(2n+1)\pi}{2\sqrt{\omega_0^2-c^2}}$ can be calculated by (29) as

$$R(ku_l;\exp(-iH_l t')) = ku_l \tag{31}$$

and

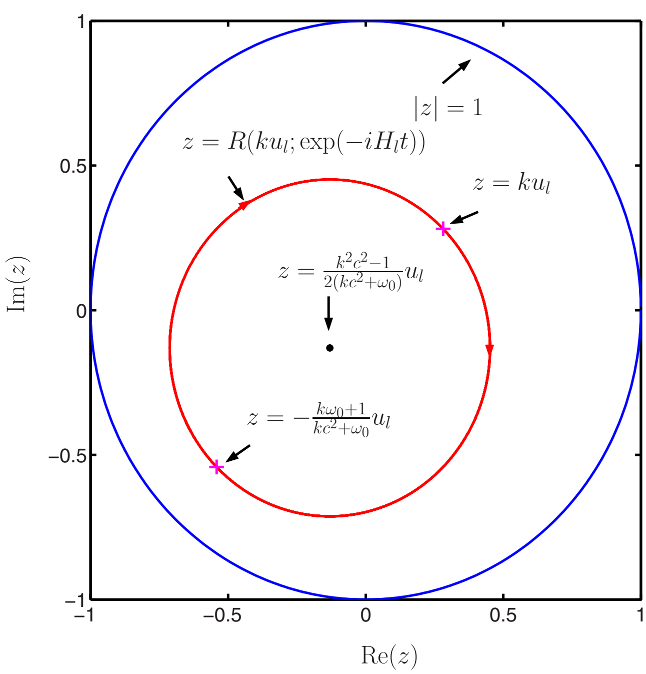

FIG. 3. (Color online) The trajectory of $R(ku_l;\exp(-iH_l t))$ on the complex plane with respect to $c < \omega_0$, where $k=\frac{3}{4}$ and $\frac{u_l}{\omega_0}=\frac{3}{8}(1+i)$.

$$R(ku_l;\exp(-iH_l t'')) = -\frac{k\omega_0 + 1}{kc^2 + \omega_0} u_l. \tag{32}$$

Notice that $kc < 1$ and $c < \omega_0$, it can be verified that

$$k < \frac{k\omega_0 + 1}{kc^2 + \omega_0} < \frac{1}{c}. \tag{33}$$

This implies that for any $t > 0$, the following inequality holds:

$$kc \le |R(ku_l;\exp(-iH_l t))| \le \frac{k\omega_0 + 1}{kc^2 + \omega_0} c. \tag{34}$$

Define the sequence $\{z_n\}$, with $z_0=0$ and

$$z_{2m+1} := R(z_{2m};\exp(-iH_1\tilde{t})),$$
$$z_{2m+2} := R(z_{2m+1};\exp(-iH_2\tilde{t})), \tag{35}$$

where $\tilde{t}=\frac{\pi}{2\sqrt{\omega_0^2-c^2}}$. Making use of (32), we can obtain

$$|z_n| = \frac{(|z_{n-1}|/c)\omega_0 + 1}{(|z_{n-1}|/c)c^2 + \omega_0} c = \frac{|z_{n-1}|\omega_0 + c}{|z_{n-1}|c + \omega_0}, \tag{36}$$

which in turn gives

$$\frac{|z_n| - |z_{n-1}|}{1 - |z_n||z_{n-1}|} = \frac{c}{\omega_0}. \tag{37}$$

Let $r_n = \operatorname{arctanh}|z_n|$, then (37) is equivalent to $r_n - r_{n-1} = \operatorname{arctanh}\frac{c}{\omega_0}$. Making use of (32), one can further obtain that $z_n = \tanh\left(n \operatorname{arctanh}\frac{c}{\omega_0}\right)e^{i(\phi+n\pi)}$.

Based on the above discussions, we are now ready to give the path between the origin $z_0=0$ and the desired target point $z_f = M(D(\alpha)) = e^{i(\pi-\varphi)}\tanh(\theta/2)$ with minimal number of switches. First, following the circular trajectory $R(0;\exp(-iH_1 t))$ from $t=0$ to $t=\frac{\pi}{2\sqrt{\omega_0^2-c^2}}$, we arrive at the point z_1, which has maximal magnitude in this trajectory. Then, following the circular trajectory $R(z_1;\exp(-iH_2 t))$ for a time period $t=\frac{\pi}{2\sqrt{\omega_0^2-c^2}}$, we arrive at the point z_2, which still has maximal magnitude correspondingly. Further, proceeding on the circular trajectory $R(z_2;\exp(-iH_1 t))$ for another time period $t=\frac{\pi}{2\sqrt{\omega_0^2-c^2}}$, we reach the point z_3. Similarly, following the circular trajectory $R(z_3;\exp(-iH_2 t))$ for a time period $t=\frac{\pi}{2\sqrt{\omega_0^2-c^2}}$, again, leads us to the point z_4. Continuing the evolution in such a spiral manner, we can construct a path with switching points $z_1, z_2, \ldots, z_n, \ldots,$ where $z_n = \tanh\left(n \operatorname{arctanh}\frac{c}{\omega_0}\right)e^{i(\phi+n\pi)}$. Let

$$N = \left\lceil \frac{\frac{\theta}{2}}{\operatorname{arctanh}\frac{c}{\omega_0}} \right\rceil, \tag{38}$$

where $\lceil a \rceil$ denotes the minimal integer number that is not less than a, then we have

$$|z_{N-1}| < |z_f| = \tanh\frac{\theta}{2} \leq |z_N|. \tag{39}$$

The above inequality implies that the trajectory will cross the circle $|z|=|z_f|$ at a point $\tilde{z}_f$ after $N-1$ switches. Accordingly, we obtain an evolution operator U_f,

$$U_f = e^{-H_2 t_N} e^{-H_1 t_{N-1}} \cdots e^{-H_2 t_2} e^{-H_1 t_1}, \tag{40}$$

or

$$U_f = e^{-H_1 t_N} e^{-H_2 t_{N-1}} \cdots e^{-H_2 t_2} e^{-H_1 t_1}, \tag{41}$$

where t_N is possibly equal to zero, such that $M(U_f)=\tilde{z}_f$. Then, following the trajectory $R(\tilde{z}_f;\exp(-iH_0 t))$ for a time period $t=t_0''$, one can finally reach the desired target point z_f. Similarly, if the argument of the control u_1 can be adjusted according to the value of z_f, the point $\tilde{z}_f$ can be obtained such that $\tilde{z}_f=z_f$. Since the operator $\exp(-iH_0 t_0'')U_f$ only differs from the desired squeezing operator $D(\alpha)$ with a possible factor $e^{-H_0 t_0'}$ on the right-hand side, we have the following proposition.

Proposition III.3. For the controls $u_1=ce^{i\phi}$ and $u_2=-ce^{i\phi}$ with $c<\omega_0$ and fixed argument ϕ, the minimal number of switches needed to realize the desired SU(1,1) CS $|\xi,k\rangle$ is $\lceil \frac{\theta}{2}/\operatorname{arctanh}\frac{c}{\omega_0} \rceil$, where $\xi=-\tanh(\theta/2)e^{-i\varphi}$. If the argument ϕ of the control u_1 is adjustable according to the value of ξ, the minimal number of switches is $\lceil \frac{\theta}{2}/\operatorname{arctanh}\frac{c}{\omega_0} \rceil - 1$.

For example, assume that $\omega_0/c=2$. Consider the problem of achieving the SU(1,1) CS $|\xi,k\rangle$, where $\xi=-e^{-i5\pi/4}\tanh\frac{3}{2}$, from the vacuum state $|0,k\rangle$ by switching the control back and forth between $u_0=0$, $u_1=ce^{i\pi/4}$, and $u_2=-ce^{i\pi/4}$.

The squeezing operator that shifts the vacuum state $|0,k\rangle$ to the target state $|\xi,k\rangle$ is $D(\alpha)=D(-\frac{3}{2}e^{-i5\pi/4})$. Accordingly, the point in the open unit disk $\mathbb{D}$ corresponding to $D(\alpha)$ is $z_f=M(D(\alpha))=e^{-i\pi/4}\tanh\frac{3}{2}$. Since $\lceil \frac{3}{2}/\operatorname{arctanh}\frac{1}{2} \rceil=3$, at least three switches are needed. One can obtain the evolution U_f such that $M(U_f)=z_f$ as

$$U_f = e^{-H_0 t_0''} e^{-H_1 t_3} e^{-H_2 t_2} e^{-H_1 t_1}, \tag{42}$$

where $t_1=t_2=\frac{\pi}{2\sqrt{3}c}$, $t_3=\frac{0.5204}{c}$, and $t_0''=\frac{2.9625}{c}$. Multiplied by a factor $e^{-H_0 t_0'}$ on the right-hand side of U_f, where $t_0'=\frac{1.1921}{c}$, we obtain the desired squeezing operator as

$$D(\alpha) = e^{-H_0 t_0''} e^{-H_1 t_3} e^{-H_2 t_2} e^{-H_1 t_1} e^{-H_0 t_0'}. \tag{43}$$

If the controls u_1 and u_2 can be selected as $u_1=ce^{i1.5019}$ and $u_2=-ce^{i1.5019}$, we can further save one time of the switch to achieve the desired squeezing operator $D(\alpha)$. Correspondingly, we have

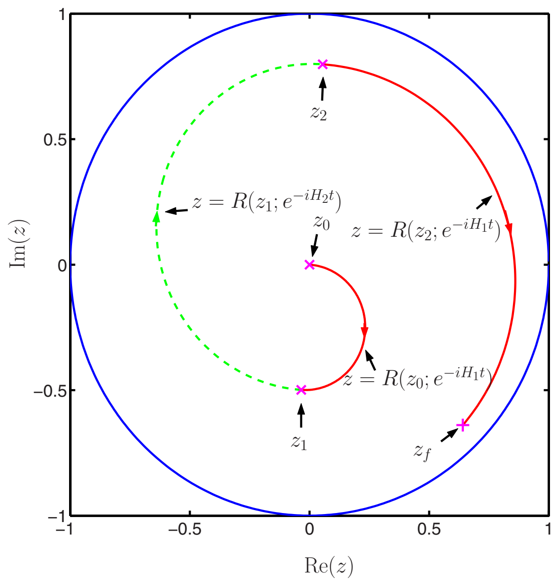

FIG. 4. (Color online) The optimal path between $z_0=0$ and $z_f=e^{-\pi/4}\tanh\frac{3}{2}$ on the complex plane, where $c/\omega_0=1/2$, $u_1=-u_2=ce^{i1.5019}$, $z_1=-e^{i1.5019}\tanh(\operatorname{arctanh}\frac{1}{2})$, and $z_2=e^{i1.5019}\tanh(2\operatorname{arctanh}\frac{1}{2})$.

$$D(\alpha) = e^{-H_1 t_3} e^{-H_2 t_2} e^{-H_1 t_1} e^{-H_0 t_0'}, \tag{44}$$

where $t_0'=\frac{1.0130}{c}$. Referring to Fig. 4, every piece of trajectory of the optimal path between $z_0=0$ and $z_f=e^{-\pi/4}\tanh\frac{3}{2}$ is provided.

IV. CONCLUSION

In this paper, we have studied the problem of achieving an arbitrary SU(1,1) CS by switching the control field back and forth between two admissible values with a minimal number of switches. By the unimodular Möbius transformation, the desired squeezing operators are one-to-one and mapped to the open unit disk in the complex plane. Accordingly, the minimal number of switches is obtained by analyzing the paths connecting the origin $z_0=0$ and the target point z_f corresponding to the desired SU(1,1) CS. The minimal number of switches needed is shown to be a function of the desired squeezing operator. The results show that, for both the hyperbolical and parabolical cases, the minimal switching number is at most two, depending on whether the argument of the involved control is adjustable or not. The elliptical case is more complicated, and the minimal number of switches also depends on the magnitude of the point corresponding to the desired squeezing operator.

The restrictions imposed on the involved control fields are practical in the experiment and hence our results are applicable. We do not see any major obstacles to extending the theory developed here to solve the optimal switching problem of states transition between two arbitrary SU(1,1) CS's, or a general time evolution on the SU(1,1) Lie group.

ACKNOWLEDGMENTS

The authors would like to thank Dr. Re-Bing Wu for his helpful suggestions. This research was supported in part by the National Natural Science Foundation of China under Grants Nos. 60674039, 60433050, and 60635040. One of the authors (T.J.T.) would also like to acknowledge partial support from the U.S. Army Research Office under Grant No. W911NF-04-1-0386.

[1] V. Ramakrishna, K. L. Flores, H. Rabitz, and R. J. Ober, Phys. Rev. A **62**, 053409 (2000).
[2] J.-W. Wu, C.-W. Li, R.-B. Wu, T.-J. Tarn, and J. Zhang, J. Phys. A **39**, 13531 (2006).
[3] F. Lowenthal, Can. J. Math. **23**, 364 (1971).
[4] F. Lowenthal, Rocky Mt. J. Math. **1**, 575 (1971).
[5] F. Lowenthal, Can. J. Math. **24**, 713 (1972).
[6] P. E. Crouch and F. S. Leite, Syst. Control Lett. **2**, 341 (1983).
[7] F. S. Leite, Rocky Mt. J. math. **21**, 879 (1991).
[8] D. D'Alessandro, Syst. Control Lett. **47**, 87 (2002).
[9] D. D'Alessandro, Automatica **40**, 1997 (2004).
[10] A. I. Solomon, J. Math. Phys. **12**, 390 (1971).
[11] C. C. Gerry, Phys. Rev. A **31**, 2721 (1985).
[12] C. C. Gerry and E. R. Vrscay, Phys. Rev. A **39**, 5717 (1989).
[13] C. C. Gerry, P. K. Ma, and E. R. Vrscay, Phys. Rev. A **39**, 668 (1989).
[14] C. C. Gerry, Phys. Rev. A **39**, 971 (1989).
[15] C. C. Gerry and J. Kiefer, Phys. Rev. A **41**, 27 (1990).
[16] C. C. Gerry, R. Grobe, and E. R. Vrscay, Phys. Rev. A **43**, 361 (1991).
[17] Z. W. Gortel and L. A. Turski, Phys. Rev. A **43**, 3221 (1991).
[18] V. Penna, Ann. Phys. **245**, 389 (1996).
[19] A. Bechler, J. Phys. A **34**, 8081 (2001).
[20] G. S. Agarwal and J. Banerji, Phys. Rev. A **64**, 023815 (2001).
[21] P. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin, 1986).