Long-distance entanglement and quantum teleportation in *XX* spin chains

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Isotropic *XX* models of one-dimensional spin-1/2 chains are investigated with the aim to elucidate the formal structure and the physical properties that allow these systems to act as channels for long-distance, high-fidelity quantum teleportation. We introduce two types of models: (i) open, dimerized *XX* chains, and (ii) open *XX* chains with small end bonds. For both models we obtain the exact expressions for the end-to-end correlations and the scaling of the energy gap with the length of the chain. We determine the end-to-end concurrence and show that model (i) supports true long-distance entanglement at zero temperature, while model (ii) supports "quasi-long-distance" entanglement that slowly falls off with the size of the chain. Due to the different scalings of the gaps, respectively exponential for model (i) and algebraic in model (ii), we demonstrate that the latter allows for efficient qubit teleportation with high fidelity in sufficiently long chains even at moderately low temperatures.

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I. INTRODUCTION

The crucial role of entanglement as a fundamental resource for quantum information tasks that transgress the classical limits is particularly evident and has been experimentally verified in such protocols as teleportation [1,2], cryptography and secure key distribution [3], and quantum communication [4]. Typically, the "most natural way" to create entanglement between parties is by means of direct interactions. Since, intuitively, large amounts of entanglement should be associated to the presence of strong correlations, low-dimensional systems, as, for instance, spin chains, offer a natural source of entanglement. However, in most systems with short-range interactions, the entanglement between a pair of particles decays rapidly with the distance. For instance, in the Ising model with transverse field [5] the two-spin concurrence vanishes for distances larger than two neighboring sites, while in the Heisenberg model [6], it is restricted only to single nearest neighbors. A first exception to this behavior was found by Amico *et al.* [7], who established that close to factorization points, the range of entanglement grows indefinitely. However, in this case, the entanglement strength between two spins rapidly vanishes as the range increases. In fact, this behavior appears to be natural, recalling that monogamy of entanglement [8] implies that if entanglement is enhanced between two given subsystems, then the entanglement between any of the two subsystems and a third one must necessarily be suppressed.

From a general quantum informatics perspective, a much desired goal would be the ability to create a large amount of entanglement between distant subsystems, simultaneously avoiding direct interactions between them and single-subsystem addressing. Along this line of thought, the first requirement can be satisfied by introducing the concept of localizable entanglement, with the aim of exploiting spin chains as quantum channels [9]. The localizable entanglement measures the average entanglement that can be concentrated on a pair of distant subsystems by performing optimal local measurements onto the rest of the system. More recently, it was shown that the ground state of some spin models with finite correlation length defined on one-dimensional chains with open ends can support large values of long-distance entanglement between the end points of the chain [10]. This approach, i.e., to look for systems whose ground state can support long-distance entanglement *prima facie*, without the need of performing operations and measurements, is clearly very appealing. However, it appears that models which exhibit true long-distance entanglement are characterized by energy gaps above the ground state that vanish exponentially as the length of the chain is increased [10]. Hence this interesting phenomenon seems doomed to survive only in the physically unattainable situation of zero temperature. In the effort to overcome this problem, it was shown that a kind of long-distance entanglement, very slowly decreasing with the length of the chain, can be supported by the ground state of spin models with infinite correlation length defined on one-dimensional open chains with small end bond interactions, and that these systems allow for robust finite-temperature teleportation across finite distances [11]. This quasi-long-distance entanglement between the end points of the chain is associated to an energy gap that vanishes algebraically with the size of the system. As a consequence, it is more resilient to thermal excitations, and can be used to engineer realistic protocols of qubit teleportation, as has been explicitly demonstrated, by numerical density matrix renormalization group (DMRG) simulations, in the case of the Heisenberg (*XXX*) chain with small end interactions [11].

In the present paper we will show that both true long-distance and quasi-long-distance entanglement can be supported by very simple isotropic *XX* models of open spin chains. These models present some interesting advantages

over the ones that have been previously studied. At variance with the Heisenberg case, they are amenable to exact analytical treatment both in the case of alternating couplings, corresponding to dimerization of the ground state and true long-distance entanglement, and in the case of small end bonds, corresponding to a ground state that supports quasi-long-distance entanglement. In this way, one achieves a full grasp of the physical mechanism responsible for long- and quasi-long-distance entanglement, and the possibility to identify unambiguously the optimal range of parameters for teleportation with maximal fidelity. Moreover, having in mind possible experimental realizations, for instance in suitably engineered optical lattices [12], *XX* chains are in principle more easily realizable or simulatable than Heisenberg interactions. The exact solvability of the open-end *XX* chains is possible thanks to the methods introduced by Lieb and co-workers [13]. We will first study the fully dimerized *XX* open spin chain with alternating couplings, discuss the exact behavior of the end-to-end concurrence in the ground state at zero temperature, and show analytically that for this kind of model the energy gap between the ground and the lowest excited state falls off exponentially with the size of the system. Therefore this system cannot be exploited for quantum teleportation in realistic situations at finite temperature. We then study the *XX* open spin chain with small end bonds and determine analytically the exact expression for the zero temperature end-to-end concurrence in some physically relevant limits. We determine the exact scaling of the energy gap with the length of the chain and show that it dies off algebraically with the size of the system. Finally, we determine the behavior of the teleportation fidelity as a function of the temperature for different strengths of the small end bonds, and show that qubit teleportation with fidelities well above the classical threshold, and in some cases close to unity, is supported even at moderately low temperatures.

The paper is organized as follows: In Sec. II we introduce the general *XX* spin chain Hamiltonian with arbitrary site-dependent couplings, and solve it analytically for the end-to-end two-point reduced density matrix, correlation, and concurrence (entanglement of formation). We calculate the fully entangled fraction and establish the analytical expression for the fidelity of teleportation of an unknown qubit state between the two end points of the chain. In Sec. III we specialize the model to the case of perfectly alternating couplings and fully dimerized ground state. We determine the single particle dispersion law, establish the exponential scaling behavior of the energy gap, and determine the analytical expressions for the ground state end-to-end concurrence and fidelity. We show that, depending on the values of the coupling strength, high or even maximal entanglement and fidelity can be achieved for chains of arbitrary length. In Sec. IV we turn to the case of open *XX* chains with small end bonds, establishing the single particle dispersion law, the algebraic scaling of the energy gap, and the analytic expressions for the concurrence and the fidelity. We then investigate the behavior of the end-to-end teleportation fidelity as a function of the temperature for different values of the small end bonds, and conclude that teleportation with unit fidelity at very low temperature and with high fidelity above the classical threshold at moderately low temperature are both supported by the *XX* channel with small end bonds. Finally, in the Conclusions we summarize our findings and discuss some outlooks on possible future developments along this line of research.

II. THE GENERAL MODEL

As already anticipated in the Introduction, we will focus our analysis on one-dimensional lattices with open ends, described by *XX* models with different types of nearest neighbor interactions. Such models are all special instances of the general *XX* Hamiltonian

$$H = \sum_{i=1}^{L-1} J_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y), \tag{1}$$

where J_i is the interaction strength between nearest neighboring sites i and $i+1$, S_i^α are the spin operators defined at site i, and L is the total number of sites (spins) or the length of the chain. The spectrum of this Hamiltonian can be determined exactly by a straightforward application of the standard methods introduced in Ref. [13]. The first step in the procedure is to perform a Jordan-Wigner transformation [14],

$$S_i^+ = c_i^\dagger e^{i\pi \sum_{j=1}^{i-1} c_j^\dagger c_j}, \quad S_i^- = e^{-i\pi \sum_{j=1}^{i-1} c_j^\dagger c_j} c_i,$$

$$S_i^z = c_i^\dagger c_i - \frac{\mathbb{1}}{2}, \tag{2}$$

where $S_j^\pm = S_j^x \pm i S_j^y$. As a result, the Hamiltonian (1) is mapped in the free fermion Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{L-1} J_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) = \mathbf{c}^\dagger M \mathbf{c}, \tag{3}$$

where $\mathbf{c}^\dagger = (c_1^\dagger, \ldots, c_L^\dagger)$ ($\mathbf{c}$) is the vector of the L creation (annihilation) operators, one for each site of the lattice, and the adjacency matrix M reads

$$M = \frac{1}{2} \begin{vmatrix} 0 & J_1 & 0 & \cdots & & 0 \\ J_1 & 0 & J_2 & & & \\ 0 & J_2 & 0 & & & \vdots \\ \vdots & & & \ddots & J_{L-2} & 0 \\ & & & J_{L-2} & 0 & J_{L-1} \\ 0 & & \cdots & 0 & J_{L-1} & 0 \end{vmatrix}. \tag{4}$$

We will investigate two particular realizations of Hamiltonian (1) that, as we will show, in the ground state allow for high or even maximal entanglement between the end spins of the chain, and thus naturally provide a channel with high or even unit fidelity for qubit teleportation. To evaluate the teleportation fidelity, one needs to determine the spin-spin concurrence (entanglement of formation) between the end points of the chain. This quantity can be computed exactly for any two-qubit state (pure or mixed), thanks to the celebrated formula of Wootters [15], and the task is left to obtain its explicit expression in the reduced state of the two end-point spins. To this purpose, we need to calculate explicitly all the possible forms of two-point correlations in the ground state.

The Hamiltonian (1) is symmetric under rotations of the spins around the z axis, so that the only nonvanishing correlations are $\langle S_i^x S_j^x\rangle = \langle S_i^y S_j^y\rangle$, $\langle S_i^z S_j^z\rangle$, and $\langle S_i^z\rangle$. In the absence of external magnetic fields, π rotations around the x and y axes are symmetries of the model, which additionally implies $\langle S_i^z\rangle = 0$ at every site. Thanks to the aforementioned symmetries, the two-point reduced density matrix $\rho_{i,j}$, obtained by tracing the full density matrix of the system over all sites except the pair $\{i,j\}$, has the form

$$\rho_{i,j} = \frac{\mathbb{1}}{4} + \langle S_i^x S_j^x\rangle(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y) + \langle S_i^z S_j^z\rangle \sigma^z \otimes \sigma^z, \quad (5)$$

where $\sigma^{x,y,z}$ are the Pauli matrices and $\langle\cdot\rangle$ is the ground state average at temperature $T=0$, or the thermal one at finite temperature $\beta=(k_B T)^{-1}$ with respect to the Gibbs state $\rho = e^{-\beta H} Z^{-1}$. We are interested in the case in which i and j are the two end points of the chain. In this instance, we have

$$S_1^+ S_L^- + S_1^- S_L^+ = -e^{i\pi\mathcal{N}}(c_1^\dagger c_L + c_L^\dagger c_1),$$

$$S_1^z S_L^z = \left(c_1^\dagger c_1 - \frac{1}{2}\right)\left(c_L^\dagger c_L - \frac{1}{2}\right), \quad (6)$$

where $\mathcal{N} = \Sigma_{i=1}^L c_i^\dagger c_i$ is the total number operator. Using Wick's theorem and taking into account that $\langle c_i^\dagger c_i\rangle = 1/2$, we obtain

$$\langle S_1^+ S_L^- + S_1^- S_L^+\rangle = -e^{i\pi L/2}(\langle c_1^\dagger c_L\rangle + \langle c_L^\dagger c_1\rangle),$$

$$\langle S_1^z S_L^z\rangle = -\langle c_1^\dagger c_L\rangle\langle c_L^\dagger c_1\rangle. \quad (7)$$

Setting $x \equiv \langle c_1^\dagger c_L\rangle$ we see that the end-to-end reduced density matrix depends uniquely on this parameter.

All the physical information about model (3) can now be obtained by diagonalizing the one-body matrix M. Let ξ_k be the eigenvector with eigenvalue Λ_k, where k is a quasimomentum label. Then, passing to new fermionic operators via the transformation $c_i = \Sigma_k \xi_k^{(i)} c_k$, the Hamiltonian takes the form

$$H = \sum_k \Lambda_k c_k^\dagger c_k. \quad (8)$$

The evaluation of the two-point correlation x is then straightforward, and one obtains

$$x = \sum_{k,q} \xi_k^{(1)} \xi_q^{(L)} \langle c_k^\dagger c_q\rangle = \begin{cases} \sum_{\Lambda_k<0} \xi_k^{(1)} \xi_k^{(L)} & \text{for } T=0 \\ \sum_k \xi_k^{(1)} \xi_k^{(L)} \dfrac{1}{1+e^{\beta\Lambda_k}} & \text{for } T>0, \end{cases} \quad (9)$$

where x depends on the coupling ratios J_i/J_1 as well as on the temperature, in case the latter is also taken into account. For reduced states of the form (5), the end-to-end concurrence is easily computed, and we have [16]

$$C_{1,L} = 2\ \max\left\{0,\ x^2 + |x| - \frac{1}{4}\right\}. \quad (10)$$

The above expression of the concurrence is nonvanishing for $|x| > (\sqrt{2}-1)/2 \simeq 0.207$, and it reaches the maximum value $C_{1,L}=1$ for $|x| \to 1/2$.

It is natural to expect that the existence of a strong quantum correlation between the two end spins of the chain can be conveniently exploited for performing tasks in quantum information, in particular considering teleportation schemes. In the standard quantum teleportation protocol, two parties A and B share a maximally entangled state (Bell state). Party A holds also a third qubit, whose unknown state is to be teleported. If the two end points of our XX chain share a highly entangled state, that in some limit may even be asymptotically close to a Bell state, they can be identified as the two parties, sender and receiver, for a long-distance, high-fidelity teleportation protocol. The efficiency of a quantum channel in teleporting an unknown state is quantified by the fidelity f between the output and the input states, averaged over all input realizations. The fidelity depends on the actual properties of the entangled resource $\rho_{1,L}$ [cf. Eq. (5)] shared by the end spins of the chain. In fact, it has been demonstrated that the optimal fidelity depends only on the "fully entangled fraction" F_{full}, according to the formula $f=(2F_{full}+1)/3$ [17]. The fully entangled fraction is defined as the fidelity between the resource $\rho_{1,L}$ and a maximally entangled state, maximized over all possible maximally entangled states. For states of the form Eq. (5) it can be easily computed, and reads $F_{full} = \frac{1}{4} + |x| + x^2$ [18]. The associated teleportation fidelity is thus

$$f = \frac{2[(1/4) + |x| + x^2] + 1}{3}. \quad (11)$$

This expression highlights the crucial interplay between entanglement and efficiency in quantum information protocols. In fact, due to the high symmetry of states of the form Eq. (5), a nonvanishing entanglement implies a nonclassical teleportation fidelity exceeding the classical threshold 2/3, and vice versa. In the limit $|x| \to 1/2$ of maximally entangled resources, the maximum teleportation fidelity reaches unity.

III. LONG-DISTANCE ENTANGLEMENT

In analogy with previous work on the dimerized Heisenberg model [10], we consider first the open end XX chain with bonds of alternating strengths $(1-\delta)$ (weak bond), and $(1+\delta)$ (strong bond), with $0 \le \delta \le 1$,

$$H = J\sum_{i=1}^{L-1} [1 + (-1)^i \delta](S_i^x S_{i+1}^x + S_i^y S_{i+1}^y). \quad (12)$$

Choosing L even and $0 \le \delta \le 1$ assures that the spins at the end of the chain interact with a weak bond of strength $(1-\delta)$ with their respective neighbors.

Let us first comment on the general features of the model (12) in the thermodynamic limit. For $\delta=0$ the model in the fermionic picture reduces to a simple tight binding with dis-

persion $\Lambda_k = J\cos(k)$. The ground state is given by a half filled band and vanishingly small excitations are present near the Fermi points. Upon the introduction of a nonvanishing δ translational invariance by one site is broken, the Brillouin zone is correspondingly halved, and a gap of width δ opens up in the single particle spectrum. Two bands develop with dispersion

$$\Lambda_{k,\pm} = \pm J\sqrt{\cos^2(k) + 4\delta^2 \sin^2(k)}. \tag{13}$$

Upon increasing the value of δ, the growing difference between the alternating coupling constants allows the creation of dimers between pairs of strongly interacting spins. The dimers, in turn, are very weakly interacting with each other. Hence because the two end spins of the chain interact very weakly with their nearest neighbors, dimerization and monogamy of entanglement force the creation of a strong quantum correlation between the end points, in analogy with what happens in the Heisenberg case [10].

The above qualitative picture can be supplemented by a detailed quantitative analysis both in the thermodynamic limit and at finite length L. The eigenvectors of the adjacency matrix M are given by [19]

$$\xi^{(j)}_{k,\mu_k} = \begin{cases} A_k \sin(kj) & j \text{ even}, \\ \mu_k A_k \sin[k(L-j+1)] & j \text{ odd}, \end{cases} \tag{14}$$

where $\mu_k = \pm 1$ is the parity (left-right symmetry) of the eigenstate ξ_k, and A_k is a normalization constant that reads

$$A_k = 2\left[2\left(L + 1 - \frac{\sin 2k(L+1)}{\sin(2k)}\right)\right]^{-1/2}. \tag{15}$$

The quasimomenta are given by the solution of the following equation:

$$\frac{\sin k(L+2)}{\sin(kL)} = -\frac{1+\delta}{1-\delta} \equiv -\frac{1}{a}. \tag{16}$$

For each quasimomentum we have two eigenvectors corresponding to opposite parity. Hence we seek for $L/2$ solutions of Eq. (16). When $a < L/(L+2)$ (roughly $\delta > 0$, when L is large) we have $L/2-1$ real solutions for which the energy dispersion has the form (13). The missing solution is complex (and plays a very important role) and has the form $k_0 = \pi/2 + ip$. For this mode the equation reads

$$\cosh(2p) + \coth(Lp)\sinh(2p) = \frac{1}{a}. \tag{17}$$

At leading order the solution is given by

$$e^{2p} = \frac{1}{a} - (1-a^2)a^{L-1}. \tag{18}$$

The appearance of a complex quasimomentum is crucial for the existence of long-distance entanglement (LDE) in the ground state and can be understood as follows. The eigenvectors $\xi_{k_0,\pm}$ are localized at the boundaries of the chain. For instance, for site j close to the left border of the chain ($j \simeq 1$), one has $|\xi^{(j)}_{k_0,\pm}| \simeq e^{(L-j+1)/\zeta}$, with a localization length ζ that, for an asymptotically large chain, reads

$$\zeta = 2/\ln[(1+\delta)/(1-\delta)]. \tag{19}$$

The energy of this mode, for both values of the parity, is exponentially small:

$$\Lambda_{k_0,\pm} \simeq \pm 2(1-a)e^{-L/\zeta}, \tag{20}$$

and we thus see that the onset of LDE in the dimerized ground state of the XX chain with open ends is strictly linked to the existence of an energy gap above the ground state that vanishes exponentially fast with the length of the chain.

At zero temperature and zero external field, all negative energy modes are filled, up to $\Lambda_{k_0,-}$. The mode k_0 is responsible for the appearance of a state localized at the end of the chain in the many-body ground state. This means that the ground state resembles $\Psi \simeq \psi_{1,L} \otimes \psi_{\text{rest}}$. As we have dimer order at sites 1 and L, we expect $\psi_{1,L}$ to be highly entangled. Because factorization will not be exact, $\psi_{1,L}$ will not be exactly a pure state but rather a weakly mixed reduction $\rho_{1,L}$ that becomes maximally entangled only in the asymptotic regime of perfect dimerization. In fact, knowing the eigenvectors of the adjacency matrix, we can evaluate exactly the end-to-end LDE between the borders of the chain. Using Eq. (14), the fermionic correlation function x reads

$$x = \langle c_1^\dagger c_L \rangle = \sum_{k,\Lambda_k<0} \mu_k A_k^2 [\sin(Lk)]^2. \tag{21}$$

To evaluate this sum we first isolate the contribution coming from the complex momentum. The remaining set of terms defines a sum with alternating signs of a periodic function analytic on the real axis. The rate of convergence to its asymptotic vanishing value is dictated by the width of the largest strip, around the imaginary axis, where the function is analytic. The inverse of this width is precisely given by the localization length Eq. (19). Such a contribution is exponentially suppressed, and the final result is

$$x = (-1)^{L/2}\frac{1}{2}(1-a^2) + O(e^{-L/\zeta}) = (-1)^{L/2}\frac{2\delta}{(1+\delta)^2} + O(e^{-L/\zeta}). \tag{22}$$

We have checked this approximation against exact numerical data and the results are plotted in Fig. 1. It turns out that the remainder has the form $(-1)^{L/2}A_0 L^2 \exp(-L/\zeta) + \cdots$, where A_0 is a positive constant. This implies that, in modulus, the asymptotic value of the end-to-end correlation is in fact approached from below: In this type of system, correlations *increase* as the length L of the chain is increased.

We see that, for any nonvanishing value of the dimerization, the ground state of the system develops a constant, nonzero correlation between the end points of arbitrarily large chains, a phenomenon known as surface order. Using Eq. (10), we deduce that this surface order, present whenever $\delta > 0$ (and $\delta < 1$), allows for LDE between the end spins as soon as $\delta > \delta_0 = (1-\sqrt{2-\sqrt{2}})/(1+\sqrt{2-\sqrt{2}}) = 0.132$, i.e., for moderately low values of the dimerization. The corresponding threshold localization length is $\zeta_0 = 7.479$. The localization length ζ rapidly decreases from the threshold value ζ_0 with increasing $\delta > \delta_0$. As a consequence, the asymptotic regime for the corresponding end-to-end concurrence (LDE) is

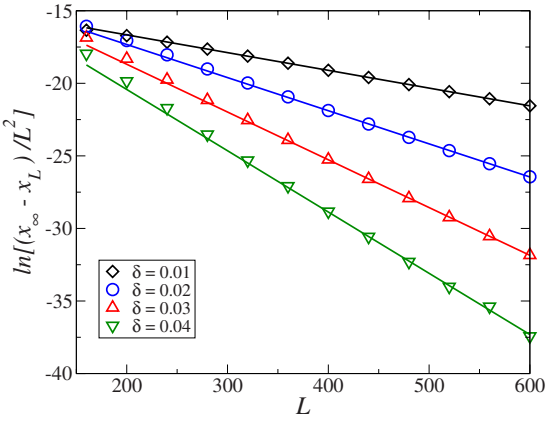

FIG. 1. (Color online) Absolute value of the zero-temperature correlation $x_L=|\langle c_1^\dagger c_L\rangle|$ as a function of the size L of the chain according to the scaling $x_L=x_\infty-A_0L^2\exp(-L/\zeta)$. The different curves reproduce $\ln[(x_\infty-x_L)/L^2]$ as a function of L for different values of the dimerization parameter δ. Symbols denote the exact numerical values. The values of ζ are obtained from Eq. (19).

reached already for chains with a size of few tens of sites.

Finally, by substituting in Eq. (10) and in Eq. (11) the expression of x, Eq. (22), we obtain the analytic expressions of the LDE (end-to-end concurrence) and of the teleportation fidelity as functions of the dimerization ratio $a=(1-\delta)/(1+\delta)$ in the asymptotic regime of chains of large size $L\gg\zeta_0$. One has

$$C_{1,L}=2\ \max\left\{0,\frac{1}{2}-a^2+\frac{a^4}{4}\right\}, \tag{23}$$

$$f=\frac{2}{3}\left[1+\left(\frac{1}{2}-a^2+\frac{a^4}{4}\right)\right]. \tag{24}$$

In Fig. 2 we report the behavior of the LDE between the end points of the chain as a function of the dimerization parameter δ for large chains. We see that as soon as the dimerization is above the threshold value δ_0, the LDE grows very rapidly with δ, reaching saturation in the limiting situation $\delta\to1$.

Before ending this section we should remark that, although extremely interesting, the true LDE picture, derived at zero temperature, does not survive, even qualitatively, at finite temperature. We have seen that the presence of LDE in the dimerized XX is connected to the presence of an eigenstate of the adjacency matrix with complex quasimomentum. From Eq. (20) it follows that the energy gap between the ground state and the lowest excited state vanishes exponentially as the size of the chain increases. Hence, even for small chains and very low temperatures, the first excited levels get significantly populated, contributing to the correlation x with values of opposite sign with respect to that of the ground state. The total effect is to lower x down to zero even at very low temperatures. In conclusion, we have determined that LDE can be supported at zero temperature by relatively simple, dimerized XX chains with open ends. However, in close analogy with more complex systems analyzed previously [10], the zero temperature LDE does not survive as soon as the temperature is switched on.

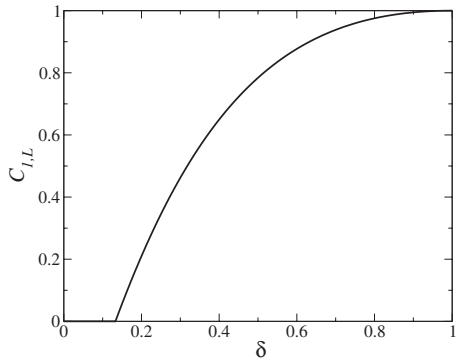

FIG. 2. End-to-end concurrence $C_{1,L}$, for long chains of length $L\gg\zeta_0$ in the dimer model (12), as a function of the dimerization parameter δ. Nonvanishing LDE is achieved and is rapidly growing as soon as $\delta>\delta_0=0.132$.

IV. QUASI-LONG-DISTANCE ENTANGLEMENT

In this section we discuss a second type of model that can support highly entangled reduced states at the end points of a spin chain. In all generality, spin models that can allow for strong end-to-end correlations are characterized by interactions between the end points and their nearest neighbors that are smaller compared to the interactions in the bulk of the chain. Otherwise, if the system does not meet this criterion, the end points would become strongly entangled with their neighbors, excluding, due to monogamy constraints [8], the possibility of LDE. We then consider a model of open XX spin chain formed by $L-2$ spins with uniform coupling strengths, plus two weakly interacting probes placed at the two end points. Such a models is described by the following Hamiltonian:

$$H=J\sum_{i=2}^{L-2}(S_i^xS_{i+1}^x+S_i^yS_{i+1}^y)+\lambda(S_1^xS_2^x+S_1^yS_2^y+S_{L-1}^xS_L^x+S_{L-1}^yS_L^y), \tag{25}$$

where $0<\lambda<1$. The presence of the λ term in Eq. (25) can be understood as a kind of generalized boundary condition, reducing to the standard XX model with uniform couplings and open ends for $\lambda=0,1$. It is then not surprising that the eigenvalues of the corresponding adjacency matrix M have the form $\Lambda_k=J\cos(k)$. The eigenvalue equation determining the quasimomenta for a generic λ reads [20]

$$\mu_k\cot(k)\left[\cot\left(\frac{L-1}{2}k\right)\right]^{\mu_k}=\frac{\lambda^2}{2-\lambda^2}, \tag{26}$$

where $\mu_k=\pm1$ is again the parity of the corresponding eigenstate. The eigenstate ξ_k associated to the quasimomentum k has the following components:

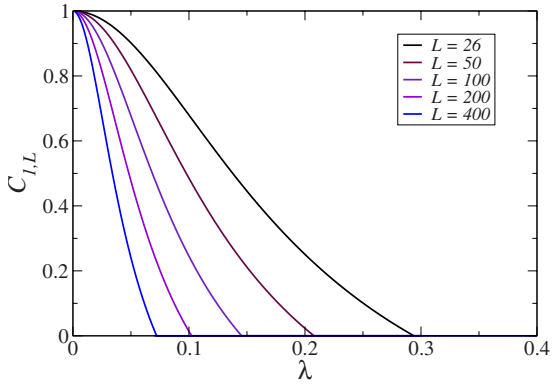

FIG. 3. (Color online) End-to-end concurrence $C_{1,L}$ at zero temperature as a function of λ for different lengths L of the chain. Curves from right to left are, respectively, for $L=26$, $L=50$, $L=100$, $L=200$, and $L=400$.

$$\xi_k^{(1)} = \frac{\lambda}{A_k}\sin(k),$$

$$\xi_k^{(i)} = \frac{1}{A_k}\{\sin[(i+1)k] + (1-\lambda^2)\sin[(i-1)k]\}, \quad 1 < i < L,$$

$$\xi_k^{(L)} = \mu\frac{\lambda}{A_k}\sin(k), \tag{27}$$

where the normalization A_k reads

$$A_k^2 = (L-1)\left(2(1-\lambda^2)\cos^2(k) + \frac{\lambda^4}{2}\right) + 2\lambda^2 - \lambda^4. \tag{28}$$

Equation (26) admits L distinct real solutions in the interval $(0,\pi)$ for $\lambda \neq 0$. Using Eq. (27) the zero temperature end-to-end correlation x reads

$$x = \sum_{\pi/2<k<\pi} \mu_k \frac{\lambda^2}{A_k^2}\sin^2 k. \tag{29}$$

The sum is limited to the $L/2$ quasimomenta that lie in the interval $\pi/2 < k < \pi$. This condition takes into account the fact that, at exactly zero temperature, all levels below the Fermi energy are occupied, while all the others are empty. The value of the end-to-end concurrence $C_{1,L}$ is readily obtained by inserting expression (29) for the end-to-end correlation x in Eq. (10). In Fig. 3 we show the behavior of the end-to-end entanglement as a function of λ, for different lengths of the chain. At fixed length L of the chain, $C_{1,L} \to 1$ as $\lambda \to 0$. At a fixed value of λ, the end-to-end concurrence slowly dies off as $L \to \infty$. The XX spin chain with small end bonds is thus characterized by what we may name "quasi-long-distance entanglement (QLDE) because, at variance with the dimerized case considered in the previous section, the end-to-end entanglement, although able to reach asymptotically maximal values in the limit of vanishing end couplings, slowly decreases as the size of the chain is increased. These considerations can be made quantitative by investigating in detail the exact expressions for x that are obtained from Eq. (29) in the two limiting cases $\lambda \to 0, 1$.

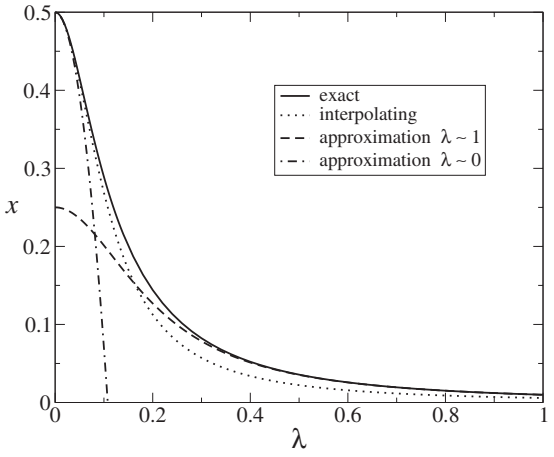

FIG. 4. Comparison between the exact numerical evaluation and the analytic approximations for the end-to-end correlation function $x=\langle c_1^+ c_L\rangle$ for a chain of $L=100$ sites. The interpolating curve is given by $x_{\rm int}=(-1)^{L/2}/(2+4cL\lambda^2)$. It is exact up to $O(\lambda^2)$ when $\lambda \to 0$, but retains its validity also for higher values of the coupling.

In the limit of vanishing small end bonds ($\lambda \to 0$), the quasimomenta assume the expressions

$$k_n \simeq \frac{n\pi}{L-1} - \lambda^2\frac{\tan[n\pi/(L-1)]}{L-1}, \quad n = 1,2,\ldots,L-2,$$

$$k_\pm \simeq \frac{\pi}{2} \pm \frac{\lambda^2}{2}. \tag{30}$$

Isolating the dominating contributions in the sum (29), and keeping terms up to second order in λ, in the limiting case $\lambda \to 0$, one obtains

$$x_{\lambda\to 0} = (-1)^{L/2}\left[\frac{1}{2} - \lambda^2 L\overbrace{\left(\frac{1}{4} + \frac{2\mathcal{C}}{\pi^2}\right)}^{c} + O(\lambda^4)\right], \tag{31}$$

where $\mathcal{C}$ is Catalan's constant $\mathcal{C}=0.915$, and the constant in brackets is $c=0.435$. The above result provides the quantitative content of the qualitative picture sketched above: At fixed size L of the chain, one can always choose λ small enough so that $|x| \to 1/2$ and the end-to-end concurrence approaches unity. Depending on the length of the chain, the condition to be satisfied to achieve large values of the end-to-end concurrence is $\lambda \ll 1/\sqrt{L}$.

For completeness, let us now consider the opposite situation of strong end bonds $\lambda \to 1$. In this case, all the L quasimomenta can be approximated by $k_n \simeq \pi n/(L+1)$ with $n = 1,\ldots,L$. Exploiting the fact that, at leading order, the alternating Riemann sum of a function is equal to (one-half of) the value of the function at the extrema, we obtain

$$x_{\lambda\to 1} \simeq (-1)^{L/2}\frac{1}{\lambda^2(L-3)+4}. \tag{32}$$

In Fig. 4 we compare the exact expression of x as a func-

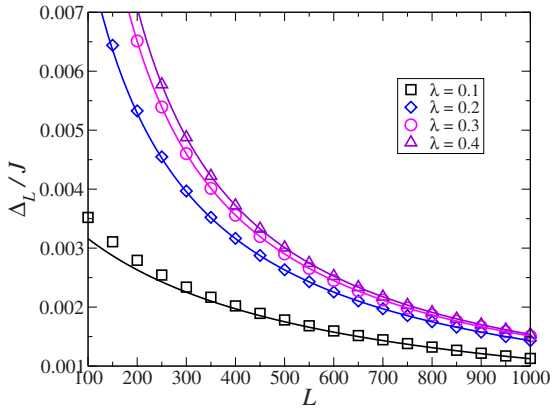

FIG. 5. (Color online) Behavior of the energy gap Δ_L as a function of the length L of the chain for different values of λ. Diamonds denote the exact numerical data. Continuous curves display the approximate expression for the gap [Eq. (33) in the text].

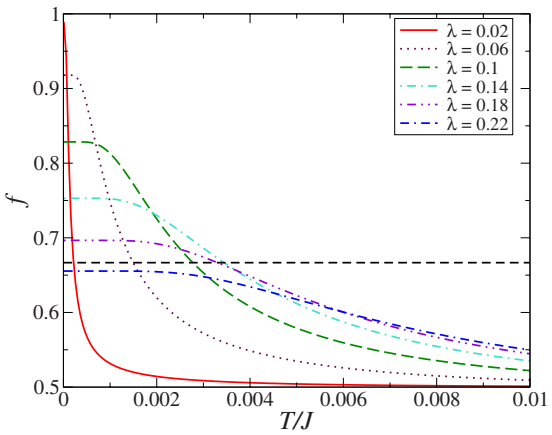

FIG. 6. (Color online) Teleportation fidelity for different values of λ as a function of the rescaled temperature T/J for an XX chain with small end bonds, supporting QLDE. The six different curves refer to increasing values of λ from 0.02 to 0.22 in steps of 0.04. From top to bottom, left to right: solid line (λ=0.02); dotted line (λ=0.06); dashed line (λ=0.1); dot-dashed line (λ=0.14); double-dot-dashed line (λ=0.18); double-dash-dotted line (λ=0.22). The length of the chain is fixed at $L=50$ sites. The horizontal dashed line $f=2/3$ is the maximum attainable fidelity using a classical teleportation channel (classical threshold).

tion of λ, obtained by direct numerical diagonalization, with the two analytic limiting expressions and an interpolating ansatz for a chain of $L=100$ sites. We will now show that the phenomenon of QLDE is intimately related to an algebraic scaling behavior of the energy gap that is radically different from the one, exponentially decreasing with the size of the chain, exhibited by the fully dimerized XX chain. This feature turns out to play a crucial role at finite temperature, allowing QLDE to be more resilient to thermal fluctuations not only at very low but even at moderately low temperatures. As already pointed out, the single-particle dispersion law has the form $\Lambda_k=J\cos k$. The lowest gap is then given by $J\cos(k^*)$ where k^* is the solution of Eq. (26) closest to $\pi/2$ with $k^*<\pi/2$. For sufficiently large values of L, the left-hand side of Eq. (26) has a vertical asymptote arbitrarily close to $\pi/2$. Expanding the function around this point we obtain for the desired solution

$$k^* \simeq \frac{\pi}{2} - \frac{\pi}{2(L-1)} + \frac{\pi}{(L-1)^2\lambda^2(2-\lambda^2)^{-1} + 2(L-1)}.$$

Then, at leading order in inverse powers of the length of the chain L, the lowest energy gap reads

$$\frac{\Delta_L}{J} \simeq \frac{\pi}{2(L-1)} - \frac{\pi}{(L-1)^2[\lambda^2/(2-\lambda^2)] + 2(L-1)}$$
$$= \frac{\pi}{2}L^{-1} - \left(\frac{\pi}{2} - \frac{\pi(2-\lambda^2)}{\lambda^2}\right)L^{-2} + O(L^{-3}). \qquad (33)$$

We have compared the asymptotic analytic behaviors for different λ and chains of length up to $L=1000$ sites with the exact numerical solutions. The results of this comparison are plotted in Fig. 5, showing excellent agreement of the exact data with the algebraic scaling (33).

The existence of a lowest energy gap slowly falling off algebraically with the size of the system is to be compared with the exponentially fast vanishing of the same quantity in the case of the dimerized XX chain. Looking at the structure of the quasimomenta in the two cases, we immediately realize that the reason for this very different behavior lies in the absence of complex quasimomenta in the spectrum of the XX model with small end bonds. On the one hand, this is an undesired feature, because it is exactly the presence of a complex quasimomentum that allows for true LDE in the dimerized XX model. On the other hand, this very same feature allows for true LDE and at the same time is responsible for the exponentially fast vanishing of the energy gap.

Moving to finite temperature, the exact numerical evaluation of the fermionic end-to-end correlation function x is obtained using Eq. (9). After having determined the value of x, we can again use Eqs. (10) and (11) to evaluate both the end-to-end concurrence and the teleportation fidelity. In Fig. 6 we report the behavior of the teleportation fidelity f for different values of λ as a function of the normalized temperature T/J for a chain of $L=50$ spins. We see that the higher the limiting value of the fidelity at vanishing temperature, the faster f falls off below the classical threshold $f=2/3$. This behavior is obviously due to the fact that a larger zero-temperature f corresponds to a smaller energy gap, and hence the system is more sensitive to the disruptive effect of thermal fluctuations.

We can compare the results reported in Fig. 6 for an XX chain with small end bonds with the ones obtained for a Heisenberg antiferromagnetic chain with two weakly interacting end probes [11]. The teleportation efficiency at finite temperature for both models has been investigated at the same fixed length of the chain $L=50$. From this comparison, one sees that the behavior of the teleportation fidelity as a function of the temperature in the two models is qualitatively similar. This fact is of relevance, because it shows that the QLDE due to the presence of weakly interacting probes at the end of a uniformly interacting chain is an effect not restricted to a particular Hamiltonian model, but can be supported by systems endowed with different bulk interactions

and different symmetries. The only important ingredient for the realization of a QLDE and a teleportation channel that are robust against thermal fluctuations is the availability of efficient control on the interactions at the end points of the channel. Similar conclusions can be drawn about LDE: In this work we have demonstrated that existence of true LDE at zero temperature is a phenomenon not restricted to a particular model and, in fact, does not crucially depend on the mechanism of dimerization. Preliminary studies indicate that gapped, anisotropic models in noncritical regimes can sustain LDE even in the absence of a clear pattern of dimerization.

Finally, it is worth noticing that even if the Heisenberg model with small end bonds analyzed in Ref. [11] sustains a teleportation fidelity that remains above the classical threshold for larger values of the end coupling parameter λ, the *XX* channel with small end bonds assures teleportation with nonclassical fidelities for larger values of the temperature. Depending on the realistic experimental settings, if the temperature can be controlled down to sufficiently low values, it is more convenient to engineer a channel based on the Heisenberg-like Hamiltonian proposed in Ref. [11]. Otherwise, if control on the external temperature cannot be implemented with high precision down to sufficiently low temperatures, it may then turn out more convenient to engineer *XX* chains of the type discussed in the present paper.

V. CONCLUSIONS AND OUTLOOK

In conclusion, we have studied models of open quantum spin chains endowed with *XX*-like Hamiltonians with nearest neighbor interactions. We have discussed two types of models, according to different engineerings of the spin-spin couplings. In the first case considered, a chain with bonds of alternating strengths, we have shown how the exact solution for the ground state implies the existence of long-distance entanglement between the end spins of the chain, independent of the size of the system and asymptotically close to unity (maximal entanglement) in the limit of exact dimerization. This system is therefore perfectly suited for *bona fide* long-distance quantum teleportation with ideal fidelity at zero temperature. However, the limiting maximal values of the fidelity are obtained at the cost of an exponentially small energy gap above the ground state. Therefore this system is *de facto* useless for efficient quantum teleportation at finite temperature. We have then moved on to discuss another class of *XX* open spin chains with uniform bulk interactions and small end bonds. In this case, we have shown that for sufficiently small values of the end couplings, the ground state of the system supports a quasi-long-distance entanglement between the end spins of the chain, asymptotically close to unity (maximal entanglement) in the limit of vanishing coupling, but slowly decreasing as the length of the chain is increased. An interesting feature of this model is that the lowest energy gap above the ground state vanishes only algebraically, as the first power of the inverse of the size of the system. Therefore in principle, it can be exploited as a quantum channel for teleportation with nonclassical fidelity at finite temperature, both very low and moderately low.

A further comment is in order concerning the behavior of these systems in the presence of disorder. This issue is important especially in view of possible experimental implementations in which the strength of the couplings can be engineered only within a certain accuracy. Since naturally the effect of disorder is that of localizing eigenstates, we expect LDE to be more robust than QLDE against disorder. In fact, in the LDE scenario the state responsible for the end-to-end entanglement is already localized at the borders whereas localization is only approximate in the QLDE case. Therefore in some instances, static imperfections may even increase the amount of LDE present in the system. On average, one expects that the presence of disorder does not greatly modify the general picture, both in the LDE and in the QLDE cases, at least as long as one deals with disorder of weak or medium strength. This conjecture is confirmed by preliminary numerical simulations. It is less obvious how the general scenario changes at finite temperature, in the presence of disorder. This problem deserves on its own a careful and detailed study that goes much beyond the scope of the present work.

Considering future research along these lines of investigation, it will be interesting to consider practical schemes for the realization of this kind of spin systems in highly controllable situations, for instance resorting to ultracold atomic mixtures in optical lattices. Another interesting open problem worth further study is to assess the existence and the possible location of a crossover between true long-distance and *prima facie* quasi-long-distance entanglement behaviors.

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