Classification of four-qubit states by means of a stochastic local operation and the classical communication invariant and semi-invariants

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We show there are at least 28 distinct true stochastic local operations and classical communication (SLOCC) entanglement classes for four qubits by means of SLOCC invariant and semi-invariants and derive the number of degenerate SLOCC classes for n qubits.

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I. INTRODUCTION

Entanglement plays a key role in quantum computing and quantum information. If two states can be obtained from each other by means of local operations and classical communication (LOCC) with nonzero probability, we say that two states have the same kind of entanglement [1]. It is well known that a pure entangled state of two qubits can be locally transformed into the Greenberger-Horne-Zeilinger (GHZ) state. Recently, many authors have investigated the equivalence classes of three-qubit states, using specified SLOCC (stochastic local operations and classical communication). Dür *et al.* showed that for pure states of three qubits there are four different degenerate SLOCC entanglement classes and two inequivalent true entanglement classes [2]. Miyake discussed the onionlike classification of SLOCC orbits and proposed the SLOCC equivalence classes using the orbits [3,4]. In [5] we gave the simple criteria for the complete SLOCC classification for three qubits and criteria for a few classes for four qubits.

Verstraete *et al.* [6] considered the entanglement classes of four qubits under SLOCC and concluded that there exist nine families of states corresponding to nine different ways of entanglement and claimed that by determinant-one SLOCC operations, a pure state of four qubits can be transformed into one of the nine families of states. Clearly, this does not say that each family is a SLOCC class. For example, Verstraete *et al.* indicated that a state consisting of two Einstein-Podolsky-Rosen pairs and the four-qubit $|\phi_4\rangle$ state [7] belong to the family G_{abcd}. Then, how many SLOCC classes by the definition in [2] are there for each family? What are the representations? After investigating the nine families by means of methods in this paper, we can say that each of the first six families includes several SLOCC classes. We list the SLOCC classes of some of the nine families as follows. For family

$$L_{ab_3}:\quad a(|0000\rangle+|1111\rangle)+\frac{a+b}{2}(|0101\rangle+|1010\rangle)+\frac{a-b}{2}(|0110\rangle+|1001\rangle)+\frac{i}{\sqrt{2}}(|0001\rangle+|0010\rangle+|0111\rangle+|1011\rangle),$$

this consists of five true SLOCC entanglement classes, which are $a=b=0$, i.e., class $|W\rangle$, $a=b\neq 0$, $a=-b\neq 0$, $a\neq \pm b \wedge 3a^2+b^2\neq 0$, and $a\neq \pm b\wedge 3a^2+b^2=0$, respectively. For family

$$L_{a_4}:\quad a(|0000\rangle+|0101\rangle+|1010\rangle+|1111\rangle)+(i|0001\rangle+|0110\rangle-i|1011\rangle),$$

this includes two true SLOCC entanglement classes: $a=0$ and $a\neq 0$. For family

$$L_{a_2 0_{3\oplus 1}}:\quad a(|0000\rangle+|1111\rangle)+|0011\rangle+|0101\rangle+|0110\rangle,$$

it includes two SLOCC classes. When $a=0$, this becomes a product state of a qubit state and three-qubit $|W\rangle$. When $a\neq 0$, this is a true entangled state.

In [8,9], the authors used the partition to investigate SLOCC classification of three qubits and four qubits. The idea for the partition was originally used to analyze the separability of n qubits and multipartite pure states in [10]. In [9], the authors declared that they found 16 true SLOCC classes of four qubits, where permutation is explicitly included in the counting. Taking into account the permutation among the qubits, there are eight true classes [9]. By means of the methods in this paper, we can illustrate how many true SLOCC entanglement classes there are for each Span$\{\ldots\}$. For example, for Span$\{O_k\Psi, O_k\Psi\}$, the canonical states are

$$|0000\rangle+|1100\rangle+a|0011\rangle+b|1111\rangle$$

and

$$|0000\rangle+|1100\rangle+a|0001\rangle+a|0010\rangle+b|1101\rangle+b|1110\rangle,$$

where $a\neq b$ [9]. It was not pointed out in [9] what relation a and b must satisfy to be a representation of an SLOCC class. It can be shown that for the former canonical state, $a=-b$ and $a\neq -b$ represent two true SLOCC classes, while for the

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latter canonical state, $ab=0$ and $ab\neq 0$ represent two true SLOCC classes. We can also explain that each of Span{000,GHZ}, Span$\{0_i\Psi, 0_j\Psi\}$, and Span$\{\text{GHZ}, W\}$ includes four true SLOCC entanglement classes and Span$\{0_k\Psi,\text{GHZ}\}$ includes more such classes. Also considering Span{000,000}, Span$\{000, 0_k\Psi\}$, and Span$\{000, W\}$, in total, the eight Spans{...} in [9] include many more true SLOCC entanglement classes.

In this paper, we find the SLOCC invariant and semi-invariants for four qubits. For the definitions of invariant and semi-invariant, see Sec. II. Using the invariant and semi-invariants, we can determine if two states belong to different SLOCC entanglement classes. We distinguish 28 distinct true entanglement classes, where permutations of the qubits are allowed. This classification is not complete. It seems that there are more true entanglement classes. The invariant and semi-invariants only require simple arithmetic operations.

II. SLOCC INVARIANT AND SEMI-INVARIANTS

We discuss a system comprised of four qubits A, B, C, and D. The states of a four-qubit system can be generally expressed as

$$|\psi\rangle = \sum_{i=0}^{15} a_i|i\rangle. \tag{1}$$

Two states $|\psi\rangle$ and $|\psi'\rangle$ are equivalent under SLOCC if and only if there exist invertible local operators α, β, γ, and δ such that

$$|\psi\rangle = \alpha\otimes\beta\otimes\gamma\otimes\delta|\psi'\rangle, \tag{2}$$

where the local operators α, β, γ, and δ can be expressed as 2×2 invertible matrices

$$\alpha=\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix},\quad \beta=\begin{pmatrix}\beta_1 & \beta_2\\ \beta_3 & \beta_4\end{pmatrix},\quad \gamma=\begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_3 & \gamma_4\end{pmatrix},$$

$$\delta=\begin{pmatrix}\delta_1 & \delta_2\\ \delta_3 & \delta_4\end{pmatrix}. \tag{3}$$

A. SLOCC invariant

Let $|\psi'\rangle=\sum_{i=0}^{15} b_i|i\rangle$ in Eq. (2). If $|\psi\rangle$ is SLOCC equivalent to $|\psi'\rangle$, then the following equation holds:

$$\mathcal{I}(\psi)=\mathcal{I}(\psi')\det(\alpha)\det(\beta)\det(\gamma)\det(\delta), \tag{4}$$

where

$$\mathcal{I}(\psi)=(a_2a_{13}-a_3a_{12})+(a_4a_{11}-a_5a_{10})-(a_0a_{15}-a_1a_{14}) -(a_6a_9-a_7a_8), \tag{5}$$

and

$$\mathcal{I}(\psi')=(b_2b_{13}-b_3b_{12})+(b_4b_{11}-b_5b_{10})-(b_0b_{15}-b_1b_{14}) -(b_6b_9-b_7b_8). \tag{6}$$

Equation (4) was derived by induction in [11]. We can also verify Eq. (4) as follows. By solving the matrix equation in Eq. (2), we obtain the amplitudes a_i of state $|\psi\rangle$ in Eq. (2). Then substituting a_i into $\mathcal{I}(\psi)$, we have Eq. (4). Notice that $\mathcal{I}(\psi)$ does not vary under determinant-one SLOCC operations (SL operations) or vanish under non-unit-determinant SLOCC operations.

If ψ is SL equivalent to ψ', then $\mathcal{I}(\psi)=\mathcal{I}(\psi')$. Equation (4) implies that each SLOCC class has infinite SL classes. This is also true for n qubits [11]. For the family $L_{a_20_{3\oplus 1}}$ in [6], let $|\psi'\rangle$ be a representative state: $a(|0000\rangle+|1111\rangle)+|0011\rangle+|0101\rangle+|0110\rangle$. Equation (4) becomes $\mathcal{I}(\psi)=-a^2\det(\alpha)\det(\beta)\det(\gamma)\det(\delta)$. For SL operations, $\mathcal{I}(\psi)=\mathcal{I}(\psi')=-a^2$. It is clear that different values of a yield different SL classes. Therefore there are an infinite number of SL classes when $a\neq 0$. However, the infinite number of SL classes all belong to a single true SLOCC entanglement class.

B. Semi-invariants F_i

Coffman defined the concurrence of three qubits [12]. We extend the definition of the concurrence of three qubits to four qubits as follows. For state $|\psi\rangle$, we define $F_i(\psi)$ as follows. Notice that $F_3(\psi)$ to $F_8(\psi)$ can be obtained from $F_1(\psi)$ and $F_2(\psi)$ by permutations of the qubits.

$$F(\psi)=4\sum_{i=1}^{10}|F_i(\psi)|,$$

where

$$F_1(\psi)=(a_0a_7-a_2a_5+a_1a_6-a_3a_4)^2 -4(a_2a_4-a_0a_6)(a_3a_5-a_1a_7),$$

$$F_2(\psi)=(a_8a_{15}-a_{11}a_{12}+a_9a_{14}-a_{10}a_{13})^2 -4(a_{11}a_{13}-a_9a_{15})(a_{10}a_{12}-a_8a_{14}),$$

$$F_3(\psi)=(a_0a_{11}-a_2a_9+a_1a_{10}-a_3a_8)^2 -4(a_2a_8-a_0a_{10})(a_3a_9-a_1a_{11}),$$

$$F_4(\psi)=(a_4a_{15}-a_6a_{13}+a_5a_{14}-a_7a_{12})^2 -4(a_6a_{12}-a_4a_{14})(a_7a_{13}-a_5a_{15}),$$

$$F_5(\psi)=(a_0a_{13}-a_4a_9+a_1a_{12}-a_5a_8)^2 -4(a_4a_8-a_0a_{12})(a_5a_9-a_1a_{13}),$$

$$F_6(\psi)=(a_2a_{15}-a_6a_{11}+a_3a_{14}-a_7a_{10})^2 -4(a_6a_{10}-a_2a_{14})(a_7a_{11}-a_3a_{15}),$$

$$F_7(\psi)=(a_0a_{14}-a_4a_{10}+a_2a_{12}-a_6a_8)^2 -4(a_4a_8-a_0a_{12})(a_6a_{10}-a_2a_{14}),$$

$$F_8(\psi)=(a_1a_{15}-a_5a_{11}+a_3a_{13}-a_7a_9)^2 -4(a_5a_9-a_1a_{13})(a_7a_{11}-a_3a_{15}),$$

TABLE I. The properties of F_i for true entanglement classes.

Classes	F_i	$F_i=0, i=$
	(a)	
$\|\mathrm{GHZ}\rangle$	[a]	
$\|C_4\rangle$	[b]	
$\|\kappa_4\rangle, \|E_4\rangle, \|L_4\rangle, \|H_4\rangle, \|\lambda_4\rangle, \|M_4\rangle$	[a]	
$\|\pi_4\rangle$	$\|F_1\|+\|F_2\| \neq 0, \|F_5\|+\|F_6\| \neq 0,$[c]	3,4,7,8
$\|\theta_4\rangle$	$\|F_3\|+\|F_4\| \neq 0, \|F_7\|+\|F_8\| \neq 0,$[d]	1,2,5,6
$\|\sigma_4\rangle$	$\|F_1\|+\|F_2\| \neq 0, \|F_3\|+\|F_4\| \neq 0,$[e]	5,6,7,8
$\|\rho_4\rangle$	$\|F_5\|+\|F_6\| \neq 0, \|F_7\|+\|F_8\| \neq 0$	1,2,3,4,9,10
$\|\xi_4\rangle$	$\|F_1\|+\|F_2\| \neq 0, \|F_7\|+\|F_8\| \neq 0,$[c]	3,4,5,6
$\|\epsilon_4\rangle$	$\|F_3\|+\|F_4\| \neq 0, \|F_5\|+\|F_6\| \neq 0,$[d]	1,2,7,8
	(b)	
$\|W\rangle$		all i
$\|\chi_4\rangle$	$\|F_5\|+\|F_6\| \neq 0, \|F_7\|+\|F_8\| \neq 0,$[a]	
$\|\upsilon_4\rangle$	$\|F_1\|+\|F_2\| \neq 0, \|F_3\|+\|F_4\| \neq 0, \|F_7\|+\|F_8\| \neq 0,$[e]	$i=5,6$
$\|\varpi_4\rangle$	$\|F_1\|+\|F_2\| \neq 0, \|F_5\|+\|F_6\| \neq 0, \|F_7\|+\|F_8\| \neq 0, F_9 = F_{10}, F_1F_2=(F_9)^2$	3,4
$\|\psi_4\rangle$	$F_9=F_{10},$[f]	
$\|\phi_4\rangle$	[a]	
$\|\mu_4\rangle$	$F_9=F_{10},$[f]	
$\|\varphi_4\rangle$	$\|F_1\|+\|F_2\| \neq 0, \|F_5\|+\|F_6\| \neq 0, F_9=F_{10}, F_1F_2=(F_9)^2$	3,4,7,8
$\|\zeta_4\rangle$	$\|F_1\|+\|F_2\| \neq 0, \|F_3\|+\|F_4\| \neq 0,$[e]	5,6,7,8
$\|\vartheta_4\rangle$	$\|F_1\|+\|F_2\| \neq 0, \|F_7\|+\|F_8\| \neq 0, F_9=F_{10}, F_1F_2=(F_9)^2$	3,4,5,6
$\|\tau_4\rangle$	$\|F_3\|+\|F_4\| \neq 0, \|F_5\|+\|F_6\| \neq 0, F_9=F_{10}, F_3F_4=(F_9)^2$	1,2,7,8
$\|\varrho_4\rangle$	$\|F_3\|+\|F_4\| \neq 0, \|F_7\|+\|F_8\| \neq 0, F_9=F_{10}, F_3F_4=(F_9)^2$	1,2,5,6
$\|\iota_4\rangle$	$\|F_5\|+\|F_6\| \neq 0, \|F_7\|+\|F_8\| \neq 0,$	1,2,3,4,9,10
$\|\omega_4\rangle$	$\|F_3\|+\|F_4\| \neq 0, \|F_5\|+\|F_6\| \neq 0, \|F_7\|+\|F_8\| \neq 0, F_9 = F_{10}, F_3F_4=(F_9)^2$	$i=1,2$

[a]If $F_1F_2=0$ and $F_3F_4=0$, then $F_9=0$ and $F_{10}\neq 0$ or $F_9\neq 0$ and $F_{10}=0$.
[b]If $F_i=F_j=F_k=0$, where $1\leq i<j<k\leq 4$, then $|F_9|+|F_{10}|\neq 0$ and $F_9F_{10}=0$.
[c]If $F_1F_2=0$, then $F_9=F_{10}\neq 0$.
[d]If $F_3F_4=0$, then $F_9=F_{10}\neq 0$.
[e]If$F_1F_2=0$ and $F_3F_4=0$, then $F_9=F_{10}=0$.
[f]If $F_i=F_j=F_k=0$, where $1\leq i<j<k\leq 4$, then $F_9\neq 0$.

$$F_9(\psi) = (a_0a_{15} - a_2a_{13} + a_1a_{14} - a_3a_{12})^2 - 4(a_0a_{14} - a_2a_{12})(a_1a_{15} - a_3a_{13}),$$

$$F_{10}(\psi) = (a_4a_{11} - a_7a_8 + a_5a_{10} - a_6a_9)^2 - 4(a_7a_9 - a_5a_{11})(a_6a_8 - a_4a_{10}).$$

For state $|\psi'\rangle$, let $F_i(\psi')$ be obtained from $F_i(\psi)$ by replacing a in $F_i(\psi)$ by b. Then by induction we can show that F_i have the following interesting properties and the properties are called semi-invariants.

In Eq. (2), let $\alpha=I$, where I is an identity. Thus, Eq. (2) becomes

$$|\psi\rangle = I \otimes \beta \otimes \gamma \otimes \delta|\psi'\rangle. \tag{7}$$

Then we have the following:

$$F_i(\psi) = F_i(\psi')\det^2(\beta)\det^2(\gamma)\det^2(\delta), \quad i = 1,2. \tag{8}$$

Equation (8) can be verified as follows. We obtain the amplitudes a_i of state $|\psi\rangle$ by solving Eq. (7). Then substituting a_i into $F_i(\psi)$, we derive Eq. (8). Also, in Eq. (2), let $\beta=I$, then $F_i(\psi)=F_i(\psi')\det^2(\alpha)\det^2(\gamma)\det^2(\delta)$, $i=3,4$. In Eq. (2), let $\gamma=I$, then $F_i(\psi)=F_i(\psi')\det^2(\alpha)\det^2(\beta)\det^2(\delta)$, $i=5,6$. In Eq. (2), let $\delta=I$, then $F_i(\psi)=F_i(\psi')\det^2(\alpha)\det^2(\beta)\det^2(\gamma)$, $i=7,8$. In Eq. (2), let $\alpha=I$ and $\beta=I$, i.e., $|\psi\rangle=I\otimes I\otimes\gamma\otimes\delta|\psi'\rangle$, then $F_i(\psi)=F_i(\psi')\det^2(\gamma)\det^2(\delta)$, $i=9,10$.

Next let $|\psi\rangle$ be SLOCC equivalent to $|\psi'\rangle$ in Eq. (2). By solving the matrix equation in Eq. (2), we obtain the amplitudes a_i of state $|\psi\rangle$ in Eq. (2). Then we can calculate F_i of state $|\psi\rangle$, i.e., the F_i of class $|\psi'\rangle$. We compute the F_i of all the degenerate entanglement classes and 28 true entangle-

TABLE II. The properties of F_i for the degenerate entanglement classes.

Classes	$F_i=0$	
$\vert\text{GHZ}\rangle_{123}\otimes(s\vert 0\rangle+t\vert 1\rangle)_4$	$i\neq 7,8$	$\vert F_7\vert+\vert F_8\vert\neq 0$
$\vert\text{GHZ}\rangle_{124}\otimes(s\vert 0\rangle+t\vert 1\rangle)_3$	$i\neq 5,6$	$\vert F_5\vert+\vert F_6\vert\neq 0$
$\vert\text{GHZ}\rangle_{134}\otimes(s\vert 0\rangle+t\vert 1\rangle)_2$	$i\neq 3,4,9,10$	$\vert F_3\vert+\vert F_4\vert\neq 0, F_9=F_{10}, F_3F_4=(F_9)^2$
$(s\vert 0\rangle+t\vert 1\rangle)_1\otimes\vert\text{GHZ}\rangle_{234}$	$i\neq 1,2,9,10$	$\vert F_1\vert+\vert F_2\vert\neq 0, F_9=F_{10}, F_1F_2=(F_9)^2$
$\vert W\rangle\otimes(s\vert 0\rangle+t\vert 1\rangle)_i$	All $F_i=0$	
$\vert\text{GHZ}\rangle_{12}\otimes\vert\text{GHZ}\rangle_{34}$	All $F_i=0$	
$\vert\text{GHZ}\rangle_{13}\otimes\vert\text{GHZ}\rangle_{24}$	$i\neq 9,10$	$F_9=F_{10}\neq 0$
$\vert\text{GHZ}\rangle_{14}\otimes\vert\text{GHZ}\rangle_{23}$	$i\neq 9,10$	$F_9=F_{10}\neq 0$
Only two qubits are entangled	All $F_i=0$	
separate states	All $F_i=0$	

ment classes. See Tables I(a) I(b), and II and [13]. If the F_i do not vanish for some classes, then we give the expressions for the F_i in [13]. We list the properties of the F_i of the 28 true entanglement classes in Tables I(a) and I(b) and of the F_i of all the degenerate entanglement classes in Table II. For the derivations of the properties of the F_i, see [13]. We also compute all the F_i of the 28 true entanglement states [see Table III(a) and III(b)].

TABLE III. The properties of D_i and F_i for the true entanglement states.

States	D_1	D_2	D_3	$F_i\neq 0$, when $i=$
		(a)		
$\vert\text{GHZ}\rangle$	0	0	0	9
$\vert C_4\rangle$	$\neq 0$	$\neq 0$	$\neq 0$	9
$\vert\kappa_4\rangle$	0	0	0	9
$\vert E_4\rangle$	0	0	0	9
$\vert L_4\rangle$	0	0	0	9
$\vert H_4\rangle$	0	0	0	9
$\vert\lambda_4\rangle$	0	$\neq 0$	0	10
$\vert M_4\rangle$	0	0	0	9
$\vert\pi_4\rangle$	$\neq 0$	0	0	1,6,9,10
$\vert\theta_4\rangle$	$\neq 0$	0	0	4,7,9,10
$\vert\sigma_4\rangle$	0	$\neq 0$	0	2,3
$\vert\rho_4\rangle$	0	$\neq 0$	0	6,7
$\vert\xi_4\rangle$	0	0	$\neq 0$	2,7,9,10
$\vert\epsilon_4\rangle$	0	0	$\neq 0$	3,6,9,10
		(b)		
$\vert W\rangle$	0	0	0	
$\vert\chi_4\rangle$	0	$\neq 0$	0	6,7,9
$\vert\upsilon_4\rangle$	0	0	0	1,3,8
$\vert\varpi_4\rangle$	0	0	0	1,5,7
$\vert\psi_4\rangle$	$\neq 0$	0	0	3,4,9,10
$\vert\phi_4\rangle$	0	$\neq 0$	0	9
$\vert\mu_4\rangle$	0	0	$\neq 0$	9,10
$\vert\varphi_4\rangle$	0	0	0	1,6
$\vert\zeta_4\rangle$	0	0	0	2,3
$\vert\vartheta_4\rangle$	0	0	0	1,8
$\vert\tau_4\rangle$	0	0	0	3,6
$\vert\varrho_4\rangle$	0	0	0	3,8
$\vert\iota_4\rangle$	0	0	0	5,8
$\vert\omega_4\rangle$	0	0	0	4,5,7

C. Semi-invariants D_1, D_2, and D_3

In [5], we computed the following expressions of D_1, D_2, and D_3 for $|\text{GHZ}\rangle$, $|W\rangle$. Using D_1, D_2, and D_3, we found a true entanglement state $|C_4\rangle$ which is distinct from $|\text{GHZ}\rangle$, $|W\rangle$, and $|\phi_4\rangle$ [7]. We define $D_i(\psi)$ for state ψ as follows:

$$D_1(\psi)=(a_1a_4-a_0a_5)(a_{11}a_{14}-a_{10}a_{15}) - (a_3a_6-a_2a_7)(a_9a_{12}-a_8a_{13}), \tag{9}$$

$$D_2(\psi)=(a_4a_7-a_5a_6)(a_8a_{11}-a_9a_{10}) - (a_0a_3-a_1a_2)(a_{12}a_{15}-a_{13}a_{14}), \tag{10}$$

$$D_3(\psi)=(a_3a_5-a_1a_7)(a_{10}a_{12}-a_8a_{14}) - (a_2a_4-a_0a_6)(a_{11}a_{13}-a_9a_{15}). \tag{11}$$

Then by induction we can demonstrate that the $D_i(\psi)$ have the following interesting properties, which are also called semi-invariants. These properties can also be verified by substituting the amplitudes a_i of state $|\psi\rangle$ in Eq. (2) into $D_i(\psi)$, $i=1,2$, and 3. For state $|\psi'\rangle$, let $D_i(\psi')$ be obtained from $D_i(\psi)$ by replacing a in $D_i(\psi)$ by b. Then, in Eq. (2), let $\alpha=I$ and $\gamma=I$, i.e. $|\psi\rangle=I\otimes\beta\otimes I\otimes\delta|\psi'\rangle$, then $D_1(\psi)=D_1(\psi')\det^2(\beta)\det^2(\delta)$. In Eq. (2), let $\alpha=I$ and $\beta=I$, then $D_2(\psi)=D_2(\psi')\det^2(\gamma)\det^2(\delta)$. In Eq. (2), let $\alpha=I$ and $\delta=I$, then $D_3(\psi)=D_3(\psi')\det^2(\beta)\det^2(\gamma)$.

Next let $|\psi\rangle$ be SLOCC equivalent to $|\psi'\rangle$ in Eq. (2). By solving the matrix equation in Eq. (2), we obtain the amplitudes a_i of state $|\psi\rangle$ in Eq. (2). Then we can calculate D_i of state $|\psi\rangle$, i.e., the D_i of class $|\psi'\rangle$. We compute the values of D_1, D_2, and D_3 of the degenerate SLOCC equivalence classes (see Table IV). We give the values of D_1, D_2, and D_3 of the 28 true entanglement classes in Tables V(a) and V(b) and of the 28 true entanglement states in Tables III(a) and III(b). We also calculate the values of D_1, D_2, and D_3 of states $|\text{GHZ}\rangle_{12}\otimes|\text{GHZ}\rangle_{34}$, $|\text{GHZ}\rangle_{13}\otimes|\text{GHZ}\rangle_{24}$, and $|\text{GHZ}\rangle_{14}\otimes|\text{GHZ}\rangle_{23}$, see Table VI. If $D_i=0$ for some i and for some

TABLE IV. The invariants of the degenerate entanglement classes.

Classes	$\mathcal{I}$	D_1	D_2	D_3	F
$\lvert\mathrm{GHZ}\rangle_{123}\otimes(s\lvert 0\rangle+t\lvert 1\rangle)_4$	0	0	0	0	>0
$\lvert\mathrm{GHZ}\rangle_{124}\otimes(s\lvert 0\rangle+t\lvert 1\rangle)_3$	0	0	0	0	>0
$\lvert\mathrm{GHZ}\rangle_{134}\otimes(s\lvert 0\rangle+t\lvert 1\rangle)_2$	0	0	0	0	>0
$(s\lvert 0\rangle+t\lvert 1\rangle)_1\otimes\lvert\mathrm{GHZ}\rangle_{234}$	0	0	0	0	>0
$\lvert W\rangle\otimes(s\lvert 0\rangle+t\lvert 1\rangle)_i$ [a]	0	0	0	0	0
$\lvert\mathrm{GHZ}\rangle_{12}\otimes\lvert\mathrm{GHZ}\rangle_{34}$	≠0	0	Δ	0	0
$\lvert\mathrm{GHZ}\rangle_{13}\otimes\lvert\mathrm{GHZ}\rangle_{24}$	≠0	Δ	0	0	>0
$\lvert\mathrm{GHZ}\rangle_{14}\otimes\lvert\mathrm{GHZ}\rangle_{23}$	≠0	0	0	Δ	>0
only two qubits are entangled	0	0	0	0	0
separate states	0	0	0	0	0

[a] In $\lvert W\rangle\otimes(s\lvert 0\rangle+t\lvert 1\rangle)_i$, $i=1,2,3,4$.

class in Tables V(a) or V(b) or IV, then it implies that $D_i=0$ for that i and for all the states of the class. If D_i is Δ for some i and for some class in Tables V(a) or V(b) or IV, then it means that $D_i=0$ for that i and for some states of the class while for other states of the class $D_i\neq 0$. If D_i is Δ, then we give the expression for D_i in [13]. For example, for class $\lvert\kappa_4\rangle$ in Table V(a), D_1 is Δ, D_2, is Δ, and $D_3=0$. It says that for some state of class $\lvert\kappa_4\rangle$ in Table V(a), $D_1\neq 0$ and $D_2\neq 0$ but $D_3=0$ for every state of class $\lvert\kappa_4\rangle$. However, for state $\lvert\kappa_4\rangle$ in Table III(a), $D_i=0$, where $i=1$, 2, and 3.

TABLE V. The SLOCC invariants of true entanglement classes.

Classes	F	D_1	D_2	D_3	$\mathcal{I}$
		(a)			
$\lvert\mathrm{GHZ}\rangle$	>0	0	0	0	≠0
$\lvert C_4\rangle$	>0	Δ	Δ	Δ	≠0
$\lvert\kappa_4\rangle$	>0	Δ	Δ	0	≠0
$\lvert E_4\rangle$	>0	Δ	0	Δ	≠0
$\lvert L_4\rangle$	>0	0	Δ	Δ	≠0
$\lvert H_4\rangle$	>0	Δ	0	0	≠0
$\lvert\lambda_4\rangle$	>0	0	Δ	0	≠0
$\lvert M_4\rangle$	>0	0	0	Δ	≠0
$\lvert\pi_4\rangle$	>0	Δ	0	0	≠0
$\lvert\theta_4\rangle$	>0	Δ	0	0	≠0
$\lvert\sigma_4\rangle$	>0	0	Δ	0	≠0
$\lvert\rho_4\rangle$	>0	0	Δ	0	≠0
$\lvert\xi_4\rangle$	>0	0	0	Δ	≠0
$\lvert\epsilon_4\rangle$	>0	0	0	Δ	≠0
		(b)			
$\lvert W\rangle$	0	0	0	0	0
$\lvert\chi_4\rangle$	>0	Δ	Δ	Δ	0
$\lvert\upsilon_4\rangle$	>0	0	Δ	Δ	0
$\lvert\varpi_4\rangle$	>0	Δ	0	Δ	0
$\lvert\psi_4\rangle$	>0	Δ	0	0	0
$\lvert\phi_4\rangle$	>0	0	Δ	0	0
$\lvert\mu_4\rangle$	>0	0	0	Δ	0
$\lvert\varphi_4\rangle$	>0	Δ	0	0	0
$\lvert\zeta_4\rangle$	>0	0	Δ	0	0
$\lvert\vartheta_4\rangle$	>0	0	0	Δ	0
$\lvert\tau_4\rangle$	>0	0	0	0	0
$\lvert\varrho_4\rangle$	>0	0	0	0	0
$\lvert\iota_4\rangle$	>0	0	0	0	0
$\lvert\omega_4\rangle$	>0	0	0	0	0

III. INVARIANT AND SEMI-INVARIANTS FOR SLOCC CLASSIFICATION

A. Representatives of true entanglement classes

It is well known that the states $\lvert\mathrm{GHZ}\rangle$, $\lvert W\rangle$, $\lvert\phi_4\rangle$, and $\lvert C_4\rangle$ are the representatives of disjoint true entanglement classes of four qubits. Utilizing the SLOCC invariant $\mathcal{I}$ and the semi-invariants F_i and D_i of four qubits, we find 28 distinct true entanglement classes. The representatives of the classes are listed below.

(1) From the construction of $\lvert\phi_4\rangle$, we do the following tests. From all the 15 true entanglement states: $(\lvert 0\rangle+\lvert i\rangle+\lvert j\rangle-\lvert 15\rangle)/2$, where $\lvert i\rangle,\lvert j\rangle\in\{\lvert 3\rangle,\lvert 5\rangle,\lvert 6\rangle,\lvert 9\rangle,\lvert 10\rangle,\lvert 12\rangle\}$, which is obtained from $\lvert C_4\rangle$, we find the representatives of seven different true entanglement classes. They are $\lvert\mathrm{GHZ}\rangle$, $\lvert\phi_4\rangle,\lvert\psi_4\rangle$, $\lvert\mu_4\rangle$, $\lvert\kappa_4\rangle$, $\lvert E_4\rangle$, and $\lvert L_4\rangle$.

(2) From [6], we consider the states of the following forms: $(\lvert 0\rangle+\lvert i\rangle+\lvert j\rangle+\lvert k\rangle+\lvert l\rangle+\lvert 15\rangle)/\sqrt{6}$, where $\lvert i\rangle,\lvert j\rangle,\lvert k\rangle,\lvert l\rangle\in\{\lvert 3\rangle,\lvert 5\rangle,\lvert 6\rangle,\lvert 9\rangle,\lvert 10\rangle,\lvert 12\rangle$. There are 15 true entanglement states, of which seven are chosen as the representatives of different true entanglement classes. They are $\lvert C_4\rangle$, $\lvert\pi_4\rangle,\lvert\sigma_4\rangle,\lvert\rho_4\rangle,\lvert\xi_4\rangle,\lvert\epsilon_4\rangle$ and $\lvert\theta_4\rangle$.

(3) From the 15 true entanglement states: $(\lvert 0\rangle+\lvert i\rangle+\lvert j\rangle+\lvert k\rangle+\lvert l\rangle-\lvert 15\rangle)/\sqrt{6}$, where $\lvert i\rangle,\lvert j\rangle,\lvert k\rangle,\lvert l\rangle\in\{\lvert 3\rangle,\lvert 5\rangle,\lvert 6\rangle,\lvert 9\rangle,\lvert 10\rangle,\lvert 12\rangle\}$, we choose $\lvert\chi_4\rangle$ as a representative.

TABLE VI. The properties of D_i for two GHZ pairs.

States	D_1	D_2	D_3
$\lvert\mathrm{GHZ}\rangle_{12}\otimes\lvert\mathrm{GHZ}\rangle_{34}$	0	≠0	0
$\lvert\mathrm{GHZ}\rangle_{13}\otimes\lvert\mathrm{GHZ}\rangle_{24}$	≠0	0	0
$\lvert\mathrm{GHZ}\rangle_{14}\otimes\lvert\mathrm{GHZ}\rangle_{23}$	0	0	≠0

(4) Consider the true entanglement states of the following forms: $|3\rangle+|x\rangle+|12\rangle$, where $|x\rangle\in\{|5\rangle,|6\rangle,|9\rangle,|10\rangle\}$; $|5\rangle+|x\rangle+|10\rangle$, where $|x\rangle\in\{|3\rangle,|6\rangle,|9\rangle,|12\rangle\}$; $|6\rangle+|x\rangle+|9\rangle$, where $|x\rangle\in\{|3\rangle,|5\rangle,|10\rangle,|12\rangle\}$. Notice that the binary representatives of 3 and 12, 5 and 10, and 6 and 9 are complementary, respectively. From the 12 true entanglement states, we find three inequivalent true entanglement states. They are $|H_4\rangle,|\lambda_4\rangle$, and $|M_4\rangle$.

(5) The classes whose representatives have three product terms.

Let $S_1=(001)^T$, $S_2=(010)^T$, $S_3=(100)^T$, $V_1=(011)^T$, $V_2=(101)^T$, and $V_3=(110)^T$. Consider the permutations: $S_iS_jV_iV_j$. For example, $S_1S_2V_1V_2$ is considered as a matrix

$$\begin{pmatrix}0&0&0&1\\0&1&1&0\\1&0&1&1\end{pmatrix}.$$

Each row of the matrix is considered a basis state of four qubits. Thus, the matrix can be considered to be the state $(|1\rangle+|6\rangle+|11\rangle)/\sqrt{3}$. From the permutations, we find representatives: $|\varphi_4\rangle$, $|\tau_4\rangle$, $|\vartheta_4\rangle$, $|\varrho_4\rangle$, $|\iota_4\rangle$, $|\varsigma_4\rangle$.

(6) $|\omega_4\rangle$ is $L_{0_{5\oplus 3}}$ in [6]. From states of the forms $(|x\rangle+|5\rangle+|y\rangle+|z\rangle)/2$, where $x+5+y+z=27$, we choose $|\upsilon_4\rangle$, $|\varpi_4\rangle$, and $|\omega_4\rangle$ as representatives.

(7) Up to permutations of the qubits, each one of the following groups is considered as one true entanglement class. However, we do not show that the different groups cannot be obtained from one another by permutations of the qubits.

We list the 28 true entanglement classes as follows.

(1) $|\mathrm{GHZ}\rangle=(|0\rangle+|15\rangle)/\sqrt{2}$.

(2) $|W\rangle=(|1\rangle+|2\rangle+|4\rangle+|8\rangle)/2$.

(3) $|C_4\rangle=(|3\rangle+|5\rangle+|6\rangle+|9\rangle+|10\rangle+|12\rangle/\sqrt{6}$.

(4) $|\kappa_4\rangle=(|0\rangle+|3\rangle+|10\rangle-|15\rangle)/2$, $|E_4\rangle=(|0\rangle+|5\rangle+|9\rangle-|15\rangle)/2$, $|L_4\rangle=(|0\rangle+|3\rangle+|9\rangle-|15\rangle)/2$.

(5) $|H_4\rangle=(|3\rangle+|6\rangle+|12\rangle/\sqrt{3}$, $|\lambda_4\rangle=(|5\rangle+|6\rangle+|10\rangle/\sqrt{3}$, $|M_4\rangle=(|3\rangle+|5\rangle+|12\rangle/\sqrt{3}$.

(6) $|\pi_4\rangle=(|0\rangle+|3\rangle+|5\rangle+|6\rangle+|10\rangle+|15\rangle)/\sqrt{6}$, $|\theta_4\rangle=(|0\rangle+|5\rangle+|6\rangle+|10\rangle+|12\rangle+|15\rangle)/\sqrt{6}$, $|\sigma_4\rangle=(|0\rangle+|3\rangle+|9\rangle+|10\rangle+|12\rangle+|15\rangle)/\sqrt{6}$, $|\rho_4\rangle=(|0\rangle+|3\rangle+|6\rangle+|10\rangle+|12\rangle+|15\rangle)/\sqrt{6}$, $|\xi_4\rangle=(|0\rangle+|6\rangle+|9\rangle+|10\rangle+|12\rangle+|15\rangle)/\sqrt{6}$, $|\epsilon_4\rangle=(|0\rangle+|3\rangle+|6\rangle+|9\rangle+|10\rangle+|15\rangle)/\sqrt{6}$.

(7) $|\chi_4\rangle=(|0\rangle+|3\rangle+|6\rangle+|10\rangle+|12\rangle-|15\rangle)/\sqrt{6}$.

(8) $|\psi_4\rangle=(|0\rangle+|5\rangle+|10\rangle-|15\rangle)/2$, $|\phi_4\rangle=(|0\rangle+|3\rangle+|12\rangle-|15\rangle)/2$, $|\mu_4\rangle=(|0\rangle+|6\rangle+|9\rangle-|15\rangle)/2$.

(9) $|\varphi_4\rangle=(|1\rangle+|6\rangle+|11\rangle)/\sqrt{3}$, $|\vartheta_4\rangle=(|2\rangle+|5\rangle+|11\rangle)/\sqrt{3}$, $|\tau_4\rangle=(|1\rangle+|7\rangle+|10\rangle)/\sqrt{3}$, $|\varrho_4\rangle=(|2\rangle+|7\rangle+|9\rangle)/\sqrt{3}$.

(10) $|\zeta_4\rangle=(|0\rangle+|11\rangle+|12\rangle)/\sqrt{3}$, $|\iota_4\rangle=(|0\rangle+|3\rangle+|13\rangle)/\sqrt{3}$.

(11) $|\upsilon_4\rangle=(|2\rangle+|5\rangle+|9\rangle+|11\rangle/2$.

(12) $|\omega_4\rangle=(|0\rangle+|5\rangle+|8\rangle+|14\rangle)/2$.

(13) $|\varpi_4\rangle=(|2\rangle+|5\rangle+|8\rangle+|12\rangle)/2$.

B. Sufficient conditions for a true entanglement state

Let $|\psi\rangle$ be a state of four qubits. From Tables I, II, IV, and V it is not difficult to see that $|\psi\rangle$ is a true entanglement state if $|\psi\rangle$ satisfies one of the following conditions.

(1) $\mathcal{I}(\psi)=0$ and $D_i(\psi)\neq 0$, where $i=1,2$ or 3.

(2) $\mathcal{I}(\psi)\neq 0$ and $F_i(\psi)\neq 0$, where $i=1,2,3,4,5,6,7$ or 8.

(3) $\mathcal{I}(\psi)\neq 0$ and $D_i(\psi)\neq 0$ and $D_j(\psi)\neq 0$.

IV. AT LEAST 28 DISTINCT TRUE ENTANGLEMENT CLASSES

A. Degenerate entanglement classes

The authors in [8] gave an upper bound for the number of degenerate $(N+1)$-entanglement classes in terms of the number of N-partite entanglement classes. In this paper, we give an exact recursive formula for the number of degenerate entanglement classes of n qubits (see Appendix). By the recursive formula, for five qubits, there are $5\times t(4)+66$ distinct degenerate SLOCC entanglement classes, where $t(4)$ is the number of true SLOCC entanglement classes for four qubits. We only use combinatorial analysis to derive the recursive formula. The authors in [9] declared there are 16 true entanglement SLOCC classes for four qubits and at most 170 degenerate entanglement classes for five qubits. If so, by our recursive formula there would be 146 degenerate SLOCC entanglement classes for five qubits. However, in this paper, we report there are at least 28 true entanglement SLOCC classes for four qubits. Thus, by the recursive formula there are at least 206 degenerate SLOCC classes for five qubits. From the recursive formula, we know that most of the degenerate entanglement classes of n qubits are $(n-1)$-qubit true entanglement classes with a separate qubit like $A-(BC\ldots Z)_{n-1}$, where $(BC\ldots Z)_{n-1}$ is truly entangled.

For four qubits, by computing, we obtain the SLOCC invariant, and the semi-invariants F_i and D_i of all the degenerate entanglement classes. See Tables IV, II, and VI. For example, the value F of three-qubit GHZ entanglement accompanied by a separable qubit does not vanish. The proof is given as follows. Other proofs are omitted. By the definition of F and Sec. 3.1 of [5], it is easy to obtain this result.

For class $|\mathrm{GHZ}\rangle_{ABC}\otimes(s|0\rangle+t|1\rangle)_D$, by Sec. 3.1 of [5], $(a_0a_{14}-a_4a_{10}+a_2a_{12}-a_6a_8)^2\neq 4(a_4a_8-a_0a_{12})(a_6a_{10}-a_2a_{14})$ or $(a_1a_{15}-a_5a_{11}+a_3a_{13}-a_7a_9)^2\neq 4(a_5a_9-a_1a_{13})(a_7a_{11}-a_3a_{15})$, and other F_i vanish.

For class $|\mathrm{GHZ}\rangle_{ABD}\otimes(s|0\rangle+t|1\rangle)_C$, $(a_0a_{13}-a_4a_9+a_1a_{12}-a_5a_8)^2\neq 4(a_4a_8-a_0a_{12})(a_5a_9-a_1a_{13})$ or $(a_2a_{15}-a_6a_{11}+a_3a_{14}-a_7a_{10})^2\neq 4(a_6a_{10}-a_2a_{14})(a_7a_{11}-a_3a_{15})$, and other F_i vanish.

For class $|\mathrm{GHZ}\rangle_{ACD}\otimes(s|0\rangle+t|1\rangle)_B$, $(a_0a_{11}-a_2a_9+a_1a_{10}-a_3a_8)^2\neq 4(a_2a_8-a_0a_{10})(a_3a_9-a_1a_{11})$ or $(a_4a_{15}-a_6a_{13}+a_5a_{14}-a_7a_{12})^2\neq 4(a_6a_{12}-a_4a_{14})(a_7a_{13}-a_5a_{15})$.

For class $(s|0\rangle+t|1\rangle)_A\otimes|\mathrm{GHZ}\rangle_{BCD}$, $(a_0a_7-a_2a_5+a_1a_6-a_3a_4)^2\neq 4(a_2a_4-a_0a_6)(a_3a_5-a_1a_7)$ or $(a_8a_{15}-a_{11}a_{12}+a_9a_{14}-a_{10}a_{13})^2\neq 4(a_{11}a_{13}-a_9a_{15})(a_{10}a_{12}-a_8a_{14})$.

B. 28 classes in Tables V(a) and V(b) are true entanglement classes

It is known that the classes $|\mathrm{GHZ}\rangle$, $|W\rangle$, $|\phi_4\rangle$ [7], and $|C_4\rangle$ [5] are inequivalent true entanglement classes.

Part 1. The classes in Table V(a) are true entanglement classes.

Since for the classes in Table V(a), $\mathcal{I}\neq 0$ and $F>0$, we only need to show that the classes in Table V(a) are distinct from the degenerate entanglement classes $|GHZ\rangle_{13}\otimes|GHZ\rangle_{24}$ and $|GHZ\rangle_{14}\otimes|GHZ\rangle_{23}$. However, the F_i in Tables III(a) do not satisfy the properties of the F_i of classes $|GHZ\rangle_{13}\otimes|GHZ\rangle_{24}$ or $|GHZ\rangle_{14}\otimes|GHZ\rangle_{23}$ in Table II. Hence, the classes in Table V(a) are true entanglement classes.

Part 2. The classes in Table V(b) are true entanglement classes.

Since $\mathcal{I}=0$ for classes in Table V(b), the classes in Table V(b) are distinct from the degenerate entanglement classes $|GHZ\rangle_{12}\otimes|GHZ\rangle_{34}$, $|GHZ\rangle_{13}\otimes|GHZ\rangle_{24}$, and $|GHZ\rangle_{14}\otimes|GHZ\rangle_{23}$. For some states of classes $|\chi_4\rangle$, $|\upsilon_4\rangle$, $|\varpi_4\rangle$, $|\psi_4\rangle$, $|\phi_4\rangle$, $|\mu_4\rangle$, $|\varphi_4\rangle$, $|s_4\rangle$, and $|\vartheta_4\rangle$, always $D_i\neq 0$ for some i (see [13]). However, all the classes in Table IV except for $|GHZ\rangle_{12}\otimes|GHZ\rangle_{34}$, $|GHZ\rangle_{13}\otimes|GHZ\rangle_{24}$, and $|GHZ\rangle_{14}\otimes|GHZ\rangle_{23}$, require $D_i=0$, where $i=1$, 2, and 3 (see Table IV). So classes $|\chi_4\rangle$, $|\upsilon_4\rangle$, $|\varpi_4\rangle$, $|\psi_4\rangle$, $|\phi_4\rangle$, $|\mu_4\rangle$, $|\varphi_4\rangle$, $|s_4\rangle$, and $|\vartheta_4\rangle$ are not degenerate entanglement classes. The classes $|\tau_4\rangle$, $|\varrho_4\rangle$, $|\iota_4\rangle$, $|\omega_4\rangle$ in Table V(b) are not degenerate entanglement classes because the properties of the F_i of classes $|\tau_4\rangle$, $|\varrho_4\rangle$, $|\iota_4\rangle$, $|\omega_4\rangle$ in Table I(b) do not satisfy the conditions of the F_i in Table II.

C. 28 Classes in Table V(a) and V(b) are distinct from each other

Clearly, the classes in Table V(a) are distinct from the ones in Table V(b) because the values of $\mathcal{I}$ of all the classes in Table V(b) are zero while the values of $\mathcal{I}$ of all the classes in Table V(a) are not zero.

Part 1. Let us show that the classes in Table V(a) are distinct from each other.

For class $|GHZ\rangle$, $D_i=0$, where $i=1$, 2, and 3 [see Table V(a)]. However, $D_i\neq 0$ always for some i and for some states of other classes in Table V(a). Consequently, the class $|GHZ\rangle$ is distinct from the other classes in Table V(a). For state $|C_4\rangle$, $D_i\neq 0$, where $i=1$, 2, and 3 [see Table III(a)]. It is easy to see from Table V(a) that for the other classes, $D_i=0$ always for some i. For example, $D_3=0$ for class $|\kappa_4\rangle$ [see Table V(a)]. It implies that $D_3=0$ for every state of class $|\kappa_4\rangle$. Therefore, class $|C_4\rangle$ is different from other classes in Table V(a).

For some states of class $|\kappa_4\rangle$, $D_1\neq 0$ and $D_2\neq 0$ (see the case for class $|\kappa_4\rangle$ in [13], and for every state of class $|\kappa_4\rangle$, $D_3=0$). Therefore class $|\kappa_4\rangle$ is different from the last 11 classes in Table V(a). Also, classes $|E_4\rangle$ and $|L_4\rangle$ are different from each other and from the last nine classes in Table V(a).

Let us demonstrate that class $|H_4\rangle$ is different from classes $|\pi_4\rangle$ and $|\theta_4\rangle$. For state $|H_4\rangle$, $F_9\neq 0$ and $F_i=0$ when $i\neq 9$ [see Table III(a)]. Thus, state $|H_4\rangle$ does not satisfy the conditions of F_i of classes $|\pi_4\rangle$ or $|\theta_4\rangle$ [see Table I(a)]. Therefore, class $|H_4\rangle$ is different from classes $|\pi_4\rangle$ and $|\theta_4\rangle$. For some states of class $|H_4\rangle$, $D_1\neq 0$ and for every state of class $|H_4\rangle$, $D_2=0$ and $D_3=0$, therefore class $|H_4\rangle$ is different from $|\lambda_4\rangle$ and $|M_4\rangle$ and the last four classes in Table V(a). Also, $|\lambda_4\rangle$ and $|M_4\rangle$ are different from each other and from the last six classes in Table V(a).

From Table I(a), it is easy to see that for the last six classes: $|\pi_4\rangle$, $|\theta_4\rangle$, $|\sigma_4\rangle$, $|\rho_4\rangle$, $|\xi_4\rangle$, and $|\epsilon_4\rangle$, the properties of F_i are disjoint, hence they are distinct from each other.

Part 2. We argue that the classes in Table V(b) are distinct from each other.

Since $F=0$ for class $|W\rangle$ and $F\neq 0$ for other classes in Table V(b), class $|W\rangle$ is distinct from the other classes in Table V(b). For some states of class $|\chi_4\rangle$, $D_i\neq 0$, where $i=1$, 2, and 3, see the cases for class $|\chi_4\rangle$ in [13]. As discussed for class $|C_4\rangle$ in Part 1, class $|\chi_4\rangle$ is distinct from the other classes in Table V(b).

For some states of class $|\upsilon_4\rangle$, $D_2\neq 0$ and $D_3\neq 0$ (see the cases for class $|\upsilon_4\rangle$ in [13]), and for every state of class $|\upsilon_4\rangle$, $D_1=0$ [see Table V(b)]. Therefore, class $|\upsilon_4\rangle$ is different from other classes in Table V(b). We omit the similar discussions, which can be found in Part 1. We need to argue that $|\psi_4\rangle$ and $|\varphi_4\rangle$ are different from each other. For state $|\psi_4\rangle$, $F_1=0$ and $F_2=0$ [see Table III(b)]. This contradicts that $|F_1|+|F_2|\neq 0$ for class $|\varphi_4\rangle$ [see Table I(b)]. This is done. Also, $|\phi_4\rangle$ and $|s_4\rangle$ are different from each other, as are $|\mu_4\rangle$ and $|\vartheta_4\rangle$. What remains is to explain that $|\tau_4\rangle$, $|\varrho_4\rangle$, $|\iota_4\rangle$ and $|\omega_4\rangle$ are distinct from each other. This is obvious because the properties of their F_i are disjoint [see Table I(b)].

Conjecture. There should be a large number of true SLOCC entanglement classes. We can show $\frac{1}{\sqrt{6}}(\sqrt{2}|15\rangle+|8\rangle+|4\rangle+|2\rangle+|1\rangle)$ in [14] is a true entanglement state, which does not belong to the 28 classes because the state has the following properties:

$$\mathcal{I}=0,\quad D_1:\quad \Delta,\quad D_2:\quad \Delta,\quad D_3:\quad \Delta, |F_1|+|F_2|\neq 0, |F_3|+|F_4|\neq 0, |F_5|+|F_6|\neq 0, \text{ and } |F_7|+|F_8|\neq 0.$$

V. SEMI-INVARIANTS OF n QUBITS

In [11], we list the following semi-invariants. When $i+j$ is odd, let $F_{odd}=(a_ia_j+a_ka_l-a_pa_q-a_ra_s)^2-4(a_ia_{j-1}-a_pa_{q-1})\times(a_ka_{l+1}-a_ra_{s+1})$. Otherwise, let $F_{even}=(a_ia_j+a_ka_l-a_pa_q-a_ra_s)^2-4(a_ia_{j-2}-a_pa_{q-2})(a_ka_{l+2}-a_ra_{s+2})$. Let

$$F=4\left(\sum_{odd(i+j)}|F_{odd}|+\sum_{even(i+j)}|F_{even}|\right). \tag{12}$$

The subscripts in the above expressions satisfy the following conditions.

$$i<j,\quad k<l,\quad p<q,\quad r<s,\quad i<k<p<r,$$

$$i+j=k+l=p+q=r+s,$$

$$i\oplus j=k\oplus l=p\oplus q=r\oplus s. \tag{13}$$

Remark. We can consider that Eq. (13) is a special partition of an integer.

Example 1. Let $(|0\rangle+|2^n-1\rangle)/2$ be the state $|GHZ\rangle$ of n qubits. Then F of state $|GHZ\rangle$ does not vanish.

Proof. This is because the following term does not vanish:

$$|[(a_0a_{2^n-1}-a_2a_{2^n-3})+(a_1a_{2^n-2}-a_3a_{2^n-4})]^2 - 4(a_0a_{2^n-2}-a_2a_{2^n-4})(a_1a_{2^n-1}-a_3a_{2^n-3})| = |a_0a_{2^n-1}| = 1/4.$$

Notice that other terms do vanish.

Example 2. Let $|W\rangle=(|0\ldots01\rangle+|0\ldots010\rangle+|0\ldots0100\rangle+\cdots)/\sqrt{n}$, where the amplitudes $a_{2^j}=1/\sqrt{n}$, where $j=0,1,\ldots,(n-1)$, and other amplitudes $a_i=0$. Then F of $|W\rangle$ vanishes.

Proof.

(1) We show that $a_ia_j=a_ka_l=a_pa_q=a_ra_s=0$. If $a_ia_j\neq 0$, then $i=2^m$ and $j=2^n$, where $m<n$. Clearly, we cannot find k or l such that $k=2^s$ and $l=2^t$ and $2^m\oplus 2^n=2^s\oplus 2^t$.

(2) We show that $(a_ia_{j-1}-a_pa_{q-1})(a_ka_{l+1}-a_ra_{s+1})=0$. It is enough to illustrate $a_ia_{j-1}=a_pa_{q-1}=a_ka_{l+1}=a_ra_{s+1}=0$. Assume that $i=2^m$ and $j-1=2^n$ and $k=2^s$ and $l+1=2^t$. Since $i+j=k+l$, $2^m+2^n+2=2^s+2^t$. Since $i\oplus j=k\oplus l$, then $i\oplus(j-1)=k\oplus(l+1)$, i.e., $2^m\oplus 2^n=2^s\oplus 2^t$. It is not possible for $2^m+2^n+2=2^s+2^t$ and $2^m\oplus 2^n=2^s\oplus 2$ to both hold.

(3) Also, we can show that $(a_ia_{j-2}-a_pa_{q-2})(a_ka_{l+2}-a_ra_{s+2})=0$.

We conclude that F vanishes. From examples 1 and 2, it is trivial that the states $|\mathrm{GHZ}\rangle$ and $|W\rangle$ of n qubits are inequivalent.

VI. SUMMARY

In this paper, we found the SLOCC invariant and semi-invariants for four qubits. By means of the invariant and semi-invariants we can determine if two states are inequivalent. Then, we show that there are an infinite number of SL classes and at least 28 distinct true SLOCC entanglement classes. The invariant and semi-invariants only require simple arithmetic operations. The ideas can be extended to five or more qubits for SLOCC classification. In this paper, we also give the exact recursive formulas for the number of degenerate SLOCC classes of n qubits. By the recursive formula, for six qubits, there are $6\times t(5)+30\times t(4)+276$ distinct degenerate SLOCC entanglement classes, where $t(5)$ is the number of true SLOCC entanglement classes for five qubits.

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APPENDIX: NUMBER OF DEGENERATE SLOCC CLASSES FOR n QUBITS

Let $d(n)$ be the number of degenerate SLOCC classes for n qubits and $t(n)$ the number of true entanglement SLOCC classes for n qubits and $t(1)=1$.

1. By Computing $d(5)$ demonstrate how to derive $d(n)$

Case 1. Only four-qubit true entanglement accompanied with a separable qubit.

For example, $ABCD-E$, where $ABCD$ is truly entangled. Note that four qubits can truly be entangled in $t(4)$ distinct ways. Clearly, there are $\binom{5}{4}t(4)$ distinct degenerate SLOCC classes. This situation can be considered as that of five balls divided into two groups. The first group contains exactly one ball and the second group contains exactly four balls. Thus, there are $\frac{5!}{1!4!}$ different ways [15]. Note that the four balls correspond to four qubits. Therefore, for this case, the number of degenerate SLOCC classes can be rewritten as $\frac{5!}{1!4!}t(1)t(4)$. We can consider this case as a partition of 5: $1+4=5$.

Case 2. Only three-qubit true entanglement accompanied with two separable qubits.

For example, $ABC-D-E$, where ABC is truly entangled. As indicated in [2], three qubits can truly be entangled in two inequivalent ways. It is easy to see that there are $2\binom{5}{3}$ distinct degenerate SLOCC classes. Let us consider that the five balls are divided into three groups. Each of the first two groups contains exactly one ball and the third group contains exactly three balls. Thus, there are $\frac{5!}{1!1!3!}$ different ways [15]. Note that $ABC-D-E$ and $ABC-E-D$ represent the same class. Hence, for this case, the number of degenerate SLOCC classes can be rewritten as $\frac{5!}{1!1!3!}\times t(1)t(1)t(3)\times\frac{1}{2!}$. We regard this case as being a partition of 5: $1+1+3=5$.

Case 3. Only two-qubit true entanglement accompanied with three separable qubits.

For example, $AB-C-D-E$, where AB is a two-qubit GHZ state. It is clear that there are $\binom{5}{2}$ distinct classes. We consider that the five balls are divided into four groups. Each of the first three groups contains exactly one ball and the fourth group contains exactly two balls. Thus, there are $\frac{5!}{1!1!1!2!}$ different ways [15]. Note that $AB-C-D-E$, $AB-C-E-D$, $AB-D-C-E$, $AB-D-E-C$, $AB-E-C-D$, and $AB-E-D-C$ belong to the same class. Then, for this case, the number of degenerate SLOCC classes can be rewritten as $\frac{5!}{1!1!1!2!}\times t(1)t(1)t(1)t(2)\times\frac{1}{3!}$. Let us consider this case as a partition of 5: $1+1+1+2=5$.

Case 4. Two GHZ pairs accompanied with a separable qubit.

For example, $AB-CD-E$, where AB and CD are two-qubit GHZ states. For this case, there are 15 degenerate SLOCC classes. Note that $AB-CD-E$ and $CD-AB-E$ represent the same class. Similarly, for this case, the number of degenerate SLOCC classes can be rewritten as $\frac{5!}{1!2!2!}\times t(1)t(2)t(2)\times\frac{1}{2!}$. We can consider this case as a partition of 5: $1+2+2=5$.

Case 5. Two-qubit GHZ$\otimes$ three-qubit GHZ and two-qubit GHZ$\otimes$ three-qubit W.

For example, $AB-CDE$. We can list the ways of entanglement as follows. $AB-CDE$; $AC-BDE$; $AD-BCE$; $AE-BCD$; $BC-ADE$; $BD-ACE$; $BE-ACD$; $CD-ABE$; $CE-ABD$; and $DE-ABC$. Hence, there are $2\binom{5}{2}$ degenerate SLOCC classes. Similarly, for this case, the number of degenerate SLOCC classes can be rewritten as $\frac{5!}{2!3!}\times t(2)t(3)$. We regard this case as being a partition of 5: $2+3=5$.

Case 6. A product state: $A-B-C-D-E$.

It is trivial that $\frac{5!}{1!1!1!1!1!}\times t(1)t(1)t(1)t(1)t(1)\times\frac{1}{5!}=1$. Let us consider this case as a partition of 5: $1+1+1+1+1=5$. In total, there are $5\times t(4)+66$ degenerate SLOCC classes for five qubits.

2. Exact recursive formula of $d(n)$

For n qubits, we consider the degenerate entanglement way $r_1\otimes r_2\otimes\cdots\otimes r_k$, where $k\geq 2$ and the r_i qubits are truly entangled, $i=1,2,\ldots,k$. As indicated in [15], there are $\frac{n!}{r_1!r_2!..r_k!}$ different ways to divide n balls into k groups such that the jth group contains exactly r_j balls, where $r_1+r_2+\cdots+r_k=n$. Note that r_i qubits can truly be entangled in $t(r_i)$ distinct ways. Let s_j be the number of occurrences of r_{i_j} in $r_1,r_2,\ldots,r_k$. Thus, for this situation, there are $\frac{n!}{r_1!r_2!\ldots r_k!}t(r_1)t(r_2)\ldots t(r_k)\frac{1}{s_1!s_2!\ldots s_l!}$ degenerate SLOCC classes. In total, $d(n)=\Sigma\frac{n!}{r_1!r_2!..r_k!}t(r_1)t(r_2)\ldots t(r_k)\frac{1}{s_1!s_2!\ldots s_l!}$, where the summation is extended over all the following Euler partitions of n: $r_1+r_2+\cdots+r_k=n$ in which $k\geq 2$ and $1\leq r_1\leq r_2\leq\cdots\leq r_k<n$.

3. Compute $d(4)$ using the recursive formula

For four qubits, the following are the partitions of 4. $1+3$; $1+1+2$; $2+2$; $1+1+1+1$.

Case 1. For the partition $1+3$, there are $\frac{4!}{1!3!}t(1)t(3)=8$ degenerate SLOCC classes. They are $A-BCD$, $B-ACD$, $C-ABD$, and $D-ABC$. Note that three qubits can truly be entangled in two inequivalent ways.

Case 2. For the partition $1+1+2$, there are $\frac{4!}{1!1!2!}t(1)t(1)t(2)\frac{1}{2!}=6$ degenerate SLOCC classes. They are $A-B-CD$, $A-C-BD$, $A-D-BC$, $B-C-AD$, $B-D-AC$, and $C-D-AB$.

Case 3. For the partition $2+2$, there are $\frac{4!}{2!2!}t(2)t(2)\frac{1}{2!}=3$ degenerate SLOCC classes. They are $AB-CD$, $AC-BD$, and $AD-BC$.

Case 4. For the partition $1+1+1+1$, we have a product state. It is trivial that $\frac{4!}{1!1!1!1!}\times t(1)t(1)t(1)t(1)\times\frac{1}{4!}=1$.

In total, there are 18 degenerate SLOCC classes (see [6]).

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