Separability of entangled qutrits in noisy channels

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We present an analysis of noisy atomic channels involving qutrits. We choose a three-level atom with V configuration to be the qutrit state. Gell-Mann matrices and a generalized Bloch vector (eight-dimensional) are used to describe the qutrit density operator. We introduce quantum quasidistributions for qutrits that provide a simple description of entanglement. Studying the time evolution for the atomic variables we find the Kraus representation of spontaneous emission quantum channel (SE channel). Furthermore, we consider a generalized Werner state of two qutrits and investigate the separability condition in the presence of spontaneous emission noise. The influence of spontaneous emission on the separability of Werner states for qutrit and qubit states is compared.

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I. INTRODUCTION

Quantum information can be stored, transmitted, and retrieved using light, cold ions, or atoms. Only in highly ideal conditions can these physical systems be regarded as isolated and immune to various sources of decoherence. Quantum channels based on atoms or photons are examples of open quantum systems interacting with an environment, causing degradation of the linear superpositions or the quantum nonseparability of correlated systems. The understanding and the control of such noisy channels is at the core of quantum communication.

Experimental teleportation of atoms [1,2] provides an example of a channel in which quantum information is transferred with a high fidelity. The quantum teleportation protocol uses as a resource entangled atoms. Entanglement of atoms can be achieved using different physical phenomena such as coherent cold collisions [3] or an optical lattice [4]. Recent experimental and theoretical investigations have shown that cold atoms and individual photons may lead the way towards chip-scale quantum information processors [5].

One of the physical processes that may deteriorate the efficiency of atomic applicability, is spontaneous emission. Dissipation induced by vacuum fluctuations in quantum channels with atoms, impacts atomic entanglement and the fidelity of quantum protocols based on atomic systems.

In most atomic applications to atomic channels, the main building blocks of information were based on two-level quantum systems, or qubits. Using N-level systems or qudits can, in principle, improve the efficiency of a quantum channel due to a larger Hilbert space. It is known that entangled qudits can provide a higher degree of efficiency in quantum protocols [6].

The simplest generalization of the qubit involves a qutrit, i.e., a quantum state spanned by three orthonormal states $|1\rangle$, $|2\rangle$, and $|3\rangle$. Qutrits can be physically implemented using three-level atoms [7], transverse spatial modes of single photons [8], or polarization states of a single-mode biphoton field [9].

The goal of this paper is to discuss the properties of noisy atomic channels involving qutrits. The physical realization of the qutrit state in a noisy channel will be based on a three-level atom with spontaneous emission. Qutrit quantum channels with vacuum fluctuations are open quantum systems. It is the purpose of this paper to study the properties of such noisy quantum channels. We shall investigate the efficiency, the fidelity of such channels, and their impact on the quantum separability on entangled qutrits.

The paper is structured as follows. In Sec. II we review the Bloch description of qutrits based on the SU(3) generators. We introduce and investigate quantum quasidistributions for qutrits that provide a simple description of qutrit entanglement. We explain why a Werner mixed qutrit state is more robust compared to the qubit situation.

In Sec. III we discuss spontaneous emission in the framework of the qutrit Bloch formalism, and derive the Kraus representation for a qutrit noisy channel. In Sec. IV we examine the influence of the spontaneous emission (SE) channel on state separability. We investigate when the impact of this noisy channel is stronger for qubits than for qutrits. The fidelity of the channel is computed and compared. In Sec. V we present a concise summary of our results.

II. FROM QUBIT TO QUTRIT STATES

It is well known that a qubit—a quantum state living in a two-dimensional Hilbert space, is used as a basic building block of quantum information [10]. Within the framework of atomic physics two-level atoms are the simplest physical realizations of qubits [11]. Many papers have been written on the subject of qubits and quantum qubit channels [10,12,13]. A natural generalization of a qubit to N-dimensional, involving qudits, has been investigated [15–17], though has received less interest. From the physical point of view the use of more complex atomic structures might be advantageous [18]. The first natural step in this generalization brings us to quantum objects that belong to a three-dimensional Hilbert space $\mathcal{H}^3$—qutrits.

It is the purpose of this section to provide a useful description of qutrit states using such tools as the concept of the Bloch vector associated with a Bloch sphere and apply quantum quasidistribution functions for the description of qutrit states. We shall exemplify our approach discussing in a parallel way qubit and qutrit properties.

A. Qutrit Bloch vectors

It is very advantageous to provide the mathematical description of qutrits in a similar way as qubits are characterized with the use of the Bloch formalism. This formalism uses in an intrinsic way the SU(2) generators and Pauli matrices as a basis for the qubit density operator

$$\rho_{\text{qubit}} = \frac{1}{2}(\mathbb{I} + \vec{n} \cdot \vec{\sigma}), \tag{1}$$

where $\vec{n}=n_i\vec{e}_j$ is a three-dimensional (real) Bloch vector. For a system of correlated qubits a and b, the corresponding density operator has the form

$$\rho_{\text{qubit}}^{ab} = \frac{1}{4}(\mathbb{I}_a \otimes \mathbb{I}_b + \vec{n}_a \cdot \vec{\sigma} \otimes \mathbb{I}_b + \mathbb{I}_a \otimes \vec{n}_b \cdot \vec{\sigma} + \mathcal{C}_{ij}\sigma_i \otimes \sigma_j), \tag{2}$$

where $\vec{n}_a$ and $\vec{n}_b$ are individual Bloch vectors of the two qubits and $\mathcal{C}_{ij}=\langle\sigma_i\otimes\sigma_j\rangle$ is the correlation matrix of the two qubits.

For a maximally entangled state of the two qubits

$$|\Psi_{\text{qubit}}\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |2,2\rangle), \tag{3}$$

written in the qubit basis $|1\rangle$ and $|2\rangle$, the mean values of the individual Bloch vectors are zero and the correlation matrix is diagonal and has the simple form

$$\mathcal{C}_{ij} = s\delta_{ij}, \tag{4}$$

where $s=(1,-1,1)$ corresponds to a sign assigned to the three corresponding components of the Kronecker delta.

It is clear that the mathematical description of a qutrit density operator involves in a natural way the SU(3) generators, called the Gell-Mann matrices λ_i [19]. Earlier applications of the SU(3) formalism to three-level atoms can be found in Refs. [20]. More recent applications of this formalism involving entanglement are presented in Ref. [21]. The density operator of the qutrit is

$$\rho = \frac{1}{3}(\mathbb{I} + \sqrt{3}\vec{n} \cdot \vec{\lambda}), \tag{5}$$

where $\vec{n}=n_i\vec{e}_i$ is now a real eight-dimensional generalized Bloch vector. The Gell-Mann matrices, like the Pauli matrices, are traceless and satisfy

$$\lambda_i\lambda_j = \frac{2}{3}\delta_{ij} + d_{ijk}\lambda_k + if_{ijk}\lambda_k,$$

where the completely antisymmetric f_{ijk} are the structure constants of the SU(3) algebra, and d_{ijk} are completely symmetric. Values of these coefficients and the explicit form of the eight Gell-Mann matrices can be found in [21].

Pure qutrit states correspond to vectors that satisfy

$$\vec{n} \cdot \vec{n} = 1, \quad \vec{n} * \vec{n} = \vec{e}_i d_{ijk} n_j n_k = \vec{n}. \tag{6}$$

These two conditions define a generalized Bloch sphere for qutrits, in analogy to a Bloch qubit sphere. Hence, pure qutrit states in a unique way refer to unit vectors $\vec{n}\in\mathcal{S}^7$, the seven-dimensional unit sphere in $\mathcal{R}^8$ (first condition). However, the second condition places three additional constraints on the real parameters defining the pure state vector.

For a system of two correlated qutrits a and b, the corresponding density operator has the form

$$\rho_{\text{qutrit}}^{ab} = \frac{1}{9}(\mathbb{I}_a \otimes \mathbb{I}_b + \sqrt{3}\vec{n}_a \cdot \vec{\lambda} \otimes \mathbb{I}_b + \sqrt{3}\mathbb{I}_a \otimes \vec{n}_b \cdot \vec{\lambda} + 3\mathcal{C}_{ij}\lambda_i \otimes \lambda_j), \tag{7}$$

where $\vec{n}_a$ and $\vec{n}_b$ are individual Bloch vectors of the two qutrits and $\mathcal{C}_{ij}=\frac{3}{4}\langle\lambda_i\otimes\lambda_j\rangle$ is the correlation matrix of the two qutrits. The maximally entangled state of the two qutrits is

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|1,1\rangle + |2,2\rangle + |3,3\rangle), \tag{8}$$

where a third state $|3\rangle$ has been added to the qubit maximally entangled state. In this case the mean values of the individual Bloch vectors are zero and the 8×8 correlation matrix is diagonal

$$\mathcal{C}_{ij} = \frac{s}{2}\delta_{ij}, \tag{9}$$

where $s=(1,-1,1,1,-1,1,-1,1)$ corresponds to a sign assigned to the eight corresponding components of the Kronecker delta.

For the Bloch vector of a qutrit, orthogonal states in $\mathcal{H}^3$ do not correspond to opposite points on $\mathcal{S}^7$, but to points of maximum opening angle of $\frac{2\pi}{3}$. A distribution of points on $\mathcal{S}^7$ that represent physical states, the generalized Bloch sphere, is highly nontrivial and the majority of points on $\mathcal{S}^7$ do not lead to any physical states (producing matrices with negative eigenvalues). Mixed qutrit states are localized within the eight-dimensional ball, though in analogy to Bloch sphere, the generalized Bloch ball has a nontrivial structure [21,22].

B. Quantum and classical quasifunctions for qutrits

A view based on local realities provide a classical interpretation of qubit or qutrit entanglement. In this description the directions on the Bloch sphere are interpreted as random local realities distributed with a classical distribution function. In this approach the correlations between systems a and b are written as a statistical average

$$\mathcal{C}_{i,j}^{cl} = \int d\vec{n}_a \int d\vec{n}_b \, P_{cl}(\vec{n}_a,\vec{n}_b) n_a^i n_a^i. \tag{10}$$

In this formula the Bloch unit directions (local realities) $\vec{n}_a$ and $\vec{n}_b$ are integrated over the qubit or the qutrit Bloch sphere with a weight function corresponding to a classical (positive everywhere) probability distribution function $P_{cl}(\vec{n}_a,\vec{n}_b)$, which has uniform marginals.

For maximally entangled states of the qubit and the qutrit the probability distributions and the corresponding correlations are

$$P_{cl}(\vec{n}_a,\vec{n}_b) = \frac{1}{4\pi}\delta^{(3)}(\vec{n}_a - s\vec{n}_b) \Rightarrow \mathcal{C}^{cl}_{ij} = \frac{s}{3}\delta_{ij},$$

$$P_{cl}(\vec{n}_a,\vec{n}_b) = \frac{2}{9\pi^2}\delta^{(8)}(\vec{n}_a - s\vec{n}_b) \Rightarrow \mathcal{C}^{cl}_{ij} = \frac{s}{8}\delta_{ij}. \quad (11)$$

Two different factors in the correlations for the qubit and qutrit state are due to different solid angles 4π for a qubit, and $\frac{9\pi^2}{2}$ for a qutrit. Calculations of the correlation functions involve the following integrals:

$$\frac{1}{4\pi}\int d\vec{n}\; n_i n_j = \frac{1}{3}\delta_{ij} \quad \text{for qubit}, \quad (12)$$

$$\frac{2}{9\pi^2}\int d\vec{n}\; n_i n_j = \frac{1}{8}\delta_{ij} \quad \text{for qutrit}. \quad (13)$$

As a result of this we see that classical correlations are $\frac{1}{3}$ and $\frac{1}{4}$ of the quantum result. We will see in the next section that these two numbers will play an essential role in the separability problem involving mixed states.

The reason why these two classical probability distributions fail to describe quantum correlations given by Eqs. (4) and (9) is the fact that a local hidden variable theory based on a positive distribution function of local realities cannot be equivalent to quantum mechanics (Bell inequality).

It is well known that in order to describe quantum correlations we have to replace the classical distributions from Eq. (10) by nonlocal positive quasidistributions or by local nonpositive quasidistributions. In the case of local and nonpositive quantum distributions, we are dealing with quantum quasidistributions similar to the Glauber P-diagonal representation for a harmonic oscillator or the atomic coherent states for N-dimensional systems. A detailed description of these quantum quasidistributions for qubits can be found in [23,24].

As a result of this approach we can write the following two quantum distribution functions with homogeneous marginals:

$$P_{qm}(\vec{n}_a,\vec{n}_b) = \frac{3}{4\pi}\delta^{(3)}(\vec{n}_a - s\vec{n}_b) - 2\left(\frac{1}{4\pi}\right)^2 \Rightarrow C_{ij} = s\delta_{ij},$$

$$P_{qm}(\vec{n}_a,\vec{n}_b) = \frac{8}{9\pi^2}\delta^{(8)}(\vec{n}_a - s\vec{n}_b) - 3\left(\frac{2}{9\pi^2}\right)^2 \Rightarrow C_{ij} = \frac{s}{2}\delta_{ij}. \quad (14)$$

As we can see these two Bloch sphere nonpositive distributions of the qubit and qutrit describe exactly quantum correlations, making the Bell inequalities void.

C. Separability of Werner qutrit states

The Werner state for two qubits is a convex combination of a maximally entangled state of two qubits with a maximally mixed state

$$\rho_W = \frac{1-p}{4}\mathbb{I}_A \otimes \mathbb{I}_B + p|\Psi_{\text{qubit}}\rangle\langle\Psi_{\text{qubit}}| \quad (15)$$

$$= \frac{1}{4}c^{\text{qubit}}_{\alpha\beta}\sigma^A_\alpha \otimes \sigma^B_\beta, \quad (16)$$

where $0 \le p \le 1$, $\alpha,\beta \in \{0,\ldots,3\}$, $\sigma_0 \equiv \mathbb{I}$. For such a state a necessary and sufficient condition for a quantum separability condition is known.

The qutrit Werner state is a convex combination of a maximally entangled state of two qutrits with a maximally mixed state

$$\rho_W = \frac{1-p}{9}\mathbb{I}_A \otimes \mathbb{I}_B + p|\Psi_{\text{qutrit}}\rangle\langle\Psi_{\text{qutrit}}| \quad (17)$$

$$= \frac{1}{9}c^{\text{qutrit}}_{\alpha\beta}\lambda^A_\alpha \otimes \lambda^B_\beta, \quad (18)$$

where $0 \le p \le 1$, $\alpha,\beta \in \{0,\ldots,8\}$, $\lambda_0 \equiv \sqrt{\frac{2}{3}}\mathbb{I}$. For such a state a necessary and sufficient condition for quantum separability is unknown. The separability condition of this state has been investigated in [21], using the SU(3) Bloch form. Despite the fundamental difference between these two states as far the mathematical criterion of separability is concerned, we shall study the qubit and the qutric cases on equal footing using as a tool the quantum distributions derived in Eq. (14) to construct the corresponding distribution functions for the Werner states. The density operators form a convex set, and as a result of this the Werner quasidistribution functions are convex combinations of the corresponding distributions

$$P_W(\vec{n}_a,\vec{n}_b) = (1-p)P_{\max}(\vec{n}_a,\vec{n}_b) + pP_\psi(\vec{n}_a,\vec{n}_b), \quad (19)$$

where $P_{\max}(\vec{n}_a,\vec{n}_b)$ is the quantum distribution corresponding to a maximally mixed state. This function is different for the qubit and the qutrit

$$P^{\text{qubit}}_{\max}(\vec{n}_a,\vec{n}_b) = \left(\frac{1}{4\pi}\right)^2,$$

$$P^{\text{qutrit}}_{\max}(\vec{n}_a,\vec{n}_b) = \left(\frac{2}{9\pi^2}\right)^2. \quad (20)$$

The quantum distribution corresponding to a maximally entangled state of the qubit or the qutrit is given as calculated in Eq. (14). As a result we obtain for the two Werner states

$$P^{\text{qubit}}_W(\vec{n}_a,\vec{n}_b) = \frac{3p}{4\pi}\delta^{(3)}(\vec{n}_a - s\vec{n}_b) + \frac{(1-3p)}{(4\pi)^2},$$

$$P^{\text{qutrit}}_W(\vec{n}_a,\vec{n}_b) = \frac{8p}{9\pi^2}\delta^{(8)}(\vec{n}_a - s\vec{n}_b) + (1-4p)\left(\frac{2}{(3\pi)^2}\right)^2. \quad (21)$$

For values of p for which these distributions are positive everywhere, the mixed Werner state has a classical interpretation and as a result is separable. We obtain that the Werner state is separable if $p \le \frac{1}{3}$ for a qubit and $p \le \frac{1}{4}$ for a qutrit.

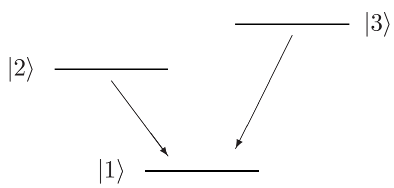

FIG. 1. Transitions allowed in a three-level atom with V configuration.

If we let the state evolve with time under the action of a channel, then as a result, we obtain the time evolution of $c_{\alpha\beta}^{\mathrm{state}}$ coefficients (qubit or qutrit state). Or equivalently, this results in a change of correlation matrix $\mathcal{C}_{ij} \to \mathcal{C}_{ij}(t)$. The condition on $P_W^{\mathrm{state}}(\vec{n}_a,\vec{n}_b)$ distributions to be positive everywhere can be translated into a condition on correlation matrix. We introduce the separability function $s_{\mathrm{state}}(t)$ for a qutrit Werner state $\rho_W(t)$,

$$s_{\mathrm{qutrit}}(t) \sim \sum_j |\mathcal{C}_{jj}(t)|, \tag{22}$$

which yields

$$s_{\mathrm{qutrit}}(t) = \frac{1}{12}\sum_{j=1}^{8} |c_{jj}^{\mathrm{qutrit}}|, \tag{23}$$

For $t=0$ we restore the previously discussed Werner state ρ_W, hence $s_{\mathrm{qutrit}}(0)=p$. In this representation, the Werner qutrit state is separable when $s_{\mathrm{qutrit}}(t) \leq \frac{1}{4}$. Since the formalism used here does not limit us only to Werner states as a set of initial states, one can, in principle, investigate other bipartite qutrit states, using the Gell-Mann matrices as a basis. For Werner qubit state evolution we can use well-known PPT criterion [14], which is exact. Hence, we can quantify the change of entanglement present in the qubit bipartite system.

III. THREE-LEVEL ATOMS AS QUTRITS

From this point of our analysis, we will consider a particular physical realization of a qutrit state, namely, three-level atoms with energies E_1, E_2, and E_3. Decoupling the level $|3\rangle$ from the remaining levels, we can easily reduce the qutrit state into a qubit. There are three configurations of three-level atoms that can be taken into account [20], and we will focus on the so-called V configuration. In the latter, only dipole transitions depicted on Fig. 1 between levels $|2\rangle \to |1\rangle$ and $|3\rangle \to |1\rangle$ are allowed

Usually, the atomic variables are populations p_i and coherences d_{ij} (corresponding to complex dipole moments between states $|i\rangle$ and $|j\rangle$, with $i,j=1,2,3$). And since $\mathrm{Tr}\{\rho\}=1$, there are only eight independent variables. These can be translated into the formalism of the qutrit Bloch vector $\vec{n}$, namely,

$$n_1 = \sqrt{3}\ \mathrm{Re}\ d_{12}^*, \quad n_2 = \sqrt{3}i\ \mathrm{Im}\ d_{12}^*,$$

$$n_4 = \sqrt{3}\ \mathrm{Re}\ d_{13}^*, \quad n_5 = \sqrt{3}i\ \mathrm{Im}\ d_{13}^*,$$

$$n_6 = \sqrt{3}\ \mathrm{Re}\ d_{23}^*, \quad n_7 = \sqrt{3}i\ \mathrm{Im}\ d_{23}^*,$$

$$n_3 = \frac{\sqrt{3}}{2}(1-2p_2-p_3), \quad n_8 = \frac{1}{2}(1-3p_3). \tag{24}$$

A. Qutrit evolution in the presence of spontaneous emission

Spontaneous emission is a dissipative process, in which the atom is coupled to electromagnetic vacuum. Equations for the evolution of the atomic variables in the presence of spontaneous emission are characterized by decay rates, Einstein coefficients A_2 and A_3, corresponding to transitions $|2\rangle \to |1\rangle$ and $|3\rangle \to |1\rangle$, respectively [20]. The coherent part of the atomic evolution is given by a free Hamiltonian as follows:

$$\mathcal{H} = \sum_i E_i |i\rangle\langle i|, \tag{25}$$

and spontaneous emission enters the equations through a noncoherent, dissipative part of the evolution. This dissipative part is described by a coupling of the atomic dipole moments with the vacuum electromagnetic field. In a rotating frame, where coherences d_{ij} oscillate with atomic detunings $E_i - E_j$ the equations for atomic variables take the following form:

$$\frac{dn_{1,2}}{dt} = -\frac{A_2}{2}n_{1,2}, \quad \frac{dn_{4,5}}{dt} = -\frac{A_3}{2}n_{4,5},$$

$$\frac{dn_{6,7}}{dt} = -\frac{A_2+A_3}{2}n_{6,7}, \tag{26}$$

for the qutrit coherences and

$$\frac{dn_3}{dt} = -A_2 n_3 - \frac{\sqrt{3}}{3}(A_3 - A_2)n_8 + \frac{\sqrt{3}}{6}(2A_2 + A_3),$$

$$\frac{dn_8}{dt} = -A_3 n_8 + \frac{A_3}{2} \tag{27}$$

for the qutrit populations. These SU(3) equations for $\vec{n}(t)$ can be written in the following matrix form:

$$\frac{d}{dt}\vec{n}(t) = \mathcal{M}\vec{n}(0) + \vec{m}_0, \tag{28}$$

where the matrix $\mathcal{M}$ and the inhomogeneous term $\vec{m}_0$ can be easily read from the previous equations. The same equations can be expressed via the Lindblad master equation for the dissipative process

$$\frac{d\rho(t)}{dt} = \sum_k \left[L_k \rho(t) L_k^\dagger - \frac{1}{2}\{\rho(t), L_k^\dagger L_k\} \right], \tag{29}$$

with two Lindblad jump operators L_k,

$$L_1 = \frac{1}{2}\sqrt{A_2}(\lambda_1 + i\lambda_2),$$

$$L_2 = \frac{1}{2}\sqrt{A_3}(\lambda_4 + i\lambda_5). \tag{30}$$

As a result, similar to the qubit case, we can write the solution as an affine transformation of the SU(3) Bloch vector

$$\vec{n}(t) = \mathcal{D}\vec{n}(0) + \vec{T}(t), \tag{31}$$

where the dumping matrix is

$$\mathcal{D} = D_{ii} + \frac{1}{\sqrt{3}}(e^{-A_3 t} - e^{-A_2 t})\delta_{i3,j8}, \tag{32}$$

with a diagonal part

$$D_{ii} = (e^{-A_2 t/2}, e^{-A_2 t/2}, e^{-A_2 t}, e^{-A_3 t/2}, e^{-A_3 t/2},\ e^{-(A_2+A_3)t/2}, e^{-(A_2+A_3)t/2}, e^{-A_3 t}). \tag{33}$$

The affine shift is a time-dependent translation

$$\vec{T}(t) = \frac{1}{2\sqrt{3}}(3 - e^{-A_3 t} - 2e^{-A_2 t})\delta_{j3} + \frac{1}{2}(1 - e^{-A_3 t})\delta_{j8}. \tag{34}$$

Thus, the density operator representing the state of an atom in the presence of spontaneous emission is of the form

$$\rho(t) = \frac{1}{3}\{\mathbb{I} + \sqrt{3}[\mathcal{D}\vec{n}(0) + \vec{T}(t)] \cdot \vec{\lambda}\}. \tag{35}$$

B. Completely positive maps and Kraus operators

The time evolution given by Eq. (35) defines a quantum channel with noise. Any channel acting on a density operator maps density operators into density operators [10,12,13,25]

$$\Phi:\ \rho_{\text{in}} \mapsto \rho_{\text{out}}. \tag{36}$$

In the case discussed in this paper ρ_{in} is the initial density operator $[\rho_{\text{in}}=\rho(0)]$ and $\rho_{\text{out}}=\rho(t)$. The interaction of the three-level atom with vacuum fluctuations is described by a unitary operation acting in a Hilbert space involving the field and the atomic degrees of freedom. The reduced dynamics, if physical, has to be described by a completely positive map that can be written in the form of the Kraus decomposition

$$\rho(t) = \sum_i \mathcal{K}_i(t)\rho_{\text{in}}\mathcal{K}_i^\dagger(t), \tag{37}$$

where $\mathcal{K}_i(t)$ are time-dependent Kraus operators satisfying the normalization condition

$$\sum_i \mathcal{K}_i^\dagger(t)\mathcal{K}_i(t) = \mathbb{I}. \tag{38}$$

C. Kraus operators for spontaneous emission channel

The action of spontaneous emission channel (SE channel) on the V atom, given by Eq. (35), can be represented by operator-sum representation. The set of Kraus operators is as follows:

$$\mathcal{K}_0(t) = k_{00}(t)\mathbb{I} + k_{03}(t)\lambda_3 + k_{08}(t)\lambda_8,$$

$$\mathcal{K}_1(t) = k_{11}(t)\lambda_1 + k_{12}(t)\lambda_2,$$

$$\mathcal{K}_2(t) = k_{24}(t)\lambda_4 + k_{25}(t)\lambda_5, \tag{39}$$

where

$$k_{00}(t) = \frac{1}{3}(1 + e^{-A_2 t/2} + e^{-A_3 t/2}),$$

$$k_{03}(t) = \frac{1}{2}(1 - e^{-A_2 t/2}),$$

$$k_{08}(t) = \frac{1}{2\sqrt{3}}(1 + e^{-A_2 t/2} - 2e^{-A_3 t/2}),$$

$$k_{11}(t) = \frac{1}{2}\sqrt{1 - e^{-A_2 t}},$$

$$k_{12}(t) = \frac{i}{2}\sqrt{1 - e^{-A_2 t}},$$

$$k_{24}(t) = \frac{1}{2}\sqrt{1 - e^{-A_3 t}},$$

$$k_{25}(t) = \frac{i}{2}\sqrt{1 - e^{-A_3 t}}. \tag{40}$$

With these values of $k_{ij}(t)$ the normalization condition (38) is satisfied.

The time dependence of the Kraus operators indicates that the infinitesimal Δt evolution is diffusive, i.e., we have

$$\mathcal{K}_1(\Delta t) = \sqrt{\Delta t}L_1,$$

$$\mathcal{K}_2(\Delta t) = \sqrt{\Delta t}L_2, \tag{41}$$

where we recognize the Lindblad jump operators. As a result the dissipative evolution is equivalent to a diffusive completely positive map that can be written in the form of the Lindblad equation (29).

IV. INFLUENCE OF SE CHANNEL ON STATE SEPARABILITY

A. SE channel action on qutrits

Action of the channel produces a time-dependent Werner state: $\Phi(\rho_W)=\rho_W(t)$. Therefore, the condition $p\leq\frac{1}{4}$ to produce a separable state will be replaced by a time-dependent condition. We can consider a channel that alters only one subsystem (for instance a),

$$\Phi_1(\rho_W)=\sum_{i=0}^{2}(\mathcal{K}_i^a\otimes\mathbb{I}^b)\rho_W(\mathcal{K}_i^a\otimes\mathbb{I}^b)^\dagger, \tag{42}$$

or a channel that independently incoherently changes both subsystems,

$$\Phi_2(\rho_W)=q\sum_{i=0}^{2}(\mathcal{K}_i^a\otimes\mathbb{I}^b)\rho_W(\mathcal{K}_i^a\otimes\mathbb{I}^b)^\dagger + (1-q)\sum_{i=0}^{2}(\mathbb{I}^a\otimes\mathcal{K}_i^b)\rho_W(\mathbb{I}^a\otimes\mathcal{K}^b)^\dagger, \tag{43}$$

where q is a probability parameter [needed to satisfy Eq. (38)]. The value of q is arbitrary, though the most natural choice would be $q=\frac{1}{2}$ corresponding to a symmetric channel.

In this case the separability condition is modified and is a function of time

$$s_{\text{qutrit}}(t)\equiv\frac{p}{8}(e^{-A_2t}+e^{-A_3t}+2e^{-1/2A_2t} + 2e^{-1/2A_3t}+2e^{-1/2(A_2+A_3)t})\leq\frac{1}{4}, \tag{44}$$

where for $t=0$ we recover the initial condition

$$s_{\text{qutrit}}(0)=p\leq\frac{1}{4}. \tag{45}$$

The function $s_{\text{qutrit}}(t)$ is shown in Fig. 2. We use dimensionless parameter A_1t instead of t itself, where A_1 is the Einstein coefficient for the qubit case (it appears in the qubit channel discussion). We introduce dimensionless parameters $A_{21}\equiv\frac{A_2}{A_1}$ and $A_{31}\equiv\frac{A_3}{A_1}$, which will illustrate the relative value of parameters A_1, A_2, and A_3. The maximally entangled state (meaning $p=1$) becomes separable with time. Two cases are shown, describing the SE channel characterized by different parameter values. Function $s_{\text{qutrit}}(t)$ is symmetric with respect to the change $A_2\leftrightarrow A_3$, however, the values of these parameters change the time in which $s_{\text{qutrit}}(t)$ reaches the threshold value $\frac{1}{4}$. Since the maximally entangled state loses its entanglement in a finite time, any less entangled state behaves in a similar way.

For initial pure qutrit state $\rho_W=|\Psi\rangle\langle\Psi|$, the fidelity of the SE channel is given by

$$\mathcal{F}_{\text{qutrit}}(t)=\langle\Psi|\Phi_2(|\Psi\rangle\langle\Psi|)|\Psi\rangle=\frac{1}{9}(1+e^{-A_2t/2}+e^{-A_3t/2})^2, \tag{46}$$

with

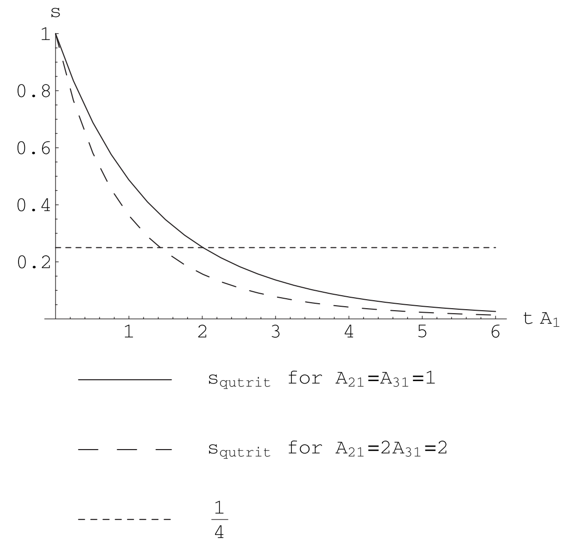

FIG. 2. Function $s_{\text{qutrit}}(A_1t)$ for $p=1$ for two cases: $A_{21}=\frac{A_2}{A_1}=A_{31}=\frac{A_3}{A_1}=1$ and $A_{21}=2A_{31}=2$. The region below $s=\frac{1}{4}$ corresponds to separable states.

$$\mathcal{F}_{\text{qutrit}}(t\to\infty)=\frac{1}{9}. \tag{47}$$

B. SE channel and Werner state for qubits

Kraus representation for spontaneous emission channel for qubits is given by (A_1 is the Einstein coefficient)

$$\mathcal{K}_0(t)=\frac{1}{2}(1+e^{-A_1t/2})\mathbb{I}-\frac{1}{2}(1-e^{-A_1t/2})\sigma_3,$$

$$\mathcal{K}_1(t)=\frac{1}{2}\sqrt{1-e^{-A_1t}}(\sigma_1-i\sigma_2). \tag{48}$$

Action of the channel produces a time-dependent qubit Werner state: $\Phi(\rho_W(0))=\rho_W(t)$. Therefore, the condition $p\leq\frac{1}{3}$ to produce a separable qubit state will change.

In case of qubit we can use PPT condition, which stands for calculating the partial transpose on the bipartite quantum state

$$\rho_A\otimes\rho_B\to\rho_A\otimes\rho_B^T. \tag{49}$$

The entanglement is present in the state when $\rho_A\otimes\rho_B^T$ has negative eigenvalues. It turns out that for the state which initially is a Werner state, the entanglement is due to the negativity of one of the eigenvalues of $\rho_A\otimes\rho_B^T$, namely:

$$\lambda=\frac{1}{4}[1-p\sqrt{x^2(1-p)^2+4p^2(1-p)}] \tag{50}$$

where $x=1-e^{-A_1t}$. In Fig. 3 we depict the region of these parameters p,x for which eigenvalue λ becomes positive, by inspecting the value of $\frac{1}{2}(\lambda+|\lambda|)$. We look only at $p>\frac{1}{3}$, for which we may have nonseparable states. The value $x=0$ corresponds to $t=0$, whereas $x=1$ corresponds to $t\to\infty$. Clearly,

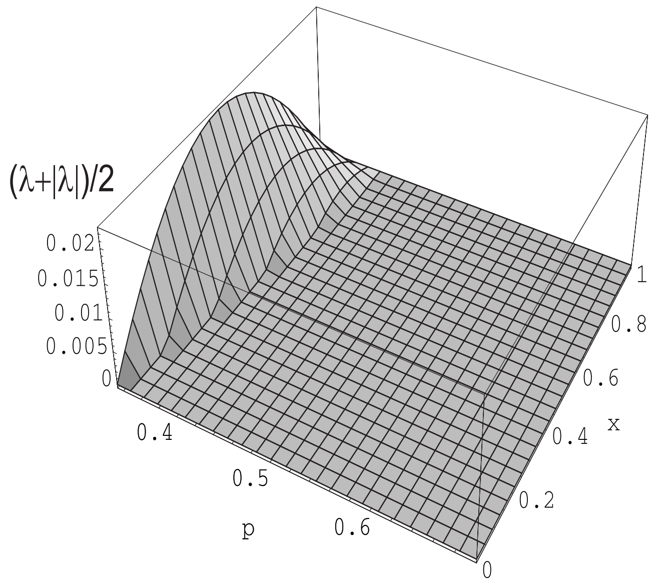

FIG. 3. Figure $\frac{1}{2}(\lambda+|\lambda|)$, showing the region of parameter values for which eigenvalue λ is positive. Positivity of $\frac{1}{2}(\lambda+|\lambda|)$ and λ corresponds to disentanglement of qubit state.

for low values of p the eigenvalue λ becomes positive, meaning that the state becomes separable. However, for states close to maximally entangled state ($p=1$), entanglement is preserved for all times.

Similar analysis of qubit disentanglement can be found in [26,27], where the authors investigate the disentanglement of two-qubit states through interaction with correlated noisy environment. Here we investigate only states which are symmetric, and hence they correspond to states with infinite entanglement lifetime (when $p=1$), in the discussion presented in [27,28].

A recent experiment confirms the finite lifetime of qubit entanglement [29]. In the next part of this section we discuss a similar effect for the qutrit.

Assuming that the initial state is a pure qubit state $\rho_W = |\Psi\rangle\langle\Psi|$, the fidelity of the SE channel is given by

$$\mathcal{F}_{\text{qubit}}(t) = \langle\Psi|\Phi_2^{\text{qubit}}(|\Psi\rangle\langle\Psi|)|\Psi\rangle = \frac{1}{4}(1+e^{-A_1 t/2})^2, \quad (51)$$

with

$$\mathcal{F}_{\text{qubit}}(t\to\infty) = \frac{1}{4}. \quad (52)$$

C. Comparison of qubit and qutrit states under the action of SE channels

Knowing how spontaneous channel acts on both qutrit and qubit states we can compare these two cases in order to state whether qutrit or qubit Werner states preserve entanglement longer. Clearly, for maximally entangled qubit states we have infinite entanglement lifetime, for maximally entangled qutrit states we detect finite lifetime. However, the measure that we employ in qutrit case is not exact (for 3×3 system we do not know the exact entanglement criterion, analog of PPT), hence the comparison between these two cases cannot be entirely precise. We can compare qutrit and qubit case for those values of p for which we detect disentanglement in qubit case. We show this comparison of SE channel action on Fig. 4, for $p=0.36$. Change of values of p, A_{21}, A_{31} leads to change in disentanglement time. In Fig. 4, parameter values are such that qubit state becomes disentangled faster.

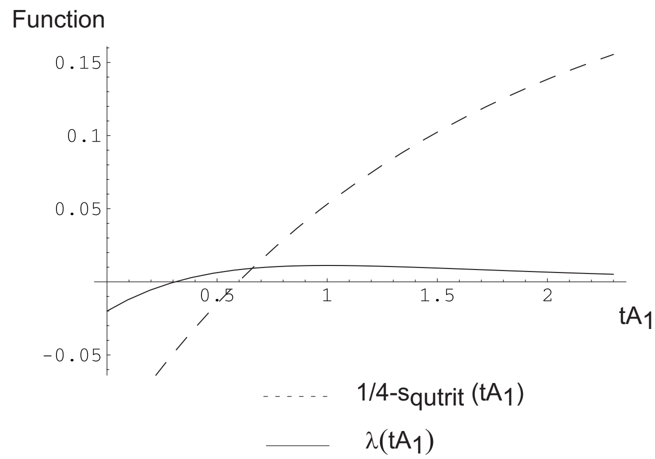

FIG. 4. Comparison of functions $(\frac{1}{4}-s_{\text{qutrit}}(A_1 t))$ and $\lambda(A_1 t)$ for $p=0.36$. When $\frac{1}{4}-s_{\text{qutrit}}(A_1 t)<0$ qutrit state is entangled, when $\lambda(A_1 t)<0$ qubit state is entangled.

In analogy, we compare fidelities of the SE channel for qubit and qutrit states. Choosing initial states to be pure states, the fidelity values converge with time to constant values, Eqs. (47) and (52). Moreover, we can state that

$$\mathcal{F}_{\text{qubit}}(t\to\infty) = \mathcal{F}_{\text{qutrit}}(t\to\infty) + \frac{5}{36}, \quad (53)$$

hence, eventually qubit transmission through the SE channel is better. However, the separability measure is more sensitive

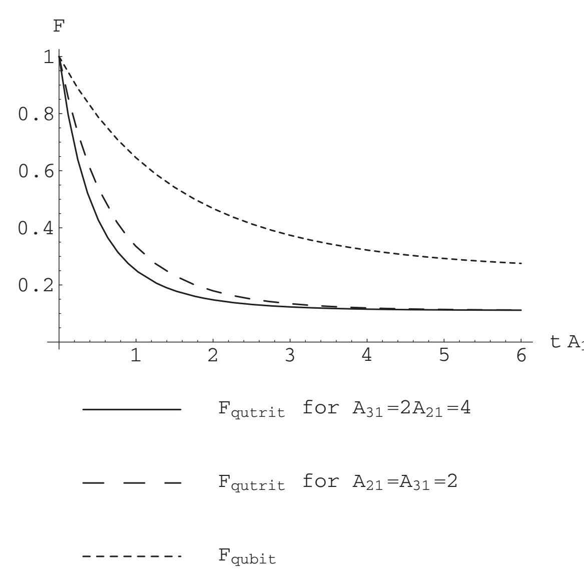

FIG. 5. Comparison of channel fidelities $\mathcal{F}_{\text{qutrit}}$ and $\mathcal{F}_{\text{qubit}}$ for initial pure qutrit and qubit states.

to relative values of parameters A_i than the fidelity function. For a proper selection of A_i qutrit state nonseparability is stronger, even though Eq. (53) holds. The fidelity comparison is shown in Fig. 5.

V. SUMMARY

We have presented an example of a qutrit state, namely, the three-level atom with V configuration, and its evolution under action of spontaneous emission channel. Separability of two qutrit states is, obviously, influenced by spontaneous emission. When compared with two qubit states, Werner qutrit states may preserve entanglement longer depending on channel and initial parameters. This result might be of some experimental importance when it comes to use of N-level atoms and multipartite entanglement. We plan to investigate further examples of qutrit channels and their influence on state separability. We aim as well at a general description of qutrit channels with respect to complete positivity.

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