Local cloning of entangled qubits

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We discuss the exact cloning of orthogonal but entangled qubits under local operations and classical communication. The amount of entanglement necessary in a blank copy is obtained for various cases. Surprisingly, this amount is more than 1 ebit for certain sets of two nonmaximal but equally entangled states of two qubits. To clone any three Bell states, at least $\log_2 3$ ebit is necessary.

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I. INTRODUCTION

Classical states can always be cloned perfectly but the quantum no cloning theorem [1] prohibits exact cloning of nonorthogonal states. However, orthogonal quantum states can always be cloned if one can perform a operation on the entire system.

A common scenario in quantum information processing is where a multipartite entangled state is distributed among a number of spatially separated parties. Each of these parties are able to perform only local operations on the subsystems they possess and can send only classical information to each other. This is known as local operation and classical communication (LOCC). If we restrict ourselves only to LOCC, further restrictions on cloning apply. For example, the very obvious first restriction will be that an entangled blank state is needed to clone an entangled state. Moreover, entanglement of the blank state should at least be equal to the entanglement of the state to be cloned or else entanglement of the entire system will increase under LOCC which is impossible. However, with a sufficient supply of entanglement; entangled states can be cloned by LOCC. For example, any arbitrary set of orthogonal states of two qubits can be cloned with the help of three ebits, any set of two orthogonal states needs only two ebits.

The concept of entanglement cloning under LOCC was first considered by Ghosh *et al.* [2] where it was shown that for LOCC cloning of two (four) orthogonal Bell states, one ebit (two ebits) of entanglement is necessary and sufficient. Since then much work has been done in this direction [3,4] which involves maximally entangled states. In this paper, we consider cloning of arbitrary but equally entangled orthogonal states under LOCC and obtain the following interesting results: (i) $\log_2 3$ ebit in the blank copy is necessary to clone any three Bell states. (ii) Local exact cloning of any two orthogonal entangled states is not possible with the help of same entanglement unless the states are maximally entangled. (iii) Even a maximally entangled state of two qubits may not help as a blank copy for cloning certain sets of two orthogonal nonmaximal equally entangled states if these states lie in the same plane.

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II. CLONING BELL STATES

The four Bell states are given as

$$|B_{mn}\rangle = \frac{1}{\sqrt{2}}\sum_{j=0}^{1} e^{2\pi i j n/2}|j\rangle|j\oplus m\rangle, \quad n,m=0,1, \tag{1}$$

where one qubit is held by Alice and the other is held by Bob. Recently, Ghosh *et al.* [2] have shown that any two Bell states can be cloned with the help of one ($\log_2 2$) ebit, whereas to copy all four Bell states, one needs at least two ($\log_2 4$) ebits of entanglement in the blank copy. Regarding three Bell states, Owari and Hayashi [4] have shown that any three Bell states cannot be cloned if only one ebit of entanglement is supplied as a resource. In this section, considering a property of entanglement, we not only prove the above but also provide the necessary amount of entanglement for such a cloning. Interestingly, this amount comes out as $\log_2 3$ ebits.

To obtain the necessary amount of entanglement needed in the blank copy for local cloning (from here on by "local cloning" or "cloning" we mean "exact cloning under LOCC") of three Bell states, we will make use of the fact that the relative entropy of entanglement cannot be increased by any LOCC operation. The relative entropy of entanglement for a bipartite quantum state ρ is defined by [5]

$$E_R(\rho) = \min_{\sigma\epsilon D(H)} S(\rho\|\sigma).$$

Here D is the set of all separable states on the Hilbert space H over which ρ is defined and $S(\rho\|\sigma)$ (the relative entropy of ρ to σ) is given by $S(\rho\|\sigma)\equiv \mathrm{tr}(\rho\log_2\rho)-\mathrm{tr}(\rho\log_2\sigma)$.

Let $\rho_1\in H^1$ and $\rho_2\in H^2$ be two quantum states and let $E_R(\rho_1)=S(\rho_1\|\sigma_1)$, $E_R(\rho_2)=S(\rho_2\|\sigma_2)$; i.e., $\sigma_1(\in H_1)$ and $\sigma_2(\in H_2)$ are the two separable states which minimize the relative entropies of ρ_1 and ρ_2, respectively. Let σ be the separable state belonging to the Hilbert space $H_1\otimes H_2$ which minimizes the relative entropy of $\rho_1\otimes\rho_2$. Then the

$$E_R(\rho_1\otimes\rho_2)\le S(\rho_1\otimes\rho_2\|\sigma_1\otimes\sigma_2) \tag{2}$$

equality holds when $\sigma_1\otimes\sigma_2=\sigma$.

It is known that [6]

$$S(\rho_1\otimes\rho_2\|\sigma_1\otimes\sigma_2) = S(\rho_1\|\sigma_1)+S(\rho_2\|\sigma_2), \tag{3}$$

hence

$$E_R(\rho_1 \otimes \rho_2) \le S(\rho_1\|\sigma_1) + S(\rho_2\|\sigma_2), \tag{4}$$

i.e.,

$$E_R(\rho_1 \otimes \rho_2) \le E_R(\rho_1) + E_R(\rho_2). \tag{5}$$

If cloning of three Bell states (e.g., $|B_{00}\rangle, |B_{01}\rangle, |B_{10}\rangle$) is possible with a known entangled state (say $|B\rangle$) as a blank copy (resource), then the state $\frac{1}{3}[|B_{00}^{\otimes 2}\rangle\langle B_{00}^{\otimes 2}| + |B_{01}^{\otimes 2}\rangle\langle B_{01}^{\otimes 2}| + |B_{10}^{\otimes 2}\rangle \times \langle B_{10}^{\otimes 2}|]$ along with the blank state $|B\rangle$ given as the input to the cloner will provide the output as

$$\rho_{\text{in}}\left(=\frac{1}{3}[|B_{00}^{\otimes 2}\rangle\langle B_{00}^{\otimes 2}| + |B_{01}^{\otimes 2}\rangle\langle B_{01}^{\otimes 2}| + |B_{10}^{\otimes 2}\rangle\langle B_{10}^{\otimes 2}|] \otimes |B\rangle\langle B|\right)$$

$$\to \rho_{\text{out}}\left(=\frac{1}{3}[|B_{00}^{\otimes 3}\rangle\langle B_{00}^{\otimes 3}| + |B_{01}^{\otimes 3}\rangle\langle B_{01}^{\otimes 3}| + |B_{10}^{\otimes 3}\rangle\langle B_{10}^{\otimes 3}|]\right).$$

We now compare the relative entropies of entanglement of ρ_{in} and ρ_{out}.

From inequality (5), we have

$$E_R(\rho_{\text{in}}) \le E_R\left(\frac{1}{3}[|B_{00}^{\otimes 2}\rangle\langle B_{00}^{\otimes 2}| + |B_{01}^{\otimes 2}\rangle\langle B_{01}^{\otimes 2}| + |B_{10}^{\otimes 2}\rangle\langle B_{10}^{\otimes 2}|]\right) + E_R(|B\rangle\langle B|)$$

as

$$E_R\left(\tfrac{1}{3}[|B_{00}^{\otimes 2}\rangle\langle B_{00}^{\otimes 2}| + |B_{01}^{\otimes 2}\rangle\langle B_{01}^{\otimes 2}| + |B_{10}^{\otimes 2}\rangle\langle B_{10}^{\otimes 2}|]\right) \le 2 - \log_2 3,$$

hence $E_R(\rho_{in}) \le 2 - \log_2 3 + E_R(|B\rangle\langle B|)$ [7]. At least two ebits of entanglement can be distilled from ρ_{out} [8,9] and the distillable entanglement is bounded above by E_R, hence $E_R(\rho_{\text{out}}) \ge 2$. However, as relative entropy of entanglement cannot increase under LOCC and, as in the output, we have at least two ebits of relative entropy of entanglement, hence, in order to make cloning possible, $\log_2 3$ ebits is necessary in the blank state. Any two qubit state (even a two qubit maximally entangled state) cannot provide this necessary amount of entanglement.

III. CLONING ARBITRARY ENTANGLED STATES

Any two equally entangled orthogonal states can lie either in same plane (I)

$$|\Psi_1\rangle = a|00\rangle + b|11\rangle,$$

$$|\Psi_2\rangle = b|00\rangle - a|11\rangle$$

or in different planes (II)

$$|\Psi_1\rangle = a|00\rangle + b|11\rangle,$$

$$|\Psi_3\rangle = a|01\rangle + b|10\rangle,$$

where $a, b$are real and unequal and $a^2 + b^2 = 1$.

In both cases, if one provides two entangled states, each having the same amount of entanglement as in the original one, cloning will become trivially possible. Here we investigate the nontrivial case when a single entangled qubit state is supplied as blank copy.

Case (I). Suppose there exists a cloning machine which can clone $|\Psi_1\rangle$ and $|\Psi_2\rangle$ when a pure entangled qubit state $|\Phi\rangle(=c|00\rangle + d|11\rangle;\ c^2 + d^2 = 1)$ is supplied to it as a blank copy. Let us supply an equal mixture of $|\Psi_1\rangle$ and $|\Psi_2\rangle$ together with the blank state $|\Phi\rangle$ to it, i.e., the state input to the cloner is

$$\rho_{\text{in}} = \left[\frac{1}{2}P(|\Psi_1\rangle) + \frac{1}{2}P(|\Psi_2\rangle)\right] \otimes P(|\Phi\rangle). \tag{6}$$

The output of the cloner

$$\rho_{\text{out}} = \frac{1}{2}P[|\Psi_1\rangle \otimes |\Psi_1\rangle] + \frac{1}{2}P[|\Psi_2\rangle \otimes |\Psi_2\rangle]. \tag{7}$$

For proving the impossibility of such a cloner, we make use of the fact that the Negativity of a bipartite quantum state ρ, $N(\rho)$ cannot increase under LOCC [10]. $N(\rho)$ is given by [11]

$$N(\rho) \equiv \|\rho^{T_B}\| - 1, \tag{8}$$

where ρ^{T_B} is the partial transpose with respect to system B and $\|\cdots\|$ denotes the trace norm which is defined as

$$\|\rho^{T_B}\| = \text{tr}(\sqrt{\rho^{T_B^\dagger}\rho^{T_B}}). \tag{9}$$

The negativity of the input state ρ_{in} is

$$N(\rho_{\text{in}}) = 2cd \le 1,$$

whereas, the negativity of the output is

$$N(\rho_{\text{out}}) = 4a^2b^2 + 4\sqrt{a^2b^2(a^2 - b^2)^2}.$$

The above cloning is not possible as long as

$$cd < 2a^2b^2 + 2\sqrt{a^2b^2(a^2 - b^2)^2}. \tag{10}$$

The above inequality has some interesting features, but the most significant feature is that even a maximally entangled state of two qubits cannot help as a blank copy for a large number of pairs of nonmaximally entangled states belonging to this class (see Fig. 1). Simple calculations show [13] that this is the case for $\sqrt{\frac{1}{2} - \frac{1}{\sqrt{5}}} < a < \sqrt{\frac{1}{2} + \frac{1}{\sqrt{5}}}$ (except for $a = \frac{1}{\sqrt{2}}$). This is surprising as recently Kay and Ericsson [12] gave a protocol by which all the pairs of states lying in different planes (II) can be cloned with the help of one free ebit. Other important features are (a) for $a = b = c = d = \frac{1}{\sqrt{2}}$, the above inequality becomes an equality. This is consistent with an earlier finding [2] that two maximally entangled bipartite states can be cloned with the help of one free ebit. (b) Inequality (10) holds even for $c = a \neq d = b$ (see Fig. 1). This in turn implies that same amount of entanglement (as in the state to be cloned) cannot help as blank copy, for any pair of nonmaximally entangled states.

Case (II). This time we suppose that our cloning machine can clone $|\Psi_1\rangle$ and $|\Psi_3\rangle$ if a pure entangled state $|\Phi\rangle(=c|00\rangle + d|11\rangle;\ c^2 + d^2 = 1)$ is used as a blank copy. Let the state supplied to this machine be

$$\rho_{\text{in}} = \frac{1}{2}[P(|\Psi_1\rangle) + P(|\Psi_3\rangle)] \otimes P[|\Phi\rangle].$$

We then have the output of the cloner

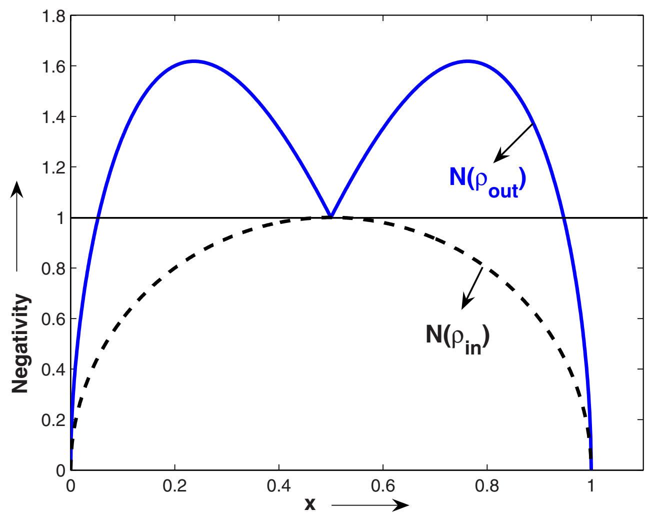

FIG. 1. (Color online) Broken line: Plot of $N(\rho_{\rm in})$ versus x, where $x=c^2(=1-d^2)$. Solid line: Plot of $N(\rho_{\rm out})$ versus x, where $x=a^2(=1-b^2)$. Please note that the negativity of the output is more than that of the input except for maximally (i.e., $x=\frac{1}{2}$) entangled ones.

$$\rho_{\rm out}=\frac{1}{2}P[|\Psi_1\rangle\otimes|\Psi_1\rangle]+\frac{1}{2}P[|\Psi_3\rangle\otimes|\Psi_3\rangle].$$

Putting $|\Psi_1\rangle$, $|\Psi_3\rangle$, and $|\phi\rangle$ in the expression for $\rho_{\rm in}$ and $\rho_{\rm out}$ and making use of Eqs. (8) and (9), we get

$$N(\rho_{\rm in})=2cd\le 1,$$

$$N(\rho_{\rm out})=2\sqrt{2(a^6b^2+a^2b^6)}.$$

From the nonincrease of negativity under LOCC it follows that as long as

$$cd<\sqrt{2(a^6b^2+a^2b^6)} \tag{11}$$

the above cloning is not possible. (a) $a=b=c=d=\frac{1}{\sqrt{2}}$ turns this inequality into an equality. This again is consistent with [2]. (b) If we put $c=a\neq d=b$ in the above inequality, i.e., if we use same amount of entanglement (as in the original states) then too cloning remains impossible as can be seen from Fig. 2. (c) Here too the inequality (11) shows that the necessary amount of entanglement in the blank copy is always greater than the entanglement of the original states unless they are maximally entangled. As an example, for $a=\sqrt{0.3}$, (i.e., entanglement of the state to be cloned is 0.8813), as long as $c<\sqrt{0.42}$, (i.e., entanglement of the blank copy <0.9815), cloning is not possible.

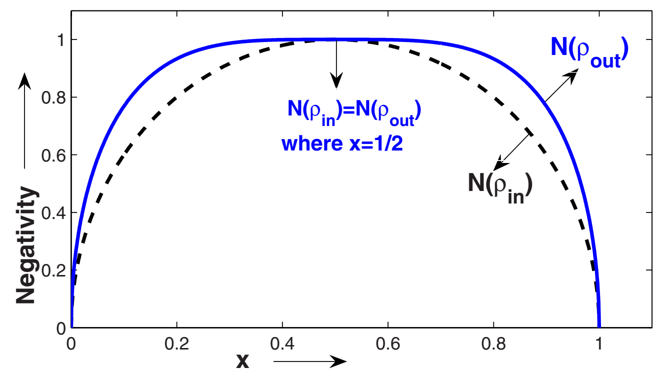

FIG. 2. (Color online) Broken line: Plot of $N(\rho_{\rm in})$ versus x, where $x=c^2(=1-d^2)$. Solid line: Plot of $N(\rho_{\rm out})$ versus x, where $x=a^2(=1-b^2)$. Please note that the negativity of the output is more than that of the input except for maximally (i.e., $x=\frac{1}{2}$) entangled ones.

IV. CONCLUSION

In this paper we addressed the problem of LOCC cloning for entangled states. To clone three Bell states, one needs at least $\log_2 3$ ebits in the blank state. So any two qubit state (pure or mixed) cannot serve this purpose. We have also shown that the blank state needed should be more entangled than the original ones for cloning any pair of nonmaximal but equally entangled orthogonal states. The necessary amount of entanglement in the blank state for such cloning to be possible is given by inequalities (10) and (11). Interestingly, this necessary amount is more than one ebit for certain sets of nonmaximal but equally entangled states contrary to certain other sets for which one ebit can serve as a blank copy.

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[8] The control-NOT operation C is defined as $C|i\rangle\otimes|j\rangle=|i\rangle\otimes|j\oplus i\rangle$, and the bilateral control-NOT operation (BXOR) defined on bipartite system as, $\mathcal{B}$ is $\mathcal{B}|i\rangle_{A1}|r\rangle_{B1}\otimes|j\rangle_{A2}|s\rangle_{B2}=|i\rangle_{A1}|r\rangle_{B1}$

$\otimes|j\oplus i\rangle_{A2}|s\oplus r\rangle_{B2}$. Denote $\mathcal{B}(m,n)$ as the BXOR operation performed on the mth pair (source) and the nth pair (target), the following operation will give $\mathcal{B}(1,3)\mathcal{B}(2,3)|B_{mn}^{\otimes 3}\rangle = |B_{m,0}^{\otimes(2)}\rangle|B_{\oplus 3m,n}\rangle$ [9]. If this operation is applied to ρ_{out}, one obtains $\frac{1}{3}[|B_{00}^{\otimes 2}\rangle\langle B_{00}^{\otimes 2}|\otimes[|B_{00}\rangle\langle B_{00}|+|B_{01}\rangle\langle B_{01}|]+|B_{10}^{\otimes 2}\rangle\langle B_{10}^{\otimes 2}| \otimes|B_{10}\rangle\langle B_{10}|]$. If Alice and Bob do the measurement in the $|0\rangle,|1\rangle$ basis on the third copy and communicate, the results will be either correlated or anticorrelated. When they are correlated the first two copies are in $|B_{00}\rangle$ and in the other case they are in state $|B_{10}\rangle$, therefore distilling two ebits in this process.

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