

Correspondence principle for the diffusive dynamics of a quartic oscillator: Deterministic aspects and the role of temperature

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The correspondence principle is investigated in the framework of deterministic predictions for individual systems. Exact analytical results are obtained for the quantum and Liouvillian dynamics of a nonlinear oscillator coupled to a phase-damping reservoir at a finite temperature. In this context, the time of critical wave function spreading—the Ehrenfest time—emerges as the characteristic time scale within which the concept of deterministic behavior is admissible in physics. A scenario of quasideterminism may then be defined, within which the motion is experimentally indistinguishable from the truly deterministic motion of Newtonian mechanics. Beyond this time scale, predictions for individual systems can be given only statistically and, in this case, it is shown that diffusive decoherence is indeed a necessary ingredient to establish the quantum-classical correspondence. Moreover, the high-temperature regime is shown to be an additional condition for the quantum-classical transition and, accordingly, a lower bound for the reservoir temperature is derived for our model.

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I. INTRODUCTION

A widely accepted explanation for the emergence of classical behavior from the quantum substratum is provided by the environment-induced decoherence (EID) program [1–4]: the interaction of the system with many uncontrollable external degrees of freedom (the environment) yields a dramatic destruction of quantum coherences, prohibiting, as a consequence, the existence of quantum superpositions in the macroscopic world. Decoherence, an experimentally demonstrated effect [5–7] to which real systems are inevitably submitted, is the essence of the arguments given by EID defenders to explain, for instance, why an astronomical body behaves classically [8,9] and to address fundamental questions, such as, for instance the Schrödinger cat paradox and the measurement problem [10].

Recent works, however, have discussed the real effectiveness of decoherence as a necessary mechanism to guarantee the quantum-classical transition. In Ref. [11], e.g., the authors show that a small amount of smoothing (due to apparatus resolution) is sufficient to ensure the classical limit. In [12], it is argued that diffusive decoherence can produce only an attenuation of quantum coherences, not their complete destruction. The authors thus conclude that the classical limit is a matter of experimental resolution. Another important objection raised in Refs. [11,13] points out the fact that decoherence is not able to reduce the spreading of the wave function. Yet, in Ref. [14] relevant questions concerning the role of the high-temperature limit and the generality of environment models have been addressed. This present debate prevents one from believing that the understanding of the quantum-classical transition is an exhausted subject.

In this work, the correspondence principle is discussed by means of an analytical analysis carried out for the dynamics of a nonlinear oscillator coupled with a phase-damping res-

ervoir composed by N harmonic oscillators in thermal equilibrium. Such a thermal bath has been shown to be effective in modeling nondissipative decoherence for a large range of finite temperatures [15]. With the help of this model, the quantum-classical correspondence is analyzed in two main directions, as indicated below.

(i) On one hand, the correspondence between the quantum centroid and the Newtonian trajectory is investigated. This analysis is motivated by Einstein’s paper to Born about the foundations of quantum mechanics [16]. Einstein concludes that the only possible interpretation for the wave function is the one based on statistical ensembles, but, on the other hand, he emphasizes the “inevitable conception that Physics must ferment a realistic description of only one system.” Indeed, “nature as a whole may be thought of as an individual system (existing only one time, with no need for repetitions) and not as an ‘ensemble of systems’” [16]. Einstein’s dissatisfaction is invoked here in terms of the following motivating question: What can quantum mechanics predict about the future of an individual system? Obviously, such a question intends to challenge the quantum formalism for realizing a task that is supposed to be an exclusive function of Newtonian mechanics. However, since physics is fundamentally concerned with experimental observations, which in turn are inevitably limited by the apparatus resolution, it is reasonable to conceive that neither the idea of an objective reality (usually associated with classical beliefs) nor a completely indeterministic scenario (usually associated with quantum world) can be maintained. In fact, as is argued in Sec. IV, once a margin of error is defined the concept of deterministic behavior can be physically accepted, at least for the short-time dynamics of initially localized wave functions. Moreover, it is shown that within such a short-time regime (the Ehrenfest regime) decoherence is not effective, in agreement with the claims of Refs. [11,13].

(ii) On the other hand, beyond this time scale within which deterministic behavior is expected to be experimentally verifiable, predictions for individual systems can be

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given only in terms of statistical averages and variances. In this case, the quantum-classical correspondence is investigated in terms of comparisons between quantum and Liouvilian results, for which the experimental resolution is also considered. In Sec. V, it is shown that decoherence is indeed a necessary mechanism to promote the quantum-classical correspondence. Our results show in addition that the reestablishment of the quantum-classical correspondence can be ensured by purely diffusive decoherence only within the regime of high temperatures.

II. THE MODEL

Recently, a model of the reservoir has been proposed that is able to describe nondissipative decoherence at finite temperatures [15]. Here, we investigate the dynamics of a system composed of such a reservoir coupled to the well-known [12,17,18] quartic oscillator. The Hamiltonian of the model reads

$$\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_I, \quad (1a)$$

where

$$\hat{H}_S = \hbar \omega_s \left(\hat{n}_s + \frac{1}{2} \right) + \hbar^2 g_s \left(\hat{n}_s + \frac{1}{2} \right)^2, \quad (1b)$$

$$\hat{H}_R = \sum_{k=1}^N \hbar \omega_k \left(\hat{n}_k + \frac{1}{2} \right), \quad (1c)$$

$$\hat{H}_I = \sum_{k=1}^N \hbar^2 g_k \left(\hat{n}_s + \frac{1}{2} \right) \left(\hat{n}_k + \frac{1}{2} \right). \quad (1d)$$

The constants g_s and g_k denote, respectively, the nonlinearity parameter and the coupling parameter, whereas ω_s and ω_k stand for the harmonic frequency of the nonlinear oscillator and the harmonic frequencies of the bath oscillators. The initial joint quantum state is assumed to be

$$\hat{\rho}(0) = |\alpha_0\rangle\langle\alpha_0| \otimes \frac{e^{-\beta\hat{H}_R}}{\text{Tr}(e^{-\beta\hat{H}_R})}, \quad (2)$$

where $\beta = (k_B T)^{-1}$ and T is the equilibrium temperature. The assumptions of separability (for the initial joint state) and purity (for the system state) are both convenient for a suitable construction of the respective classical distribution. The corresponding classical dynamics is described in terms of the Hamiltonian

$$H = H_S + H_R + H_I, \quad (3a)$$

where

$$H_S = \omega_s \left(\frac{p_s^2 + q_s^2}{2} \right) + g_s \left(\frac{p_s^2 + q_s^2}{2} \right)^2, \quad (3b)$$

$$H_R = \sum_{k=1}^N \omega_k \left(\frac{p_k^2 + q_k^2}{2} \right), \quad (3c)$$

$$H_I = \sum_{k=1}^N g_k \left(\frac{p_s^2 + q_s^2}{2} \right) \left(\frac{p_k^2 + q_k^2}{2} \right). \quad (3d)$$

For convenience, new canonical variables $q_k = X_k \sqrt{m_k \omega_k}$ and $p_k = P_k / \sqrt{m_k \omega_k}$ have been used instead of the usual position-momentum pair (X_k, P_k) . The quantum and classical Hamiltonians are correctly related by the scheme of bosonic quantization proposed in Ref. [19]. A Gaussian spectral distribution given by

$$g_k = \frac{\Omega}{\sqrt{N}} \exp\left(-\frac{\pi(k-k_0)^2}{2N^2}\right), \quad (4)$$

where $k_0 = N/2$ and Ω is constant, will be assumed in our calculations in the next sections. As has been shown in Ref. [15], it allows for the appropriate implementation of the thermodynamic limit ($N \rightarrow \infty$). For simplicity and convenience, the assumption $\omega_k = \omega_0$ will also be adopted here. As discussed in Ref. [15], although restrictive, this choice does not change the main properties of the model, namely, the Gaussian decay for the quantum coherences and the structural form of the decoherence time.

For the Liouvilian dynamics the following phase space distribution is used:

$$\rho(0) = \frac{e^{-(q_s - q_0)^2/\hbar} e^{-(p_s - p_0)^2/\hbar}}{\sqrt{\pi\hbar} \sqrt{\pi\hbar}} \prod_{k=1}^N \left(\frac{e^{-(p_k^2 + q_k^2)/z\hbar}}{z\pi\hbar} \right), \quad (5)$$

where

$$z^{-1} = \tanh\left(\frac{\beta\hbar\omega}{2}\right). \quad (6)$$

This distribution corresponds to the Wigner function of $\hat{\rho}(0)$ given by (2). Note that the Wigner function for the reservoir state reduces to the traditional Boltzmannian function $e^{-\beta H_R} / \text{Tr}(e^{-\beta H_R})$ only in the high-temperature regime ($\beta\hbar\omega \ll 1$).

There are several reasons supporting the choice of this model, namely, (i) as has been shown in [15], the phase-damping reservoir induces nondissipative decoherence at finite temperature, this being one important difference from other purely diffusive environments [12,20]; (ii) the nonlinear oscillator used as the system of interest constitutes a paradigm of nontrivial quantum dynamics which exhibits interference and quantum superposition of coherent states (cat state) formation [17,18]; and (iii) classical and quantum solutions are analytical and exact for all regimes of parameters, with no need for either Markovian or weak coupling assumptions.

III. ANALYTICAL RESULTS

A. Newtonian results

The Newtonian solution is obtained by integrating the system of $2(N+1)$ differential equations defined by Hamilton's equations $\dot{q}_k = \partial_{p_k} H$ and $\dot{p}_k = -\partial_{q_k} H$. The final result for the Newtonian trajectory of the system can be conveniently written in matrix notation as follows:

$$R_N(\tau) = \mathbb{M}_1[\phi_N]R_0, \quad (7a)$$

where

$$\phi_N(\tau) \equiv \frac{\omega_s \tau}{\hbar g_s} + \left(\frac{R_0^T R_0}{\hbar} \right) \tau + \sum_k \frac{g_k}{g_s} \left(\frac{p_k^2 + q_k^2}{\hbar} \right) \tau, \quad (7b)$$

$$\mathbb{M}_1[\chi] \equiv \begin{bmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{bmatrix}, \quad (7c)$$

where $R_N(\tau) \equiv \begin{pmatrix} q_s(\tau) \\ p_s(\tau) \end{pmatrix}$, $R_0 \equiv \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$, and $R_0^T R_0 = p_0^2 + q_0^2$. For convenience, all results have been written in terms of the following dimensionless parameter:

$$\tau = \hbar g_s t. \quad (8)$$

Note that the Newtonian solution does not depend at all on \hbar . The effect of the nonlinearity becomes evident in the equations above: the rotation matrix $\mathbb{M}[\phi_N]$ becomes dependent on the initial condition R_0 . This fact can be interpreted as the basic mechanism responsible for the twist found in the dynamics of classical distributions [12,17] and for the dynamical formation of cat states as shown in Ref. [18].

B. Quantum results

The analytical solution for the reduced density matrix associated with the system of interest is obtained by taking the trace of the global density matrix $\hat{\rho}(t) = e^{-i\hat{H}t/\hbar} \hat{\rho}(0) e^{i\hat{H}t/\hbar}$ over the reservoir degrees of freedom, i.e., $\hat{\rho}_S(t) = \text{Tr}_R \hat{\rho}(t)$. As shown in Ref. [15], the result can be written in the Fock basis as

$$\begin{aligned} \hat{\rho}_S(\tau) &= \sum_{n,n'=0}^{\infty} e^{-|\alpha_0|^2} \frac{\alpha_0^n}{\sqrt{n!}} \frac{(\alpha_0^*)^{n'}}{\sqrt{n'!}} \\ &\times e^{-i(n^2-n'^2)\tau} e^{-i(n-n')(1+\omega_s/\hbar g_s + G_1/2g_s)\tau} \\ &\times C_{n-n'}(\tau) |n\rangle \langle n'|, \end{aligned} \quad (9)$$

where $G_1 \equiv \sum_k g_k$ and

$$C_{n-n'}(\tau) \equiv \prod_{k=1}^N \frac{1 - e^{-\beta \hbar \omega_k}}{1 - e^{-\beta \hbar \omega_k} e^{-i(g_k/g_s)(n-n')\tau}}. \quad (10)$$

From (10) we see that

$$|C_{n-n'}| = \prod_k \left(1 + \frac{\sin^2[(g_k/g_s)(n-n')\tau/2]}{\sinh^2(\beta \hbar \omega_k/2)} \right)^{-1/2}. \quad (11)$$

For short times ($\tau \ll 1$) this function behaves as

$$|C_{n-n'}| \simeq e^{-(n-n')^2 \tau^2 / \tau_{DQ}^2}, \quad (12)$$

where τ_{DQ} , which denotes the characteristic decoherence time of the phase-damping reservoir, is given by

$$\tau_{DQ} = \sqrt{8} \frac{g_s}{G_2} \sinh\left(\frac{\beta \hbar \omega}{2}\right), \quad (13)$$

with $G_2 \equiv \sum_k g_k^2$. (In the thermodynamic limit $G_2 \simeq 0.89\Omega$.) The result (12) shows the character of Gaussian decay in-

duced by the phase-damping reservoir over the off-diagonal terms ($n-n' \neq 0$) of the density operator given by (9). Hereafter the assumption $\omega_k = \omega$, as suggested in [15], will be implemented for simplicity.

The column matrix $R_Q(\tau) \equiv \begin{pmatrix} \langle \hat{q}_s \rangle \\ \langle \hat{p}_s \rangle \end{pmatrix}$, defined in terms of the expectation values of the position and the momentum, is calculated with the help of the usual relations $\langle \hat{q}_s \rangle = \sqrt{\hbar/2} (\langle \hat{a}_s^\dagger \rangle + \langle \hat{a}_s \rangle)$ and $\langle \hat{p}_s \rangle = i\sqrt{\hbar/2} (\langle \hat{a}_s^\dagger \rangle - \langle \hat{a}_s \rangle)$, where $\langle \hat{a}_s \rangle = \text{Tr}[\hat{a}_s \hat{\rho}_S(\tau)]$ and $\langle \hat{a}_s^\dagger \rangle = \text{Tr}[\hat{a}_s^\dagger \hat{\rho}_S(\tau)]$. The calculations of these terms are too lengthy (though straightforward) in virtue of the many degrees of freedom of the system, and so they will be omitted. The final result can be written as

$$R_Q(\tau) = (\Gamma_{1Q} \mathbb{M}_1[\theta_{1Q}]) (\Lambda_{1Q} \mathbb{M}_1[\phi_{1Q}]) R_0, \quad (14a)$$

where

$$\Gamma_{1Q}(\tau) \equiv |C_{-1}(\tau)|, \quad (14b)$$

$$\Lambda_{1Q}(\tau) \equiv \exp\left[-\left(\frac{R_0^T R_0}{\hbar}\right) \sin^2 \tau\right], \quad (14c)$$

$$\theta_{1Q}(\tau) \equiv i \ln\left(\frac{|C_{-1}(\tau)|}{C_{-1}(\tau)}\right) + \frac{G_1 \tau}{2g_s}, \quad (14d)$$

$$\phi_{1Q}(\tau) \equiv 2\tau + \frac{\omega_s \tau}{\hbar g_s} + \left(\frac{R_0^T R_0}{2\hbar}\right) \sin(2\tau). \quad (14e)$$

The definition of the auxiliary dimensionless functions Γ_{1Q} , Λ_{1Q} , θ_{1Q} , and ϕ_{1Q} , in Eqs. (14b)–(14e) allowed us to write the final result (14a) in such a suitable compact form, which will be useful for further comparisons. Interestingly, in Eq. (14a) the term $\Lambda_{1Q} \mathbb{M}_1[\phi_{1Q}]$, depending solely on parameters associated with the quartic oscillator, and the term $\Gamma_{1Q} \mathbb{M}_1[\theta_{1Q}]$, related to the interaction with the reservoir, have been factorized. Note that Γ_{1Q} is an attenuation factor describing the environmental decoherence [see Eq. (12)], whereas the factor Λ_{1Q} is the one responsible for the well-known revivals occurring in the closed dynamics of the quartic oscillator [17,18,21]. In fact, in the absence of the reservoir ($\Omega=0$ and $\Gamma_{1Q} \mathbb{M}_1[\theta_{1Q}]=1$), one may verify from (14c) the occurrence of revivals at the instants

$$\tau_R = n\pi \quad (n = 1, 2, 3, \dots). \quad (15)$$

This phenomenon, which has been shown to be a signature of the self-interference mechanism [18], has no classical analog, as can be seen in the corresponding Liouvillian result given by Eq. (22e). In Refs. [12,17], numerical simulations comparing quantum (Wigner and Husimi) with classical distributions indeed demonstrate the absence of interference in the classical dynamics.

Note by (7a)–(7c) and (14a)–(14e) that one may rigorously verify that

$$\lim_{\hbar \rightarrow 0} R_Q = R_N \quad (16)$$

in the absence of an environmental reservoir ($\Omega=0$). In contrast, for $\Omega \neq 0$ this relation no longer holds, which can be

demonstrated from (11) by showing that $\lim_{\hbar \rightarrow 0} |C_{n-n'}| < 1$, for $t > 0$. It is important to realize that this is not an effect of the many degrees of freedom composing the reservoir. In fact, the inequality applies even for only one oscillator in the reservoir. Rather, it occurs because the entanglement among the subsystems does not vanish asymptotically as \hbar tends to zero (see, e.g., [22]).

For the quadratic terms, $R_{2Q} \equiv \begin{pmatrix} \langle \hat{q}_s^2 \rangle \\ \langle \hat{p}_s^2 \rangle \end{pmatrix}$, a really tedious but equally straightforward calculation yields

$$R_{2Q}(\tau) = \begin{pmatrix} R_0^T R_0 + \hbar \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \Gamma_{2Q} \Lambda_{2Q} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{R_0^T}{\sqrt{2}} \mathbb{M}_2[\theta_{2Q} + \phi_{2Q}] \frac{R_0}{2}, \quad (17a)$$

where

$$\Gamma_{2Q}(\tau) \equiv |C_{-2}(\tau)|, \quad (17b)$$

$$\Lambda_{2Q}(\tau) \equiv \exp \left[- \left(\frac{R_0^T R_0}{\hbar} \right) \sin^2(2\tau) \right], \quad (17c)$$

$$\theta_{2Q}(\tau) \equiv \iota \ln \left(\frac{|C_{-2}(\tau)|}{C_{-2}(\tau)} \right) + \frac{G_1 \tau}{g_s}, \quad (17d)$$

$$\phi_{2Q}(\tau) \equiv 6\tau + \frac{2\omega_s \tau}{\hbar g_s} + \frac{R_0^T R_0}{2\hbar} \sin(4\tau), \quad (17e)$$

$$\mathbb{M}_2[\chi] \equiv \begin{bmatrix} \cos \chi & \sin \chi \\ \sin \chi & -\cos \chi \end{bmatrix}. \quad (17f)$$

Once again, several dimensionless functions have been defined to guarantee compactness of the main result (17a)–(17f). Also, Γ_{2Q} and Λ_{2Q} are terms associated with the reservoir and with the quartic oscillator, respectively.

Then, for the quantum variances, written as

$$(\Delta R_Q)^2 \equiv \begin{pmatrix} \Delta q_Q^2 \\ \Delta p_Q^2 \end{pmatrix} = \begin{pmatrix} \langle \hat{q}_s^2 \rangle - \langle \hat{q}_s \rangle^2 \\ \langle \hat{p}_s^2 \rangle - \langle \hat{p}_s \rangle^2 \end{pmatrix}, \quad (18)$$

we have

$$\begin{aligned} (\Delta R_Q)^2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(\frac{\hbar}{2} + \frac{R_0^T R_0}{2} (1 - \Gamma_{1Q}^2 \Lambda_{1Q}^2) \right) \\ &+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{R_0^T}{\sqrt{2}} (\Gamma_{2Q} \Lambda_{2Q} \mathbb{M}_2[\psi_{2Q}] \\ &- \Gamma_{1Q}^2 \Lambda_{1Q}^2 \mathbb{M}_2[2\psi_{1Q}]) \frac{R_0}{\sqrt{2}}, \end{aligned} \quad (19)$$

where $\psi_{iQ} \equiv \phi_{iQ} + \theta_{iQ}$, with $i=1, 2$.

C. Liouvillian results

The time-evolved Liouvillian distribution is obtained by integrating the Liouville equation, as shown in Ref. [23]. The formal solution is given by

$$\rho(q_k, p_k, t) = \rho(q_k(-t), p_k(-t), 0). \quad (20)$$

This formula establishes that the result can be obtained by replacing the arguments q_k and p_k of the initial distribution $\rho(q_k, p_k, 0)$ given by Eq. (5) by the Newtonian trajectories evolved backward in time, $q_k(-t)$ and $p_k(-t)$, respectively.

Classical statistical averages are calculated as

$$\langle F \rangle_L(t) = \int dV F \rho(q_k, p_k, t), \quad (21)$$

where $dV = dq_s dp_s dq_1 dp_1 \cdots dq_N dp_N$.

With formulas (20) and (21), it is possible to calculate the Liouvillian averages $\langle q_s \rangle$, $\langle q_s^2 \rangle$, $\langle p_s \rangle$, and $\langle p_s^2 \rangle$, and also the related variances $\Delta q_L^2 = \langle q_s^2 \rangle - \langle q_s \rangle^2$ and $\Delta p_L^2 = \langle p_s^2 \rangle - \langle p_s \rangle^2$. As for the quantum analysis, although straightforward, the calculations are too lengthy and do not add any further information to the results, so they can be omitted. The Liouvillian results may be written as

$$R_L(\tau) = (\Gamma_{1L} \mathbb{M}_1[\theta_{1L}]) (\Lambda_{1L} \mathbb{M}_1[\phi_{1L}]) \mathbb{N} R_0, \quad (22a)$$

where

$$D_m(\tau) \equiv \prod_k \left[1 - \text{im} \left(\frac{g_k}{g_s} \right) \frac{z\tau}{2} \right]^{-1}, \quad (22b)$$

$$\Gamma_{1L}(\tau) \equiv |D_1(\tau)|, \quad (22c)$$

$$\eta_1(\tau) \equiv 1 + \tau^2, \quad (22d)$$

$$\Lambda_{1L}(\tau) \equiv \eta_1^{-1} \exp \left[- \left(\frac{R_0^T R_0}{\hbar} \right) \frac{\tau^2}{\eta_1(\tau)} \right], \quad (22e)$$

$$\theta_{1L}(\tau) \equiv \iota \ln \left(\frac{|D_1(\tau)|}{D_1(\tau)} \right), \quad (22f)$$

$$\phi_{1L}(\tau) \equiv \frac{\omega_s \tau}{\hbar g_s} + \frac{\tau}{\eta_1(\tau)} \left(\frac{R_0^T R_0}{\hbar} \right), \quad (22g)$$

$$\mathbb{N}(\tau) \equiv \frac{1}{\eta_1} \begin{bmatrix} 1 - \tau^2 & 2\tau \\ -2\tau & 1 - \tau^2 \end{bmatrix}, \quad (22h)$$

with $\mathbb{N}^T \mathbb{N} = \mathbb{N} \mathbb{N}^T = 1$. \mathbb{M}_1 is the rotation matrix defined by (7c), $R_L \equiv \begin{pmatrix} q_s \\ p_s \end{pmatrix}$, and $R_0 = \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$. In Eq. (22b), the index m is such that $m=1, 2$ (see, e.g., Eqs. (22c) and (25b)).

In the short-time regime the classical attenuation factor reduces to

$$|D_m(\tau)| = e^{-m^2 \tau^2 / \tau_{DL}^2}, \quad (23)$$

where τ_{DL} , the Liouvillian analog of the decoherence time (13), is given by

$$\tau_{DL} = \sqrt{8} \frac{g_s}{G_2} \tanh \left(\frac{\beta \hbar \omega}{2} \right). \quad (24)$$

Once again the reservoir contributions could be factorized in the analytical solution for the centroid. Here, however, an

extra rotation associated with the nonlinearity of the oscillator is imposed upon the initial condition R_0 through the matrix \mathbb{N} .

It is worth emphasizing that the attenuation factor Λ_{1L} does not exhibit any revival, as has been mentioned. While quantum distributions are able to suffer self-interference, which in turn produces relocalization and revival in the amplitude of the centroid, classical distributions do not interfere, as shown in [12,17].

For the quadratic terms $R_{2L} \equiv \begin{pmatrix} \langle q_s^2 \rangle \\ \langle p_s^2 \rangle \end{pmatrix}$, we have

$$R_{2L}(\tau) = \begin{pmatrix} R_0^T R_0 + \hbar \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \Gamma_{2L} \Lambda_{2L} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{R_0^T}{\sqrt{2}} \mathbb{M}_2 [\theta_{2L} + \phi_{2L}] \mathbb{N}_2 \frac{R_0}{\sqrt{2}}, \quad (25a)$$

where

$$\Gamma_{2L}(\tau) \equiv |D_2(\tau)|, \quad (25b)$$

$$\Lambda_{2L}(\tau) \equiv \eta_2^{-1} \exp \left[-4 \left(\frac{R_0^T R_0}{\hbar} \right) \frac{\tau^2}{\eta_2(\tau)} \right], \quad (25c)$$

$$\eta_2(\tau) \equiv 1 + 4\tau^2, \quad (25d)$$

$$\theta_{2L}(\tau) \equiv \iota \ln \left(\frac{|D_2|}{D_2} \right), \quad (25e)$$

$$\phi_{2L}(\tau) \equiv \frac{2\omega_s \tau}{\hbar g_s} + 2 \left(\frac{R_0^T R_0}{\hbar} \right) \frac{\tau}{\eta_2}, \quad (25f)$$

$$\mathbb{N}_2 \equiv \frac{1}{\eta_2^2} \begin{bmatrix} 1 - 12\tau^2 & 2\tau(3 - 4\tau^2) \\ -2\tau(3 - 4\tau^2) & 1 - 12\tau^2 \end{bmatrix}. \quad (25g)$$

For the Liouvillian variances, written as

$$(\Delta R_L)^2 \equiv \begin{pmatrix} \Delta q_L^2 \\ \Delta p_L^2 \end{pmatrix} = \begin{pmatrix} \langle q_s^2 \rangle - \langle q_s \rangle^2 \\ \langle p_s^2 \rangle - \langle p_s \rangle^2 \end{pmatrix}, \quad (26)$$

the following result has been obtained:

$$\begin{aligned} (\Delta R_L)^2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\frac{\hbar}{2} + \frac{R_0^T R_0}{2} (1 - \Gamma_{1L}^2 \Lambda_{1L}^2) \right] \\ &+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{R_0^T}{\sqrt{2}} (\Gamma_{2L} \Lambda_{2L} \mathbb{M}_2 [\psi_{2L}] \mathbb{N}_2 \\ &- \Gamma_{1L}^2 \Lambda_{1L}^2 \mathbb{M}_2 [2\psi_{1L}] \mathbb{N}_1) \frac{R_0}{\sqrt{2}}, \end{aligned} \quad (27)$$

where $\psi_{iL} \equiv \phi_{iL} + \theta_{iL}$ with $i=1,2$.

Figures 1 and 2 illustrate the main characteristics of quantities such as $R_N^T R_N = R_0^T R_0$ (Newtonian), $R_Q^T R_Q = \Gamma_{1Q}^2 \Lambda_{1Q}^2 R_0^T R_0$ (quantum), and $R_L^T R_L = \Gamma_{1L}^2 \Lambda_{1L}^2 R_0^T R_0$ (Liouvillian) for the closed ($\Omega=0$) and the open ($\Omega>0$) dynamics of the quartic oscillator. Quadratic terms such as $R^T R$, constructed with the results (7a), (14a), and (22a), are convenient to plot because they do not display the oscillations due

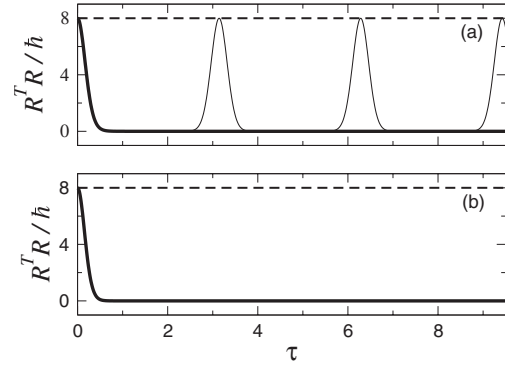


FIG. 1. Behavior of $R^T(\tau)R(\tau)/\hbar$ as a function of τ for Newtonian (dashed lines), quantum (thin solid lines), and Liouvillian (thick solid lines) theories. These numerical results have been obtained for $R_0^T R_0/\hbar=8.0$ and $g_s/\Omega=5.0$ (weak coupling). In (a) the simulations are shown for the closed dynamics ($\Omega=0$), for which the revival phenomenon can be observed in the quantum dynamics at the instants $n\pi$. In (b) we see the results for the open dynamics ($\Omega=0.1$ and $N=50$) in a regime of high temperature ($\beta\hbar\omega=0.1$). The quantum result becomes classical, in the Liouvillian sense, due to environmental decoherence.

to the rotation matrix \mathbb{M}_1 . The effect of diffusive decoherence in destructing quantum revivals becomes evident in the figures. Next, the correspondence principle is discussed with the help of these results.

IV. QUASIDETERMINISM

The concept of *determinism* is mainly concerned with philosophical beliefs, according to which there are laws (hidden or not) univocally connecting the past and the future of every dynamical system, including animals and human be-

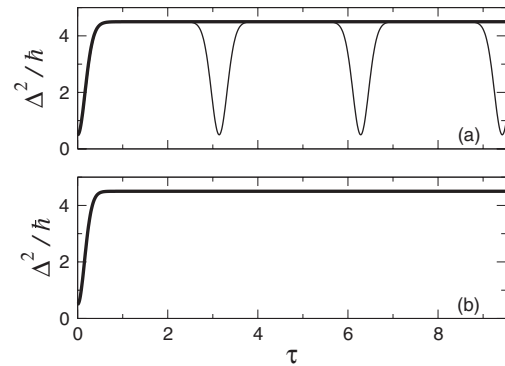


FIG. 2. Behavior of the mean variance defined by $\Delta^2 \equiv (\Delta q^2 + \Delta p^2)/2$ as a function of τ for quantum (thin solid line) and Liouvillian (thick solid line) theories in units of \hbar . These numerical results have been obtained for the same parameter values as in Fig. 1. The simulations for the closed dynamics ($\Omega=0$) are shown in (a), where it is possible to see the relocalization of the quantum wave function at the instants $n\pi$. In (b) we see the results for the open dynamics ($\Omega=0.1$ and $N=50$) in a regime of high temperature ($\beta\hbar\omega=0.1$). Decoherence induces diffusion, which makes the quantum variances similar to the Liouvillian ones.

ings. There is no chance for free choice. Note that the success of Newtonian theory in making predictions for the macroscopic world gives scientific support for such a philosophical concept. On the other hand, any attempt to apply the concept to the microscopic world will fail, once the notion of a trajectory linking the past and the future does not exist in the quantum realm. In fact, variances are always present, as predicted by Heisenberg's uncertainty principle. Despite this fact, it is still possible to defend the idea of determinism by speculating on the existence of some sort of hidden laws (or *hidden variables*).

This paper, however, does not intend to settle this controversial question from an ontological viewpoint, asserting at the end whether determinisms exist or not. Rather, the aim is to use the tools of physical science, namely, fundamental laws and experimental investigations, to evaluate under which conditions the concept of determinism is acceptable in physics. In particular, the following test will be employed here: if the position and the momentum of a particle can be correctly predicted at a given instant for an individual system, then the concept of determinism will be regarded as a meaningful one in physics. This is what we call a scenario of *quasideterminism*, since the motion will be experimentally indistinguishable from the truly deterministic Newtonian one. It is important to keep in mind this specific connotation attributed here to the word determinism, since it will be used from now on in the discussion about the central problem: the correspondence principle.

The argument motivating the historical statement of the correspondence principle is the one according to which quantum results must match classical ones because the macroscopic world is correctly described by classical physics. However, from a rigorous point of view, this reasoning cannot be considered as completely correct. Actually, it strongly depends on our capability to make an experimental check. Extremely accurate measurements on the center-of-mass position of a macroscopic projectile near the surface of the Earth would indicate some dispersion around the Newtonian parabolic trajectory. In fact, once we admit that the classical world is governed by quantum laws at a microscopic level, we have to conceive the center-of-mass motion as being determined by the quantum dynamics of all quantum particles composing the projectile. Accordingly, the center of mass is expected to behave quantum mechanically, with variances such as $\Delta x_{c.m.} = (\langle \hat{x}_{c.m.}^2 \rangle - \langle \hat{x}_{c.m.} \rangle^2)^{1/2}$ being associated with its motion. That is, in principle, the quantum nature of the center of mass, and thus the deviation from Newtonian predictions, could be verified by such accurate measurements. In this context, the status of the Newtonian mechanics as the basis for enunciating the correspondence principle is questionable.

To circumvent these difficulties, the following approach is proposed. We assume that the quantum-classical correspondence is achieved when quantum predictions are consistent with the concept of determinism (such as defined above). That is, if quantum mechanics is capable of making correct predictions for an individual system (which is expected to occur in a macroscopic regime), then the correspondence principle will be satisfied. As a consequence, since in this case the wave function dispersion is expected not to be de-

tectable by virtue of the insufficient experimental resolution, Newtonian predictions can be successfully applied as well. Then, the approach is the following: for insufficient experimental accuracy, quantum dispersion is not apparent, Newtonian predictions are adequate, and the concept of deterministic behavior survives in physics. In this case, by virtue of Ehrenfest's theorem, quantum mechanics will also succeed in making predictions for individual systems, at least for the short-time dynamics of initially localized wave functions (see, e.g., the famous example of the chaotic moon of Saturn [8,9], for which the estimated Ehrenfest time is about two decades), and, finally, Einstein's requirements will be satisfied.

Let us consider, as a first example, the quantum dynamics of a free particle initially prepared in a minimal-uncertainty state. A well-known calculation [24] shows that the product of the quantum variances evolves as

$$\Delta x(t)\Delta p(t) = \frac{\hbar}{2} \sqrt{1 + \left(\frac{\hbar t}{2m\Delta x_0}\right)^2}. \quad (28)$$

Consider now an experience in which the phase space resolution, defined by the measurement device, is given by the action δS , which here will be given by $\delta S = M\hbar/2$, M being an arbitrary real number. The product given in (28) achieves the critical value δS for

$$t_{\text{det}} = t_E M \sqrt{1 - \frac{1}{M^2}}, \quad (29)$$

where

$$t_E = \frac{2m\Delta x_0^2}{\hbar}. \quad (30)$$

t_E is the time scale first derived by Ehrenfest [25] for the critical spreading of the free-particle wave function. (In his work, Ehrenfest showed that t_E can reach large values for macroscopic particles.) Now, within the time scale defined by t_{det} , the quantum dispersion of the wave function is smaller than the experimental resolution, so it is not possible to distinguish the quantum character of the phenomenon. In this case, Newtonian mechanics can be successfully applied to make predictions about the future phase space state (x, p) of the particle. In other words, up to the instant t_{det} , the idea of a deterministic behavior is physically acceptable (within the experimental precision considered). Then, within this time scale, quantum mechanics satisfies Einstein's principle as a theory capable of making predictions for individual systems, and, in turn, the quantum-classical correspondence (as we have defined it) has been achieved. Beyond this time scale, quantum mechanics indeed fails in making predictions for individual systems, but exactly the same thing occurs with Newtonian mechanics. The determinism has been lost and the correspondence breaks off.

Now these ideas are applied to the dynamics of the nonlinear oscillator in the presence of a reservoir. Is decoherence capable of extending the time scale of determinism? It has already been pointed out in some works [11,13] that decoherence is not able to reduce the wave function spreading. Since the spreading is the basic mechanism responsible for

the breakdown in determinism, as has been suggested by the example above, one can anticipate that decoherence is not able to realize such a task. In fact, that can be demonstrated in our model as follows.

First, it is easy to see that decoherence does not restrain spreading. Using $g_s=0$ (the system of interest is now a harmonic oscillator) one obtains, by (12) and (19), the following long-time ($\tau > \tau_{DQ}$) variance product:

$$\Delta q_Q(\infty)\Delta p_Q(\infty) = \frac{\hbar}{2} + \frac{R_0^T R_0}{2}. \quad (31)$$

This result is larger than $\hbar/2$, which is the value obtained for the closed dynamics of a harmonic oscillator. Clearly, purely diffusive decoherence has induced diffusion.

Second, let us consider the nonlinear oscillator under decoherence in a situation in which the experimental resolution is written as $\delta S = M\hbar/2$. Since q and p play essentially the same role in our model, we chose for convenience an average phase space variance $\Delta^2 \equiv (\Delta q_Q^2 + \Delta p_Q^2)/2$ instead of the product $\Delta q_Q \Delta p_Q$. Defining the determinism break time τ_{det} as the instant at which the mean variance becomes comparable to the experimental resolution ($\Delta^2 = \delta S$), one may use (19) to obtain

$$1 - \Gamma_{1Q}^2(\tau_{\text{det}})\Lambda_{1Q}^2(\tau_{\text{det}}) = \frac{\hbar}{R_0^T R_0}(M - 1). \quad (32)$$

Interestingly, a quite similar result is obtained when one compares the quantum centroid (14a) with the Newtonian trajectory (7a) through the prescription $\frac{1}{2}(R_N^T R_N - R_Q^T R_Q) = \delta S$. This attests to the consistency of our approach: the determinism break time τ_{det} , which corresponds to the time of critical spreading, is directly related to the instant at which the quantum centroid and Newtonian trajectory stop agreeing.

Expanding Eq. (32) for $\tau_{\text{det}} \ll 1$ one obtains

$$\tau_{\text{det}} = \frac{\tau_{\text{det}}^{(c)}}{\sqrt{1 + (\tau_{\text{det}}^{(c)}/\tau_{DQ})^2 2R_0^T R_0/(M-1)\hbar}}, \quad (33)$$

where

$$\tau_{\text{det}}^{(c)} = \sqrt{\frac{(M-1)\hbar^2}{2(R_0^T R_0)^2}}. \quad (34)$$

In these expressions $\tau_{\text{det}}^{(c)}$ denotes the break time for the *closed* dynamics of the nonlinear oscillator and τ_{DQ} the decoherence time given by Eq. (13). Equation (33) demonstrates that decoherence does not extend the determinism time scale, since $\tau_{\text{det}} < \tau_{\text{det}}^{(c)}$. Therefore, in the context of Einstein's requirements for individual systems, decoherence plays no essential role.

In addition, by (8) one may write $\tau_{\text{det}} = \hbar g_s t_{\text{det}}$ to show, with the help of the results above, that $t_{\text{det}} \propto \hbar^{-1/2} \propto t_E$, where t_E is the Ehrenfest time for the quartic oscillator [19,26]. This is another indication that the time scale of determinism is indeed equivalent to the Ehrenfest one.

V. THE QUANTUM TO CLASSICAL TRANSITION

Once we accept the validity of the uncertainty principle at the basis of microscopic phenomena, we are forced to regard

the Newtonian mechanics as an incomplete theory. In fact, since the initial phase space state (q_0, p_0) of a particle can never be determined with arbitrary precision, the future phase space state can be predicted only statistically. Hence, the feeling of an objective reality disappears. In this scenario, the Liouvillian theory turns out to be the more general classical theory, whereas the Newtonian theory works as an approximation for the short-time regime of narrow distributions, as suggested in the previous section. Thus, the correspondence principle is now discussed by comparing quantum with Liouvillian results, for times beyond the determinism time scale defined in the previous section.

First, one should note from our analytical and exact results that there is no regime of parameters (with or without the reservoir) capable of making quantum and classical results mathematically identical. Although apparently naive, this sentence emphasizes, in agreement with [11–13], the approximate character of the quantum-classical correspondence and, in turn, the role of the experimental resolution in defining it. Accordingly, once again the theoretical action $\delta S = M\hbar/2$, with M real, will play the role of the experimental phase space resolution in our calculations.

The future phase space state of an individual system subjected to either quantum or classical statistical fluctuations can be predicted only in terms of averages and variances as $\langle\langle q \rangle\rangle, \langle\langle p \rangle\rangle \pm \frac{1}{2}(\Delta q, \Delta p)$. Then, the quantum-classical correspondence will be investigated in terms of the following differences:

$$D(\tau) = \frac{1}{2} |\langle R \rangle_Q^T(\tau) \langle R \rangle_Q(\tau) - \langle R \rangle_L^T(\tau) \langle R \rangle_L(\tau)| \quad (35)$$

for the centroids and

$$D(\tau) = \left| \frac{\Delta q_Q^2(\tau) + \Delta p_Q^2(\tau)}{2} - \frac{\Delta q_L^2(\tau) + \Delta p_L^2(\tau)}{2} \right| \quad (36)$$

for the variances.

These definitions disregard, for simplicity, the influence of the rotation matrices $\mathbb{M}_{1,2}$ and \mathbb{N} , but capture the attenuation effects due to both the reservoir and the nonlinearity. Using the results (14a), (22a), (19), and (27), it is possible to show that both equations above reduce to

$$D(\tau) = |\Gamma_{1Q}^2(\tau)\Lambda_{1Q}^2(\tau) - \Gamma_{1L}^2(\tau)\Lambda_{1L}^2(\tau)| \frac{R_0^T R_0}{2}. \quad (37)$$

This result, demonstrating the equivalence between (35) and (36), suggests the existence in this model of a common dynamics for the quantum-classical differences associated with the centroids and with the variances. As a consequence, one may conceive the existence of a single time scale describing the breakdown in the correspondence, this being a counterexample contributing to the discussion raised in Ref. [27].

In Fig. 3 the behavior of the function D is shown as a function of τ for the closed and open dynamics of the nonlinear oscillator in two different regimes of temperature. Note that diffusive decoherence, modeled in terms of our phase-damping reservoir, is indeed effective in restoring the quantum-classical correspondence. However, a word concerning the role of temperature is in order.

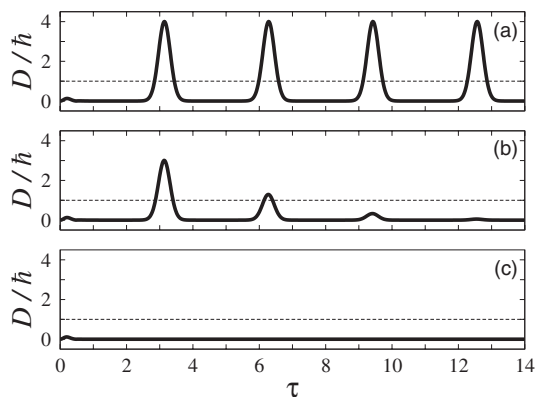


FIG. 3. Numerical results of $D(\tau)/\hbar$ obtained for the same parameter values as in Fig. 1. The horizontal dashed lines indicate the critical resolution in units of \hbar (here, $\delta S/\hbar=1$). In (a), where the simulation for the closed dynamics ($\Omega=0$) is plotted, the differences occur due to the quantum revival. In (b) and (c) we see the results for the open dynamics ($\Omega=0.1$ and $N=50$), for which $\beta\hbar\omega=1.0$ in (b) and 0.1 in (c). Note that the quantum results become classical only in the high-temperature regime.

Results (13) and (24) indicate that the attenuation effects due to the reservoir may occur also for low temperatures. However, in such a regime the differences between quantum and Liouvillian decay times become more accentuated. Indeed, from (13) and (24) it is easy to show that

$$\tau_{DQ} = \tau_{DL} \cosh\left(\frac{\beta\hbar\omega}{2}\right). \quad (38)$$

Now, the differences in the decays for the low-temperature regime ($\beta\hbar\omega \gg 1$) become evident. This result turns out to be an indication that the high-temperature regime is an additional *necessary* condition for the establishment of the quantum-classical transition. However, this is not the whole truth, since the condition $\tau_{DQ} = \tau_{DL}$ does not automatically imply that $\tau_{DQ} < \tau_R$, where τ_R is the revival time given by (15). In fact, since τ_R is the time scale that determines the appearance of purely quantum effects, such as quantum interference, the classical limit is guaranteed by imposing the requirement that decoherence occurs early. In general, one may state that the quantum-classical transition is achieved as long as the decoherence time is smaller than the time scale for which quantum phenomena (e.g., self-interference) take place in the dynamics. In our model this condition implies, by (13) and (15), that

$$\frac{k_B T}{\hbar\omega} > \frac{g_s}{2\Omega}, \quad (39)$$

where the high-temperature regime ($\beta\hbar\omega \ll 1$) and the thermodynamic limit ($N \rightarrow \infty$, $G_2 \rightarrow 0.89\Omega$) have already been considered. This simple relation establishes the lower bound for the value of the reservoir temperature capable of yielding

the quantum-classical transition. In addition, this result shows that nondissipative decoherence inducing the quantum-classical transition can occur at finite temperatures.

VI. SUMMARY

In this paper, the exact quantum and Liouvillian dynamics of a quartic oscillator coupled with a purely diffusive reservoir at arbitrary temperature have been solved analytically in terms of expectation values and variances associated with the phase space variables. The results demonstrate the effectiveness of our reservoir model in describing nondissipative decoherence and inducing quantum-classical transition.

Concerning the correspondence principle, this paper intends to contribute in two main directions. On one hand, examples were given showing that the Ehrenfest time is intimately connected with the time scale within which the concept of determinism (suitably defined) is acceptable in physics. In fact, within the Ehrenfest time scale, initially localized wave functions remain sufficiently narrow in such a way that its dispersion cannot overcome the limits imposed by the experimental resolution. In this case, Newtonian and quantum mechanics both succeed well in making predictions for individual systems, thus satisfying Einstein's requirements. Beyond the Ehrenfest time scale, quantum predictions deviate from Newtonian ones, but the latter are no longer correct. This analysis gives an answer to Einstein's dissatisfaction and also attributes a new physical meaning to the Ehrenfest time. In addition, it has been shown that diffusive decoherence plays no essential role within such a time scale, this being a point of agreement with the claims of Refs. [11,13].

On the other hand, in a second direction, this paper shows that, although decoherence is indeed necessary to reestablish quantum-classical correspondence beyond the determinism time scale, a certain amount of coarse graining (related to experimental resolution) is also required in order to characterize the correspondence in a more precise way. This is necessary since the quantum-classical correspondence cannot be fully achieved at a mathematical level. Also, our results point to the fact that the high-temperature regime is another fundamental ingredient in the scenario of the quantum-classical correspondence. A quantification of how high the reservoir temperature must be, in order to guarantee the correspondence, is given for our model in terms of a lower bound for the temperature. Although the assumption of $T \rightarrow \infty$ appears in several approaches (see, e.g., [12,20]) as an artifact to simulate the effects of diffusive decoherence from the traditional dissipative master equations, here it is justified in terms of physical conditions. Yet our approach points to the possibility of modeling diffusive decoherence at finite temperatures.

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