Complementarity and entanglement in bipartite qudit systems

Matthias Jakob[1,2] and János A. Bergou[3]

[1]*ARC Seibersdorf Research GmbH, Tech Gate Vienna, Donau-City-Strasse 1, A-1220 Vienna, Austria*

[2]*Department of Physics, Sofia University, James Bourchier 5 Boulevard, 1164 Sofia, Bulgaria*

[3]*Department of Physics, Hunter College, City University of New York, 695 Park Avenue, New York, New York 10021, USA*

(Received 25 June 2007; published 15 November 2007)

We consider complementarity in a bipartite quantum system of arbitrary dimensions. Single-partite and bipartite properties turn out as mutually exclusive quantities. The single-partite properties can be related to a generalized predictability and visibility which compose two complementary realities for themselves. These properties combined become mutually exclusive to the genuine quantum mechanical bipartite correlations of the system which can be quantified with the generalized I concurrence that defines a proper entanglement measure. Consequently, the complementary relation quantifies entanglement in the bipartite system. The concept of complementarity determines entanglement as a property which mutually excludes any single-partite reality. As an application, we provide a proper definition of distinguishability in an n-port interferometer.

DOI: 10.1103/PhysRevA.76.052107 PACS number(s): 03.65.Ta, 03.65.Ud, 03.67.−a, 03.67.Mn

I. INTRODUCTION

The principle of complementarity of Bohr [1] was originally a *qualitative* statement about mutually exclusive but equally real properties of a *single* quantum system. It took until 1979 when a *quantitative* version of its best-known representative, the wave-particle duality concept, was derived [2]. By considering unbalanced two-beam interferometers, where the intensities of the two beams are not equal, an *a priori* predictability $\mathcal{P}$, about which path an interfering quantum object takes, is obtained. This predictability limits the amount of visibility $\mathcal{V}$ that can be achieved in an interferometer according to the complementarity relation $\mathcal{P}^2+\mathcal{V}^2 \leq 1$ [3]. This inequality is saturated if the quantum object is described by a *pure* state. Thus, wave-particle duality becomes a quantitative statement. This relation has been generalized to which-way detection [4–6] and quantum erasing schemes [7]. A topical review can be found in [8]. Although which-way detection and quantum erasing schemes both consider *composite* quantum systems, an explicit quantitative relation between complementarity and entanglement measures has not been given in Refs. [4–7]. The existence of such a relation was conjectured but still not found in Ref. [8].

A systematic approach to quantitative complementarity relations in *composite* quantum systems has recently been developed in Refs. [9–15]. In these generalized complementarity relations multipartite realities mutually exclude single-partite properties of the subsystems and can be quantified by entanglement measures. Entanglement measures in form of the concurrence and the 3-tangle, depending on the number of subsystems in the composite quantum system under consideration, become key entries in these complementarity relations [9,10]. It is the entanglement which mutually excludes any single-partite, i.e., wavelike and particlelike properties. Thus, quantum correlations between the subsystems explicitly enforce complementarity. Not quite unexpectedly, the concept of complementarity allows us to quantify entanglement if we accept the existence of genuine *multipartite* realities in composite quantum systems.

The obstacle so far was the restriction to multipartite *qubit* systems. It was conjectured that the concept of an entanglement measure cannot be extended to systems of arbitrary dimensions $d>2$. One of the reasons is, that the complementary aspect of a single quantum particle exposed to a d-port interferometer with which-path detection schemes is not fully understood yet. In particular, the proposed relation is of the form of an inequality with unity on the right-hand side and it might lead to conflicting statements since the inequality is never saturated [16–18]. Therefore, a thorough comprehension of such a relation is essential in order to extend it to composite quantum systems and to explore its connection to entanglement measures. This relation, of course, must then be saturated for pure composite quantum systems in order to avoid any inconsistencies.

In this paper we carry out a detailed quantitative study of the complementarity in $n\otimes m$-dimensional, bipartite quantum systems. We define proper generalizations of the visibility $\mathcal{V}$ and predictability $\mathcal{P}$ in n dimensions. The quadratic sum of these expressions will form the single-partite properties, $S_A^2=\mathcal{P}_A^2+\mathcal{V}_A^2$ and $S_B^2=\mathcal{P}_B^2+\mathcal{V}_B^2$, of the subsystems A and B. The predictability, $\mathcal{P}$, and visibility, $\mathcal{V}$, coincide with the relevant quantities that have been introduced by Dürr [16], up to a dimension-dependent factor. Their quadratic sum can be related to the purities of the relevant subsystems A and B of a composite system AB, which, in turn, can be considered as a measure of the information content of the subsystems. In agreement with previous results in composite qubit systems [9,10], a genuine bipartite quantum property naturally emerges in composite $n\otimes m$-dimensional systems. This property describes the phase relations between the two parties which can be revealed in correlated measurements only. It establishes thus a bipartite reality which exclusively exists when the composed quantum system is considered as a *single* quantum object in which the subsystems completely lose their identity. As it turns out, this property can be quantitatively expressed by the I concurrence that has been introduced by Rungta *et al.* [19] as a generalization of the concurrence in bipartite qubits. This quantitative entanglement measure will then be mutually exclusive but equally real to the single-partite properties of the subsystems. If the system is fully entangled, single-partite properties must cease to exists and vice versa. Intermediate cases in which both properties coexist are quantitatively described by the generalized complementarity relation.

The newly defined complementarity relation leads to a proper definition of distinguishability in an n-port interferometer accompanied by which-path measurement schemes. The relevant complementarity relation will then be saturated for pure quantum states in contrast to the relation obtained by Dürr [16]. Consequently, inconsistencies about the complementary aspect between distinguishability and visibility are eliminated.

The paper is organized as follows: In Sec. II we derive the proper definitions of predictability and visibility in n-dimensional quantum systems from the generators of SU(n). We show that these definitions depend on a dimension-dependent scaling factor which is of crucial importance in Sec. III where we discuss composite quantum systems. We will derive the main result in this section which states that the generalized concurrence of the bipartite quantum system of arbitrary dimension $n\otimes m$ is mutually exclusive to the single-partite properties of the individual subsystems. In Sec. IV we show that the generalized concurrence can be considered as an entanglement measure on its own by relating it to the entropy of entanglement. In addition, we will discuss effects of mixture of the composite quantum system on the complementarity relation. In Sec. V we consider n-port interferometers and define predictability and visibility from the interferometric viewpoint adopted by Dürr [16]. We show that these definitions differ from those of Sec. II by a dimension-dependent scaling factor. As an application, we derive a proper definition of a generalized distinguishability for an n-port interferometer with which-path detection in Sec. VI. Again, the dimension-dependent scaling of the generalized predictability and visibility plays an important role in the derivation of the complementarity relation. We show that this definition leads to a saturated complementarity relation between distinguishability and visibility in the case of pure composite quantum systems. We conclude with a discussion of our results in Sec. VII.

II. DERIVATION OF PREDICTABILITY AND VISIBILITY FROM SU(n)

The quantitative complementarity relation between visibility and predictability has been derived from the traditional concept of wave-particle duality [2,3]. It is commonly studied in Young-type *double-slit* interferometers where predictability and visibility acquire obvious meanings. The predictability is related to the difference between the probabilities of the quantum object to be in one of the two path alternatives while the visibility is just given by the contrast of the interference fringes. In multiport interferometers, however, the concepts of visibility and predictability are not as clear [16,20] and may lead to conflicting statements [17,18]. We will discuss these concepts from an interferometric point of view in the next section. Here, however, we take a different approach.

It is well known that the (n^2-1)-dimensional Bloch vector is uniquely related to a complete determination of an n-dimensional quantum state on the basis of actual measurements [21–28]. The components of the Bloch vector represent expectations of observables that are tomographically complete. In Ref. [22] a specific set of operators has been derived that form three groups $\{\hat{\lambda}\}=(\{\hat{u}\},\{\hat{v}\},\{\hat{w}\})$, denoted by the symbols u, v, and w, and are defined, in the computational basis, as

$$\hat{u}_{jk} = |j\rangle\langle k| + |k\rangle\langle j|, \tag{2.1}$$

$$\hat{v}_{jk} = -i(|j\rangle\langle k| - |k\rangle\langle j|), \tag{2.2}$$

$$\hat{w}_l = \sqrt{\frac{2}{l(l+1)}}\left(\sum_{j=1}^{l}|j\rangle\langle j| - l|l+1\rangle\langle l+1|\right), \tag{2.3}$$

where the relation between the $\hat{\lambda}_i$, and the $\hat{u}$'s, $\hat{v}$'s, and $\hat{w}$'s, is given by the ordered array

$$(\hat{\lambda}_1, \ldots, \hat{\lambda}_{n^2-1}) = (\hat{u}_{12}, \ldots, \hat{v}_{12}, \ldots, \hat{w}_{n-1}). \tag{2.4}$$

Here, $1\leq j<k\leq n$ and $1\leq l\leq n-1$. Consequently, there are n^2-1 operators $\hat{\lambda}_k$. It is important to realize that these operators generate the algebra of SU(n). Therefore, they obey certain commutation relations [22,27] given by

$$[\hat{\lambda}_j,\hat{\lambda}_k] = 2if_{jkl}\hat{\lambda}_l, \tag{2.5}$$

where f_{jkl} is the completely antisymmetric structure constant of the SU(n) group. Moreover, the generators $\hat{\lambda}_k$ of SU(n) satisfy the conditions [26,27]

$$\hat{\lambda}_k = \hat{\lambda}_k^{\dagger}, \tag{2.6}$$

$$\mathrm{Tr}(\hat{\lambda}_k) = 0, \tag{2.7}$$

$$\mathrm{Tr}(\hat{\lambda}_j\hat{\lambda}_k) = 2\delta_{jk}. \tag{2.8}$$

Thus all of the generators are self-adjoint, traceless, and pairwise orthogonal. With this, we can express the density matrix as [22,27]

$$\rho = \frac{1}{n}\hat{\mathbb{1}}_n + \frac{1}{2}\sum_{k=1}^{n^2-1}\langle\hat{\lambda}_k\rangle\hat{\lambda}_k, \tag{2.9}$$

where $\langle\hat{\lambda}_k\rangle=\mathrm{Tr}(\rho\hat{\lambda}_k)$, and the associated Bloch vector is given by

$$\boldsymbol{\lambda} = (\langle\hat{\lambda}_1\rangle, \ldots, \langle\hat{\lambda}_{n^2-1}\rangle). \tag{2.10}$$

The length of the Bloch vector $|\boldsymbol{\lambda}|^2$ represents a measure of the information content of the system

$$|\boldsymbol{\lambda}|^2 = \sum_{k=1}^{n^2-1}\langle\hat{\lambda}_k\rangle^2 = 2\left(\mathrm{Tr}(\rho^2) - \frac{1}{n}\right) \leq \frac{2(n-1)}{n}, \tag{2.11}$$

since it is directly related to the purity P of the system

$$P = \frac{n}{n-1}\left(\mathrm{Tr}(\rho^2) - \frac{1}{n}\right) = \frac{n}{2(n-1)}|\boldsymbol{\lambda}|^2. \tag{2.12}$$

In Eq. (2.11) the equal sign is obtained in pure quantum systems for which $\mathrm{Tr}(\rho^2)=1$ is satisfied. Please note, that not all Bloch vectors of maximum length on the n^2-1 dimensional hypersphere represent a quantum state. This follows from the following condition which the coherences of the density operator must obey: $|\rho_{jk}|^2 \leq \rho_{jj}\rho_{kk}$, where $j \neq k$.

In order to proceed let us rewrite (2.11) explicitly as the sum of the squared expectations of the operators (2.1)–(2.3) which, of course, also gives the square length of the Bloch vector,

$$\sum_{l=1}^{n-1} |\langle \hat{w}_l \rangle|^2 + \sum_{j,k=1}^{n} (|\langle \hat{u}_{jk} \rangle|^2 + |\langle \hat{v}_{jk} \rangle|^2) = 2\left(\mathrm{Tr}(\rho^2) - \frac{1}{n}\right) \leq \frac{2(n-1)}{n}. \tag{2.13}$$

As before, the coefficients satisfy $1 \leq j < k \leq n$ and $\langle \hat{w}_l \rangle = \mathrm{Tr}(\rho \hat{w}_l)$, $\langle \hat{u}_{jk} \rangle = \mathrm{Tr}(\rho \hat{u}_{jk})$, and $\langle \hat{v}_{jk} \rangle = \mathrm{Tr}(\rho \hat{v}_{jk})$ are the expectations of the generators.

We are interested in the question whether these generators can be related to the conventional notion of wave-particle duality. From Eqs. (2.1)–(2.3) it is obvious that the predictability should be linked to the expectations of the group of the w generators since they detect populations. In contrast, the visibility should be associated with the expectations of the group of generators $\{u\}$ and $\{v\}$ which measure correlations. Therefore, we define the following measure of visibility $\mathcal{V}$:

$$\mathcal{V}^2 = \sum_{j,k;j<k} (|\langle \hat{u}_{jk} \rangle|^2 + |\langle \hat{v}_{jk} \rangle|^2) = 2 \sum_{j,k;j\neq k} |\rho_{jk}|^2. \tag{2.14}$$

The predictability $\mathcal{P}$ is then expressed as

$$\mathcal{P}^2 = \sum_{l=1}^{n-1} |\langle \hat{w}_l \rangle|^2 = 2\left(\sum_{j=1}^{n} \rho_{jj}^2 - \frac{1}{n}\right), \tag{2.15}$$

where the last equation follows from Eqs. (2.11) and (2.13). From this we obtain the following complementarity relation between visibility and predictability:

$$\mathcal{P}^2 + \mathcal{V}^2 = 2\left(\mathrm{Tr}(\rho^2) - \frac{1}{n}\right) \leq \frac{2(n-1)}{n}. \tag{2.16}$$

The equality holds if and only if ρ represents a pure state. The predictability and visibility are both limited from above by $2(n-1)/n$ and represent mutually exclusive properties. If the predictability is large the coherence properties or, equivalently, the visibility must be small and vice versa. The upper bound depends on the dimensionality n of the system and it is larger than unity whenever $n>2$. This follows from the scaling of these properties to the length of the Bloch vector. Consequently, the maximum possible visibility or predictability coincide with the length of the Bloch vector which explicitly depends on the dimensionality of the system under consideration. Another advantage of this definition emerges when we consider composite quantum systems. We will come back to this in Sec. III.

The above definitions of the predictability and visibility coincide with those of Greenberger and Yasin [3] introduced for the case of the double-slit interferometer representing the case of dimensionality $n=2$. In particular, for $n=2$ the visibility and predictability are given by

$$\mathcal{V} = 2|\rho_{12}|, \tag{2.17}$$

$$\mathcal{P} = \sqrt{2(\rho_{11}^2 + \rho_{22}^2) - \frac{1}{2}} = |\rho_{11} - \rho_{22}|, \tag{2.18}$$

which describe the contrast of the interference fringes $(I_{\max}-I_{\min})/(I_{\max}+I_{\min})$ and the difference between the probabilities to be in one of the two path alternatives of the interferometer [3]. Moreover, they are consistent with natural requirements on these properties. In particular, the maximum predictability is obtained if and only if one of the states in the computational basis is occupied. In contrast, maximum visibility can be achieved only if all the coherently superposed states in the computational basis are equally populated.

The traditional wave-particle duality relation which has been derived for the double-slit interferometer can be easily extended to n-dimensional quantum systems. The generators of the SU(n) group can be suitably arranged such that their expectations determine the usual visibility and predictability properties of the quantum system. In order to determine these properties a tomographically complete set of measurement operators is necessary. This is another equally important feature of complementarity [29]. However, a large number of expectations of different operators contribute to visibility or predictability. Thus a formal distinction within the group of expectations that all contribute to visibility, for instance, is not necessary. If we consider, for example, the case $n=2$ the Pauli-spin operators σ_x, σ_y, and σ_z, together with the identity operator $\mathbb{1}$, define the generators of SU(2). The expectation of the σ_z operator determines the predictability properties completely while the expected values of the σ_x and σ_y operators, suitably grouped together, determine the visibility properties. Thus, a formal distinction between the expectations of the σ_x and σ_y operators is not necessary from the conventional wave-particle duality point of view. It is the projection of the Bloch vector on the x-y plane which contributes to the visibility properties, see Fig. 1.

It is an important new feature that the complementarity relation (2.16) is bounded from above by the length of the Bloch vector (2.11) in the case of qudits. The predictability and the visibility completely determine the information content of the quantum system. These properties define the single-partite property $\mathcal{S}$ of a quantum system

$$\mathcal{S}^2 = \mathcal{P}^2 + \mathcal{V}^2 = 2\left(\mathrm{Tr}(\rho^2) - \frac{1}{n}\right) \leq \frac{2(n-1)}{n}. \tag{2.19}$$

The single-partite property $\mathcal{S}$ can be considered as a measure of the information content of a quantum system since it is bounded by the length of the Bloch vector which, in turn, defines the intrinsic information content according to Eq.

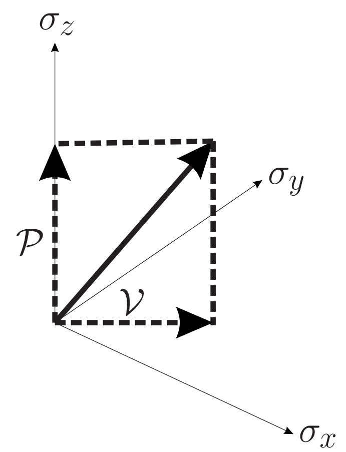

FIG. 1. Bloch vector in a two-dimensional system. The projection on the *x*-*y* plane represents the visibility while the projection on the *z* plane represents the predictability. A formal distinction between the σ_x and σ_y expectation is not necessary from the wave-particle duality point of view. Both contribute equally to the visibility. Thus expected values of the generators can be effectively arranged in the orthodox wave and particle properties.

(2.12). Consequently, it is *invariant* under *unitary* transformations. In contrast, its constituents, predictability and visibility, depend on the basis chosen and can be changed under unitary transformations. The invariance of $\mathcal{S}$ under unitary transformations indicates that the total information content of a closed quantum system is conserved [30–33].

III. COMPOSITE QUANTUM SYSTEMS

In the preceding sections we have considered *single* *n*-dimensional quantum systems from the conventional viewpoint of wave-particle duality. We have pointed out that the generalized visibility and predictability form measures of quantum information whose upper bounds depend on the dimensionality n of the quantum system under consideration. In this section we study *bipartite* quantum systems AB of arbitrary dimensionality $n_A \otimes n_B$. In these systems a genuine quantum property, viz. entanglement emerges naturally. While predictability and visibility are concepts adopted from classical theory, entanglement does not have a classical counterpart.

From this point of view entanglement appears as a measure of that part of quantum information that is shared among the two parties of a bipartite quantum system in such a way that it cannot be accessed by addressing the individual subsystems. In contrast to classical correlations, entanglement enforces the individual systems to reduce their information content. Thus, entanglement must be considered as a property which mutually excludes the single-partite features of the individuals. Based on this observation we have recently derived a complementarity relation for composite *qubit* systems [9,10]. We found that concurrence, which, in two-dimensional systems, constitutes an entanglement measure related to the entanglement of formation [34,35], becomes an essential element in this relation. It is the concurrence that mutually excludes the single-partite properties of the individuals which are composed of visibility and predictability [9–12].

As in Secs. II and V, here, too, we characterize the amount of single-partite information content $\mathcal{S}_k$ of the subsystems $k=A$ or $k=B$ of the composite quantum system AB by the length of the corresponding Bloch vector. The maximum length of the Bloch vector, $2(n_k-1)/n_k$, obviously depends on the dimensionality n_k of the subsystems under consideration. Note that these dimensionalities might be different for the different subsystems. From Eq. (2.19) we obtain the following expression for the single-partite properties of the individuals:

$$\mathcal{S}_k^2 = \mathcal{P}_k^2 + \mathcal{V}_k^2 = 2\left(\mathrm{Tr}(\rho_k^2) - \frac{1}{n_k}\right) \leq \frac{2(n_k-1)}{n_k}. \qquad (3.1)$$

Here, $\rho_k \equiv \mathrm{Tr}_l(\rho_{AB})$, for $k,l=A,B$ and $k\neq l$, defines the reduced density matrix of subsystem k and ρ_{AB} is the density matrix of the composite quantum system AB.

Let us propose the following complementarity relation in the bipartite system, based on the assumption that entanglement in the form of concurrence mutually excludes the single-partite information of the individuals:

$$\mathcal{P}_k^2 + \mathcal{V}_k^2 + (\mathcal{C}_{AB}^{(n)})^2 = \mathcal{S}_k^2 + (\mathcal{C}_{AB}^{(n)})^2 \leq \frac{2(n_k-1)}{n_k}. \qquad (3.2)$$

Here, $\mathcal{C}_{AB}^{(n)}$ defines a proper n-dimensional generalization of the concurrence in $2\otimes 2$ systems. The above inequality is saturated if and only if the composite system AB is represented by a *pure* quantum state. Clearly, in mixed systems, the amount of single-partite information and concurrence which in pure systems, when taken together, generate the total information content, are reduced due to dephasing or decoherence. This explains qualitatively the inequality. We can rewrite Eq. (3.2) equivalently as

$$\mathcal{P}_A^2 + \mathcal{V}_A^2 + \mathcal{P}_B^2 + \mathcal{V}_B^2 + 2(\mathcal{C}_{AB}^{(n)})^2 = \mathcal{S}_A^2 + \mathcal{S}_B^2 + 2(\mathcal{C}_{AB}^{(n)})^2 \leq \frac{2(n_A-1)}{n_A} + \frac{2(n_B-1)}{n_B}, \qquad (3.3)$$

which explicitly states that the generalized concurrence $\mathcal{C}_{AB}^{(n)}$ mutually excludes the single-partite properties of the individual systems. Clearly, the upper bound of (3.3) is proportional to the total information content which is inherent in a pure composite quantum system. This information content is the sum of the maximum possible lengths of the Bloch vectors of systems A and B. As already stated, this bound is reached in case of *pure* composite quantum systems for which the system is completely determined, i.e., all the information is inherent in the system. This information content cannot be exceeded in a composite system since its entropy is always bounded from above by the sum of the entropies of the individuals. In other words, we cannot extract more information from the composite system than from both parts of the system together and the maximum of this information content is represented by the sum of the maximum possible lengths of the Bloch vectors of the individuals.

Assume a *pure* composite quantum system AB in the following in which case the complementarity relation (3.2), or equivalently (3.3), is saturated. In accordance with Eqs. (3.2) and (3.3), the generalized concurrence $(\mathcal{C}_{AB}^{(n)})^2$ is then defined as

$$(\mathcal{C}_{AB}^{(n)})^2 = \frac{2(n_k-1)}{n_k} - \mathcal{S}_k^2 = \frac{2(n_k-1)}{n_k} - 2\left(\mathrm{Tr}(\rho_k^2) - \frac{1}{n_k}\right) = 2[1-\mathrm{Tr}(\rho_k^2)]. \quad (3.4)$$

Thus, in pure systems, any reduction of the single-partite properties must be accompanied by an increase of the bipartite property concurrence. Here, the total information content, i.e., the sum of the two properties, remains *invariant* and adds up to the squared length of the Bloch vector. Note, that the following relation between the subsystems A and B holds in pure composite quantum systems:

$$\mathrm{Tr}(\rho_A^2) \equiv \mathrm{Tr}(\rho_B^2). \quad (3.5)$$

Consequently, the generalized concurrence covers the range

$$0 \le (\mathcal{C}_{AB}^{(n)})^2 \le \frac{2(n-1)}{n}, \quad (3.6)$$

where $n=\min[n_A,n_B]$. Note also, that the upper limit of the generalized concurrence depends on the dimensionality n of the *lower dimensional* subsystem. This feature is important for a bipartite property since the concurrence, when considered as a measure of information, should not exceed the maximally possible information content of the subsystem of lower dimensionality.

We, further, emphasize that the absolute value of the generalized concurrence cannot be changed by adding an unused dimension to the subsystems A or B since it explicitly depends on the degree of mixing of the subsystems, $1-\mathrm{Tr}(\rho_k^2)$. Consequently, the concurrence depends on the properties of the reduced density matrices only. Note, that the actual value of the mixing or, equivalently, the purity of a quantum system does not depend on its dimensionality. This is an important feature of a genuine bipartite property and especially of an entanglement measure. It must not change if one adds an unused dimension to one of the subsystems. This is the main reason why we have defined the single-partite properties such, that they explicitly depend on the dimensionality of the subsystems. It allows us to define the generalized concurrence independent from the dimensionality of the subsystems.

So far we have taken for granted that the generalized concurrence introduced in Eq. (3.4) defines a proper entanglement measure. Next we show that this is indeed the case by considering a proper generalization of the concurrence in $2\otimes 2$ systems to arbitrary dimensional bipartite $n_A\otimes n_B$ systems. This has been accomplished by Rungta *et al.* [19], who introduced a universal inverter $\mathcal{S}_D$ that acts on the density operator ρ of a D-dimensional *qudit* state as

$$\mathcal{S}_D = \nu_D(\mathbb{1}-\rho). \quad (3.7)$$

Here ν_D is a positive constant which is usually chosen to be 1, see later, and $\mathbb{1}$ is the identity operator in D-dimensional systems. We note that the universal inverter $\mathcal{S}_D$ has been previously applied in studies of the separability of mixed states by Horodecki and Horodecki [36]. In two dimensions the universal inverter has a simple interpretation. In particular, the super-operator $\mathcal{S}_2$ acts on a qubit density operator $\rho=\frac{1}{2}(\mathbb{1}+\mathbf{n}\vec{\sigma})$, where $\mathbf{n}$ is a unit vector in three dimensions and $\vec{\sigma}=\{\sigma_x,\sigma_y,\sigma_z\}$ are the Pauli spin matrices, as

$$\mathcal{S}_2(\rho) = \sigma_y\rho^*\sigma_y = \frac{1}{2}(\mathbb{1}-\mathbf{n}\vec{\sigma}) = \nu_2(\mathbb{1}-\rho), \quad (3.8)$$

where $\nu_2=1$ and ρ^* is the complex conjugate (or transpose) of ρ. Consequently, $\mathcal{S}_2$ flips the spin of the qubit explaining its notation as an inverter. Equation (3.7) is therefore just a generalization of (3.8) to arbitrary dimensions.

Based on the observation that the concurrence of a *pure* bipartite qubit state, $|\Psi_{AB}\rangle$, can be defined as [34,35]

$$\begin{aligned}\mathcal{C}_{AB}^{(2)} &= \sqrt{\langle\Psi_{AB}|\mathcal{S}_2\otimes\mathcal{S}_2(|\Psi_{AB}\rangle\langle\Psi_{AB}|)|\Psi_{AB}\rangle}\\ &\equiv |\langle\Psi_{AB}|\sigma_y^A\otimes\sigma_y^B|\Psi_{AB}^*\rangle|\\ &= \sqrt{2[1-\mathrm{Tr}(\rho_A^2)]}\\ &= \sqrt{2[1-\mathrm{Tr}(\rho_B^2)]},\end{aligned} \quad (3.9)$$

by setting $\nu_2=1$, Rungta *et al.* [19] defined the generalized concurrence of a joint pure state $|\Phi_{AB}\rangle$ of a $D_A\otimes D_B$ system as

$$\begin{aligned}\mathcal{C}_{AB}^{(n)} &\equiv \sqrt{\langle\Phi_{AB}|\mathcal{S}_{D_A}\otimes\mathcal{S}_{D_B}(|\Phi_{AB}\rangle\langle\Phi_{AB}|)|\Phi_{AB}\rangle}\\ &= \sqrt{2\nu_{D_A}\nu_{D_B}[1-\mathrm{Tr}(\rho_A^2)]}\\ &= \sqrt{2\nu_{D_A}\nu_{D_B}[1-\mathrm{Tr}(\rho_B^2)]}.\end{aligned} \quad (3.10)$$

in analogy with Eq. (3.9). Here, as before, $\rho_k=\mathrm{Tr}_l(|\Psi_{AB}\rangle\times\langle\Psi_{AB}|)$ in the case of the $2\otimes 2$ system and $\rho_k=\mathrm{Tr}_l(|\Phi_{AB}\rangle\times\langle\Phi_{AB}|)$ in the case of the $D_A\otimes D_B$ system ($k,l=A,B$ and $k\neq l$) describe the reduced density matrices of the subsystems $k=A,B$, respectively. The last equations in (3.9) and (3.10) follow from the action of $\mathcal{S}_{D_A}\mathcal{S}_{D_B}(\rho_{AB})$ on a *pure* density operator ρ_{AB} which is defined as [19]

$$\begin{aligned}\mathcal{S}_{D_A}\mathcal{S}_{D_B}(\rho_{AB}) &= \nu_{D_A}\nu_{D_B}[1-\mathrm{Tr}(\rho_A^2)-\mathrm{Tr}(\rho_B^2)+\mathrm{Tr}(\rho_{AB}^2)]\\ &= 2\nu_{D_A}\nu_{D_B}[1-\mathrm{Tr}(\rho_A^2)].\end{aligned} \quad (3.11)$$

Here, the last equation follows from $\mathrm{Tr}(\rho_{AB}^2)=1$ and $\mathrm{Tr}(\rho_A^2)=\mathrm{Tr}(\rho_B^2)$ which is valid in the case of pure density operators ρ_{AB}. A proper choice of the constants ν_{D_A} and ν_{D_B} that is consistent with the concurrence for qubits, is $\nu_{D_A}=\nu_{D_B}=1$. This ensures that the concurrence cannot be changed simply by adding extra, unused dimensions to one or both of the subsystems. Consequently, for pure states, the concurrence defined this way depends only on the purity of the marginal density operators. Since the generalized concurrence (3.10) is defined from the universal inverter, Rungta *et al.* termed it as the *I concurrence*.

We find that the generalized concurrence (3.4) that has been derived from the complementarity relation (3.2) is completely analogous to the I concurrence (3.10) defined from the universal inverter (3.7). Consequently, the complementarity between single-partite and bipartite properties defines the quantitative entanglement measure I concurrence. The genuine bipartite quantum property I concurrence mutually excludes any single-partite properties that are composed of visibility and predictability. Bipartite quantum properties force single-partite properties to cease to exist.

By considering the generators of SU(n) we can derive an operational expression for the I concurrence of $n\otimes m$ bipartite systems. It is directly related to the qubit version of concurrence which can be expressed by means of the Pauli spin matrices σ_y as shown in (3.9). The n-dimensional generalization of the σ_y operator is given by the group of $\hat{v}_{jk}$ operators introduced in Eq. (2.2). Consequently, we express the I concurrence of a pure composite quantum state $|\Psi_{AB}\rangle$ of dimensionality $n\otimes m$ as

$$(\mathcal{C}_{AB}^{(n)})^2 = \sum_{i<j}\sum_{k<l} |\langle\Psi_{AB}|\hat{v}_{ij}^{A}\otimes\hat{v}_{kl}^{B}|\Psi_{AB}^{*}\rangle|^2. \tag{3.12}$$

By direct calculation one finds that this quantity, indeed resembles the previously defined concurrence

$$(\mathcal{C}_{AB}^{(n)})^2 = 2[1-\mathrm{Tr}(\rho_A^2)] \equiv 2[1-\mathrm{Tr}(\rho_B^2)]. \tag{3.13}$$

Thus, the I concurrence derived from the generators of SU(n) coincides with the I concurrence derived from the universal inverter. Further, it is an evident generalization of the concurrence formula for bipartite qubit states since the Pauli spin operator σ_y is replaced by its n-dimensional extensions.

At this point, we mention that it is possible to derive a lower bound for the entanglement of formation from the extension of the I concurrence to mixed bipartite quantum systems ρ_{AB} of arbitrary dimensions $n_A\otimes n_B$ and from finding a lower bound of this concurrence [37–41]. A particularly simple lower bound of the concurrence $\mathcal{C}(\rho_{AB})$ is given by

$$\mathcal{C}^2(\rho_{AB}) \geq \sum_{j>i}\sum_{l>k} [\mathcal{C}_{ij}^{kl}(\rho_{AB})]^2, \tag{3.14}$$

where

$$\mathcal{C}_{ij}^{kl}(\rho_{AB}) = \max\left(0, \lambda_{ijkl}^{1} - \sum_{n}\lambda_{ijkl}^{n}\right), \tag{3.15}$$

and λ_{ijkl}^{n} are the square roots of the eigenvalues of the matrix $\rho_{AB}(\tilde{\rho}_{AB})_{ij}^{kl}$ with

$$(\tilde{\rho}_{AB})_{ij}^{kl} = \hat{v}_{ij}^{A}\otimes\hat{v}_{kl}^{B}\rho_{AB}^{*}\hat{v}_{ij}^{A}\otimes\hat{v}_{kl}^{B}, \tag{3.16}$$

in descending order. Note, that the matrix (3.16) has a rank not greater than 4, i.e., most of the eigenvalues are identical zero. Thus, Eq. (3.15) is directly related to the standard definition of the concurrence in mixed qubit systems [34,35]. The lower bound (3.14) can be derived utilizing the methods of Ref. [37] extended to a $n_A\otimes n_B$-dimensional bipartite qudit system exercising the properties of the minimum value of a function subject to constraints. It is, however, too stringent and does not permit the detection of the entanglement of formation in bound entangled states. In order to detect bound entanglement more complicated formulas must be employed, see Refs. [39,42]. These formulas, again, adopt the generators of SU(n), see especially Ref. [42]. They require, however, an optimization procedure which depends on a continuous parameter.

In summary, complementarity in $n\otimes m$-dimensional bipartite quantum systems properly quantifies entanglement by the generalized concurrence. The mere existence of entanglement mutually excludes any single-partite properties of the subsystems that are composed of predictability and visibility. If entanglement is present, the subsystems will lose their information content and they do not qualify for objective individual reality. It is the entanglement which enforces complementarity. Both, entanglement and the single-partite properties define quantities that are *invariant* under local unitary transformations. On the one hand, this is important for the generalized concurrence to define an entanglement monotone [43]. On the other hand, this invariance is necessary for the proper quantification of the complementarity between single-partite and genuine bipartite properties in the form of entanglement.

IV. RELATION BETWEEN GENERALIZED CONCURRENCE AND ENTROPY OF ENTANGLEMENT

In this section we will explore the relation between the generalized concurrence and the entropy of entanglement in *pure* composite quantum systems. In other words, we will address the question, whether it is possible to express the entropy of entanglement as a monotonically increasing functional of the generalized concurrence. When this is answered in the positive it will allow us to consider the generalized concurrence as a measure of entanglement on its own. The standard definition of the entropy of entanglement uses the von Neumann entropy [44,45] of the subsystems. Here we will apply the Fano entropy [46] which is a special class of Rényi entropies [47,48]. The Fano entropy is defined from the purity $\mathrm{Tr}(\rho^2)$ of the system

$$S_F = -\ln[\mathrm{Tr}(\rho^2)], \tag{4.1}$$

and is much simpler to calculate than the von Neumann entropy. Moreover, the Fano entropy of subsystems in composite continuous quantum systems has been applied as a measure of entanglement [49–55] since its dependence on the purity of the subsystems is similar to that of the von Neumann entropy. In addition, it plays an important role in quantum thermodynamics. A proper modification of the Fano entropy by introducing a metric operator into the scalar product of density operators, i.e., the purity, provides a Lyapunov functional that behaves strictly monotonically in time. This functional can be considered as a generalized entropy definition [56–59] and it has important consequences to the question of time reversal in open quantum systems [60,61].

We define the entropy of entanglement $E(|\Psi\rangle)$ of a *pure* composite quantum state $|\Psi_{AB}\rangle$ as the Fano entropy (4.1) of either of the two subsystems A and B,

$$E(|\Psi\rangle) = -\ln[\mathrm{Tr}(\rho_A^2)] \equiv -\ln[\mathrm{Tr}(\rho_B^2)]. \tag{4.2}$$

From Eq. (3.4) it follows that

$$E(|\Psi\rangle) \equiv E(\mathcal{C}_{AB}^{(n)}) = -\ln\left(1 - \frac{(\mathcal{C}_{AB}^{(n)})^2}{2}\right), \quad (4.3)$$

which implies that

$$\frac{\partial E(\mathcal{C}_{AB}^{(n)})}{\partial \mathcal{C}_{AB}^{(n)}} = \frac{2\mathcal{C}_{AB}^{(n)}}{2 - (\mathcal{C}_{AB}^{(n)})^2} \geq 0. \quad (4.4)$$

Here, the inequality follows from the boundaries of the generalized concurrence given in Eq. (3.6) which states that for arbitrary dimensions n, even in the limit $n \to \infty$, the concurrence cannot exceed two. Thus, the entropy of entanglement (4.2) of any $n_A \otimes n_B$-dimensional pure state $|\Psi_{AB}\rangle$ is a monotonically increasing functional of the generalized concurrence $\mathcal{C}_{AB}^{(n)}$. This, however, allows us to consider the generalized concurrence as an entanglement measure on its own. We note that this is obvious when we recognize that the generalized concurrence is just the linear entropy $S_L = 1 - \mathrm{Tr}(\rho_k)^2$ of subsystem k up to a scaling factor of 2.

Recently, Mintert *et al.* [39] derived a lower bound for the generalized concurrence of *mixed* quantum systems of arbitrary dimensions. In general, the concurrence of mixed states ρ is given as the convex roof,

$$\mathcal{C}_{AB}^{(n)}(\rho) = \inf \sum_i p_i \mathcal{C}_{AB}^{(n)}(\Psi_i), \quad \rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|, \quad p_i \geq 0,$$

$$\mathcal{C}_{AB}^{(n)}(\Psi_i) = \sqrt{2\{1 - \mathrm{Tr}[(\rho_A^{(i)})^2]\}}, \quad (4.5)$$

of all possible decompositions into pure states $\rho^{(i)} = |\Psi_i\rangle\langle\Psi_i|$. The state ρ is separable if and only if the concurrence $\mathcal{C}_{AB}^{(n)} \times(\rho)$ vanishes. In this case ρ exhibits only classical correlations and can be represented as a convex sum over product states $\rho = \Sigma_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}$. We mention, at this point, that it is not generally accepted to call separable states as classically correlated. In particular, Ollivier and Zurek introduced a quantity termed *quantum discord* as a measure of quantum correlations that differs from zero even for separable states [62]. However, this quantity is connected with measurements and the appearance of these quantum correlations does not mean that the state is necessarily entangled. Also related with this concept is the quantification of quantum correlations by thermodynamical properties [63] which does not involve measurements and the concept of local versus nonlocal information in quantum information theory [14]. Mintert *et al.* succeeded in finding a lower bound of $\mathcal{C}_{AB}^{(n)}(\rho) \geq \mathcal{B}$. For details of the construction of this bound $\mathcal{B}$ we refer to the literature [39]. Here, we point out only that this bound establishes a lower bound for the complementarity relations (3.2) and (3.3) in *mixed* systems,

$$\mathcal{S}_A^2 + \mathcal{S}_B^2 + 2\mathcal{B}^2 \leq \mathcal{S}_A^2 + \mathcal{S}_B^2 + 2[\mathcal{C}_{AB}^{(n)}(\rho)]^2 \leq \frac{2(n_A - 1)}{n_A} + \frac{2(n_B - 1)}{n_B}. \quad (4.6)$$

In *pure* composite quantum systems correlations that do not follow trivially from the single-partite properties of the subsystems exist only in form of the genuine bipartite quantum property concurrence, i.e., only in form of entanglement. Thus, the total information content of the composite quantum system is exhausted completely by the single-partite properties and the concurrence.

In *mixed* systems, however, *classical* correlations between the subsystems might appear although the individual subsystems do not possess any single-partite information content. Consider, for instance, the mixed bipartite qubit state

$$\rho_{AB} = \frac{1}{2}[|0_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 0_B| + |1_A\rangle\langle 1_A| \otimes |1_B\rangle\langle 1_B|]. \quad (4.7)$$

Clearly, the individuals do not have any single-partite information content since they are completely mixed, i.e., their entropy is maximized. However, there is information in form of *classical* correlations present. Both qubits, if considered as spins, are parallel. This information describes classical correlations and is present since the entropy of the composite system is not maximized. The von Neumann entropy S_{AB} of state (4.7) is unity, implying that one bit of information is present in the system. It is equal to the entropy of either of the subsystems and saturates the lower bound of the inequality

$$\max[S_A, S_B] \leq S_{AB} \leq S_A + S_B, \quad (4.8)$$

which must be satisfied in classical systems. This again expresses the fact that the correlations in (4.7) are of classical origin. Consequently, in *mixed* systems the total information content is not exhausted completely by the single-partite and bipartite properties, predictability, visibility, and concurrence. In addition to these properties, the information from classical correlations might contribute.

This information, however, should not contribute to a complementarity relation since classical correlations are merely a consequence of single-partite and bipartite properties. Let us consider a simple example based on the *pure* quantum state

$$|\Psi\rangle = |1_A\rangle|1_B\rangle. \quad (4.9)$$

From an information point of view the state is completely described by the predictabilities of the individual subsystems. However, since the two spins are parallel, there is seemingly an additional piece of information present in the form of classical correlations. This information, however, is not independent as it follows directly from the single-partite information of the constituents. Thus, in pure states, classical correlations are redundant information which is not independent from the information content of the individuals. Consequently, information on classical correlations does not contribute to the complementarity relation. We will discuss the complementarity relation in mixed quantum systems in more detail elsewhere.

V. INTERFEROMETRIC DERIVATION OF PREDICTABILITY AND VISIBILITY

In this section we derive predictability and visibility properties by considering an n-beam interferometer. This situation has been discussed by Dürr [16] who compiled a list of

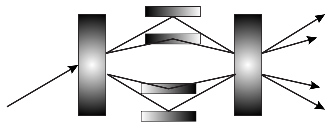

FIG. 2. Scheme of a four-beam interferometer. The incoming quantum object is split into four beams at the first four-port beam splitter. The beams are redirected with the help of mirrors and recombined such that all beams impinge on the second four-port beam splitter which produces four output beams.

desirable properties of any multipath generalization of visibility and predictability of Young's double-slit interferometer. We note, however, that there are alternative proposals which do not have all of these properties [20]. We demonstrate that the generalized multipath visibility and predictability, as derived by Dürr, coincide with the definitions proposed in the preceding section up to a dimensional-dependent scaling factor.

We want to consider in more detail the situation that has been discussed previously by Dürr [16]. The n-dimensional interferometer consists of two n-port beam splitters. The beams are redirected with the help of mirrors in between these beam splitters such that all beams impinge on the second one, as illustrated in Fig. 2 for the case of a four-beam interferometer. While the first beam splitter is assumed to have arbitrary splitting ratios, the second one has a $1/n$ splitting ratio. Consequently, an output beam of the second beam splitter has the generic form

$$|b\rangle = \sqrt{\frac{1}{n}}[e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{\phi_n}]^T, \tag{5.1}$$

where T denotes the transpose. We have chosen the n beams in front of the second beam splitter as an orthogonal basis of quantum states for a matrix representation [16]. The phases ϕ_j can be varied arbitrarily by modifying the second beam splitter with variable phase shifters.

The intensity I or equivalently the probability that the quantum object leaves the interferometer in output beam $|b\rangle$ depends on the input density operator ρ in front of the second beam splitter. It can be derived as [16]

$$I = \langle b|\rho|b\rangle = \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\rho_{jk}e^{-i(\phi_j-\phi_k)}$$
$$= \frac{1}{n}\left(1 + \sum_j\sum_{k\neq j}|\rho_{jk}|\cos(\phi_j - \phi_k - \arg\rho_{jk})\right), \tag{5.2}$$

where ρ_{ij} is the density matrix in the computational basis and $\arg\rho_{jk}$ denotes the phase of the complex matrix element ρ_{jk}. Based on a number of desirable properties on the generalized n-dimensional visibility [16], Dürr proposed a visibility measure $\tilde{\mathcal{V}}$ that is constructed from moments of the intensity function I. These moments are obtained by phase averaging the considered function $f(\phi_1, \ldots, \phi_n)$ over all phases $\langle\cdots\rangle_\phi$,

$$\langle f\rangle_\phi = \frac{1}{(2\pi)^n}\int_0^{2\pi}d\phi_1\cdots\int_0^{2\pi}d\phi_n f(\phi_1, \ldots, \phi_n). \tag{5.3}$$

From this we obtain for the first and second moment of the intensity (5.2),

$$\langle I\rangle_\phi = \frac{1}{n}, \tag{5.4}$$

$$\sqrt{\langle(\Delta I)^2\rangle_\phi} = \frac{1}{n}\sqrt{\sum_j\sum_{k\neq j}|\rho_{jk}|^2}, \tag{5.5}$$

where $\Delta I = I - \langle I\rangle_\phi$ is the standard deviation of I from its mean value. It is obvious from Sec. II that the root-mean-square (rms) spread $\sqrt{\langle(\Delta I)^2\rangle_\phi}$ describes some measure of visibility since it is directly proportional to the visibility defined in Eq. (2.14) that was derived from the generalized Bloch vector. The visibility $\tilde{\mathcal{V}}$, defined by Dürr, is normalized to unity and given as [16]

$$\tilde{\mathcal{V}} = \left(\frac{n^3}{n-1}\langle(\Delta I)^2\rangle_\phi\right)^{1/2} = \left(\frac{n}{n-1}\sum_j\sum_{k\neq j}|\rho_{jk}|^{1/2}\right)^{1/2}, \tag{5.6}$$

where we applied Eq. (5.5). Depending on the properties of the quantum object and the interferometer the visibility covers the full range

$$0 \leq \tilde{\mathcal{V}} \leq 1. \tag{5.7}$$

The generalized n-beam visibility $\tilde{\mathcal{V}}$ is directly proportional to the generalized visibility $\mathcal{V}$ that has been derived from the Bloch vector in Sec. II,

$$\tilde{\mathcal{V}}^2 = \frac{n}{2(n-1)}\mathcal{V}^2. \tag{5.8}$$

Thus, up to a dimension-dependent factor the two definitions coincide. However, the normalization of visibility to unity, i.e., the definition $\tilde{\mathcal{V}}$, has a number of drawbacks which clearly emerged in Sec. III where we considered composite quantum systems in which entanglement plays a crucial role. The generalized visibility $\mathcal{V}$, normalized to the length of the Bloch vector, is much better suited for this case. In addition, this definition allows us to consider the visibility as a measure of quantum information. It is thus conceptually more appropriate to use this definition, which forms an invariant, when considering the information content of a closed quantum system [30–33]. This has also been realized in Ref. [16]. Consequently, we will use $\mathcal{V}$ as the definition of the generalized visibility whose maximum possible value $\sqrt{2(n-1)/n}$, i.e., the length of the Bloch vector, explicitly depends on the dimensionality n of the quantum object.

Similarly, a generalized predictability can be defined in n-dimensional interferometers. Based on a number of desirable properties, Dürr proposed the generalized predictability $\tilde{\mathcal{P}}$ by considering moments of probability distributions, along the same lines as in the case of the generalized visibility [16]. Here, the average over a distribution f is defined as

$$\langle f\rangle_j = \frac{1}{n}\sum_{j=1}^{n} f_j, \tag{5.9}$$

where f_j is a set of n numbers. From normalization of the density operator, $\mathrm{Tr}(\rho)=1$, it follows that the mean value of the probabilities is given by

$$\langle \rho_{jj}\rangle_j = \frac{1}{n}\sum_{j=1}^{n} \rho_{jj} = \frac{1}{n}. \tag{5.10}$$

The generalized predictability $\tilde{\mathcal{P}}$ is based on the rms spread $\sqrt{\langle(\rho_{jj}-1/n)^2\rangle_j}$ and is obtained as [16]

$$\tilde{\mathcal{P}} = \left[\frac{n}{n-1}\sum_{j=1}^{n}\left(\rho_{jj}-\frac{1}{n}\right)^2\right]^{1/2} = \left[\frac{n}{n-1}\left(-\frac{1}{n}+\sum_{j=1}^{n}\rho_{jj}^2\right)\right]^{1/2}, \tag{5.11}$$

where we employed $\mathrm{Tr}(\rho)=1$ to obtain the last equation. As before, the generalized predictability $\tilde{\mathcal{P}}$ is normalized to unity and covers the range

$$0 \le \tilde{\mathcal{P}} \le 1, \tag{5.12}$$

depending on the properties of the interferometer.

The generalized predictability $\tilde{\mathcal{P}}$, derived here from interferometric considerations, is directly proportional to the generalized predictability $\mathcal{P}$, obtained from the Bloch vector in the preceding section,

$$\tilde{\mathcal{P}}^2 = \frac{n}{2(n-1)}\mathcal{P}. \tag{5.13}$$

The constant $n/[2(n-1)]$ is the same factor that appears in Eq. (5.8) and is just the inverse of the length of the Bloch vector. However, as already discussed, it is of advantage to normalize the predictability to the length of the Bloch vector. Consequently, we use the generalized predictability $\mathcal{P}$ as the properly defined predictability. It covers the range

$$0 \le \mathcal{P}^2 \le \frac{2(n-1)}{n}. \tag{5.14}$$

Again, the upper bound of the predictability depends on the dimensionality n of the quantum object. Similar to the visibility, the predictability also represents a measure of the quantum information content of the qudit state. The generalized predictability together with the generalized visibility completely determine the total information content $\mathcal{S}^2=\mathcal{P}^2+\mathcal{V}^2$ of the quantum system as discussed after Eq. (2.19). The properly defined visibility and predictability whose upper possible bound of their information content depends on the dimensionality will also play a central role in the next section where we discuss the generalized distinguishability in a n-port interferometer. Both properties are measures of the information content of a qudit and, consequently, these measures must depend on the dimensionality of the qudit.

VI. GENERALIZED DISTINGUISHABILITY IN *n*-PORT INTERFEROMETER

The complementarity relation derived in Sec. III is particularly useful for the clarification of some issues connected with complementarity in n-port interferometers that are accompanied by which-path detection schemes. Complementarity in those systems has been discussed by Dürr [16] and Bimonte and Musto [17,18]. Dürr proposed a complementarity relation for distinguishability and visibility that is, however, not saturated even in pure systems as it was demonstrated by Bimonte and Musto. As a consequence, this complementarity relation by Dürr does not convey the very idea of complementarity according to which an increase of fringe visibility is at the cost of a loss of which-path information and vice versa. In particular, both quantities might increase or decrease simultaneously [17,18]. Consequently, they are not mutually exclusive. The intention of the following section is, therefore, to find proper definitions of distinguishability and visibility such that a complementarity relation between them indeed conveys correctly the concept of complementarity.

Let us first give a short summary about Dürr's approach to the complementarity between distinguishability and visibility and their definitions as proposed in Ref. [16]. In order to understand the concept of distinguishability we must analyze interferometric schemes that involve which-path detectors. Such detectors can be considered as a second quantum system which interacts with the principal quantum object exposed to the interferometer. This interaction changes the state of the which-path detector depending on the path chosen by the principal system. Consequently, it leads to an *entanglement* between the principal quantum system and the which-path detector and which-way information can be gained by a suitable measurement of the states of the detector. The detector, thus, stores the information about the path that the quantum object has chosen.

When the quantum object (quanton for short) is exposed to the n-port interferometer it can be described by the arbitrary quantum state

$$|\Psi\rangle_Q = \sum_{j=1}^{n} c_j|\Psi_j\rangle_Q. \tag{6.1}$$

Here, the $|\Psi_j\rangle_Q$ are orthonormalized states that describe the situation when the quanton (Q) travels solely through the jth slit. Note, that the coefficients c_j depend on the properties of the interferometer. The interaction with the which-path detector can be designed such that it changes the state of the total system $|\Psi_{\mathrm{total}}\rangle$, i.e., the quanton plus which-path detector, to

$$|\Psi_{\mathrm{total}}\rangle = \sum_j c_j|\Psi_j\rangle_Q \otimes |\xi_j\rangle_D, \tag{6.2}$$

where the final states of the which-path detector (D) $|\xi_j\rangle_D$ are normalized but not necessarily orthogonal and depend on j. The output state $|\Psi_{\mathrm{total}}\rangle$ is consequently an entangled state in general. These correlations between the quanton and the detector make the storage of which-path information possible.

In order to read out this information, a measurement of a suitable observable $\hat{W}_D$ on the detector must be performed. We denote the eigenvalues of $\hat{W}_D$ by w_k and the corresponding eigenstates by $|w_k\rangle_D$, as in [16], and take for granted that the measurement performed is a standard von Neumann measurement which ensures that the outcome is one of the eigenvalues of $\hat{W}_D$ and the detector state is projected onto the corresponding eigenstates. Further, we assume the eigenvalues to be nondegenerate in order to simplify matters. The projectors corresponding to the eigenvalues w_k are denoted as $P_D^{(k)}=|w_k\rangle_D\langle w_k|$. They are, of course, mutually orthogonal and complete,

$$P_D^{(k)}P_D^{(l)} = \delta_{kl}P_D^{(k)}, \quad \sum_k P_D^{(k)} = \mathbb{1}_D, \tag{6.3}$$

where $\mathbb{1}_D$ is the unit operator of the detector subspace. When a measurement is performed the outcome w_k is found with the probability

$$p_k = \mathrm{Tr}_{DQ}(P_D^{(k)}\rho_{DQ}), \tag{6.4}$$

where we have introduced the density operator of the total system $\rho_{DQ}=|\Psi_{\mathrm{total}}\rangle\langle\Psi_{\mathrm{total}}|$ which, of course, is pure. The density operator of the quanton $\rho_Q^{(k)}$ that is conditioned on the outcome w_k of the $\hat{W}_D$ measurement is given by

$$\rho_Q^{(k)} = \frac{\mathrm{Tr}_D(P_D^{(k)}\rho_{DQ})}{p_k}. \tag{6.5}$$

Note, that $\mathrm{Tr}_Q[(\rho_Q^{(k)})^2]=1$, i.e., the conditioned density operator describes a pure state. Thus, after a measurement of the detector observable $\hat{W}_D$ (without reading out the measurement outcome) the density operator of the quanton must be described by the statistical ensemble

$$\rho_Q = \sum_k p_k\rho_Q^{(k)}, \tag{6.6}$$

which is not describing a pure state. This ensemble is sorted into subensembles $\rho_Q^{(k)}$ depending on the measurement outcome of the detector observable and these subensembles are weighted by the probability p_k that this outcome appears. The sorting into the subensembles, of course, depend on the choice of the detector observable which has been described in great detail in Ref. [8].

Let us define now the conditioned which-way knowledge $\mathcal{K}_k$ as the predictability $\widetilde{\mathcal{P}}$, as it is defined in Eq. (5.11), of the subensemble (6.5)

$$\mathcal{K}_k = \left(\frac{n}{n-1}\sum_{j=1}^{n}[(\rho_Q^k)_{jj} - 1/n]^2\right)^{1/2}. \tag{6.7}$$

In the equation above, $(\rho_Q^k)_{jj}=\langle\Psi_j|\rho_Q^k|\Psi_j\rangle$, describes the diagonal elements of the density matrix $\{\rho_Q^k\}$ in the state representation $|\Psi_j\rangle_Q$ which corresponds to the populations of ρ_Q^k to be in the state $|\Psi_j\rangle_Q$. As it is pointed out in Ref. [16], it is important to catch the difference between the predictability $\mathcal{P}$ and the conditional which-path knowledge $\mathcal{K}_k$. In particular, the predictability describes *a priori* knowledge which depends on the properties of the interferometer only. In contrast, the conditioned which-way knowledge $\mathcal{K}_k$ represents *a posteriori* which-path knowledge which only emerges as a result of the measurement of the which-path detector.

The complete which-way knowledge is now defined by the statistical average over all possible outcomes w_k depending on the measurement observable $\hat{W}_D$,

$$\mathcal{K}_W = \sum_k p_k\mathcal{K}_k. \tag{6.8}$$

This quantity represents the amount of which-way information obtained on average, given that the observable $\hat{W}_D$ was measured. It is not difficult to show that the following inequality is valid:

$$\mathcal{K}_W \geq \widetilde{\mathcal{P}}, \tag{6.9}$$

since the right-hand side, i.e., the predictability (5.11), is obtained without any measurement on the detector and just describes the *a priori* which-path knowledge of the quanton. It is obvious, that $\mathcal{K}_W$ depends on the choice of the measurement observable $\hat{W}_D$ which crucially influences the sorting of ρ_{Q} into subensembles, see Eq. (6.6). Therefore, there is a largest possible $\mathcal{K}_W$ and this is defined as the *distinguishability* $\widetilde{\mathcal{D}}$ of the paths that the quanton has taken [5,6,64],

$$\widetilde{\mathcal{D}} = \max_W\{\mathcal{K}_W\}, \tag{6.10}$$

resulting from the best choice of the measurement observable $\hat{W}_D$ [8,16]. The proposed complementarity relation reads then [16]

$$\widetilde{\mathcal{D}}^2 + \widetilde{\mathcal{V}}^2 \leq 1, \tag{6.11}$$

where the visibility is given by Eq. (5.6) and the distinguishability $\widetilde{\mathcal{D}}$ is defined in (6.10). This inequality generalizes the wave-particle duality to cases including which-path detectors and to $n>2$-port interferometer, i.e., the double-slits configuration where this inequality has been derived by Englert [6]. For the proof of (6.10) we refer to Refs. [6,16].

We stress that there is an important difference between the inequality of Englert, derived for the double-slits configuration, and that of Dürr, derived for the n-port interferometer in the case when $n>2$. In particular, while in the double-slits configuration accompanied by which-path detectors this inequality is always *saturated* when the combined quantum state of the interfering object and the detector is pure, this might not be the case in the $n(>2)$-port interferometer [17,18]. This can happen when the dimensionality of the Hilbert space of the detector states is smaller than that of the interfering quanton as we will demonstrate by considering the specific example of Bimonte and Musto [17,18], below.

Unfortunately, this narrows the validity of the proposed complementarity relation (6.10). Note, further, that both $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{V}}$ cover the range

$$0 \leq \widetilde{\mathcal{V}} \leq 1, \tag{6.12}$$

$$0 \leq \tilde{\mathcal{D}} \leq 1. \tag{6.13}$$

We have already discussed in Sec. V that such a normalization is not appropriate in composite quantum systems of dimensionality $n>2$. We expect, therefore, the definition of the distinguishability, Eq. (6.10), to be inapplicable in some cases. Indeed, if we consult the original papers about distinguishability in two-path interferometers [6,8], the argumentation there was that distinguishability is defined as the maximum possible which-way information *nature* can grant us. In other words, it is to some extent independent from measurable observables and can exceed the distinguishability that can be inferred from a measurement [7]. Thus, we should define distinguishability as an *intrinsic* property which is inherent in the composite quantum state of the system and the detector. In other words, distinguishability should rather depend on the *quantum correlations* between the system and the detector.

This motivated Englert and Bergou [8] to conjecture a quantitative relation between some measure of the entanglement and the distinguishability. That such a quantitative relation actually exists was shown in Ref. [9] in the case of a two-port interferometer with which-path detection scheme. In its simplest form, this system can be considered as a bipartite entangled qubit system AB where qubit A represents the two path alternatives of the interfering quantum object and qubit B represents the two orthogonal states of the which-path measurement device. From this it can be shown that the distinguishability of qubit A can be defined as [9]

$$\mathcal{D}_A^2 = \mathcal{C}_{AB}^2 + \mathcal{P}_A^2, \tag{6.14}$$

where $\mathcal{C}_{AB}$ is the concurrence of the bipartite qubits AB and $\mathcal{P}_A$ is the predictability of qubit A in the computational basis defined in (2.18). As a natural generalization, we propose the following definition of the distinguishability $\mathcal{D}_A^{(n)}$ for a quantum object A exposed to an n_A-port interferometer with a which-path measurement scheme B of Hilbert space dimensionality n_B:

$$(\mathcal{D}_A^{(n_A)})^2 = (\mathcal{C}_{AB}^{(n)})^2 + (\mathcal{P}_A^{(n_A)})^2, \tag{6.15}$$

where $\mathcal{C}_{AB}^{(n)}$ is the generalized concurrence defined in Eq. (3.4), $n=\min\{n_A,n_B\}$, and $\mathcal{P}_A^{(n_A)}$ is the predictability of the quantum object A defined in Eq. (2.15) for the dimensionality n_A. Note, that we have assumed the quantum state of the composite system AB to be pure. Taking into account the generalized complementarity relation (3.2) in the case of *pure* states, for which it is saturated, we obtain

$$(\mathcal{D}_A^{(n_A)})^2 + (\mathcal{V}_A^{(n_A)})^2 = \frac{2(n_A-1)}{n_A}. \tag{6.16}$$

This expression states that the complementarity relation is saturated for pure composite quantum systems. Indeed, the properly defined distinguishability is mutually exclusive to the visibility and this remains true in interferometers of arbitrary dimensionality n accompanied by which-path detection schemes of arbitrary dimensions m. Note, that the distinguishability (6.15) contains two contributions. One is a single-partite property, i.e., the predictability, while the other is a genuine bipartite quantum property, i.e., the concurrence. It is the concurrence which makes the distinguishability different from the single-partite property predictability. Consequently, the distinguishability is a combination of single-partite and bipartite properties and cannot be considered as a property of the interfering quantum object alone.

It is obvious from Eq. (6.16) that the distinguishability can cover the full range

$$0 \leq (\mathcal{D}_A^{(n_A)})^2 \leq \frac{2(n_A-1)}{n_A}, \tag{6.17}$$

depending on the amount of visibility. The contribution of the entanglement, i.e., the concurrence, to the distinguishability, however, depends on the properties of both systems and it can cover the range

$$0 \leq (\mathcal{C}_{AB}^{(n)})^2 \leq \frac{2(n-1)}{n}, \tag{6.18}$$

where $n=\min\{n_A,n_B\}$. Consequently, if the dimensionality n_B of the Hilbert space of the which-path detector is smaller than the Hilbert space dimensionality, n_A, of the interfering quantum object, the contribution of the concurrence to the distinguishability cannot cover the full range. In other words, if the distinguishability is at its maximum there is always a contribution from predictability present. This has an important consequence for quantum erasing schemes [65] which, in general, can only affect the bipartite contribution. Thus, it is the concurrence which is erased in a quantum eraser in order to restore visibility [7,8,66]. There are, however, exceptions in case of a *conditional* quantum eraser which can additionally erase *a priori* which-path information, i.e., the predictability, as shown in Refs. [66,67].

In the following, we discuss the test case of Bimonte and Musto [17,18] who considered a simple three-beam interferometer with equally populated beams and a which-path detector scheme of Hilbert space dimensionality two, for which the original definition of the distinguishability (6.10) given by Dürr [16] leads to unsatisfactory results. In particular, they showed that even in some cases involving pure quantum states the complementarity relation between distinguishability and visibility is not saturated. Consequently, there are cases when visibility and distinguishability both increase or decrease at the same time, a result which is quite unsatisfactory from the viewpoint of complementarity.

Let us consider the example of Bimonte and Musto by adopting the proper definition of distinguishability (6.15). The pure state of the interfering quantum object is given by

$$|\Phi\rangle_A = \sqrt{\frac{1}{3}}\sum_i |\Psi_i\rangle \Leftrightarrow \rho_A = \frac{1}{3}\sum_{i,j=1}^{3} |\Psi_i\rangle\langle\Psi_j|. \tag{6.19}$$

The interaction of the quantum object with the which-path detector leads to entanglement between the two systems and their joint state ρ_{AB} is described by

$$|\Phi\rangle_{AB} = \sqrt{\frac{1}{3}}\sum_i |\Psi_i\rangle|\xi_i\rangle \Leftrightarrow \rho_{AB} = \frac{1}{3}\sum_{i,j=1}^{3} |\Psi_i\rangle\langle\Psi_j| \otimes |\xi_i\rangle\langle\xi_j|, \tag{6.20}$$

where $|\xi_i\rangle$ are normalized, but not necessarily orthogonal, detector states. Bimonte and Musto assumed for simplicity that the Hilbert space of the detector state $\mathcal{H}_D$ is two dimensional and described the detector states by using the Bloch parametrization

$$|\xi\rangle\langle\xi| = \frac{\mathbb{1} + \mathbf{n}\cdot\vec{\sigma}}{2}. \tag{6.21}$$

Here, $\mathbf{n}$ are unit vectors in three dimensions and $\vec{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$ are the Pauli spin matrices in $\mathcal{H}_D$. They suggested the following nonorthogonal detector states where the phase θ can be varied at will:

$$|\xi_1\rangle\langle\xi_1| = \frac{\mathbb{1} + \mathbf{n}_1\cdot\vec{\sigma}}{2}, \quad \mathbf{n}_1 = (0,0,1), \quad |\xi_1\rangle = |2\rangle, \tag{6.22}$$

$$|\xi_2\rangle\langle\xi_2| = \frac{\mathbb{1} + \mathbf{n}_2\cdot\vec{\sigma}}{2}, \quad \mathbf{n}_2 = (\sin\theta, 0, \cos\theta), \tag{6.23}$$

$$|\xi_2\rangle = \frac{1}{\sqrt{2}}(\sqrt{1+\cos\theta}|2\rangle + \sqrt{1-\cos\theta}|1\rangle),$$

$$|\xi_3\rangle\langle\xi_3| = \frac{\mathbb{1} + \mathbf{n}_3\cdot\vec{\sigma}}{2}, \quad \mathbf{n}_3 = (-\sin\theta, 0, \cos\theta), \tag{6.24}$$

$$|\xi_3\rangle = \frac{1}{\sqrt{2}}(\sqrt{1+\cos\theta}|2\rangle - \sqrt{1-\cos\theta}|1\rangle).$$

The visibility and the distinguishability depend on the phase θ of the detector state. We obtain the following for the generalized visibility of system A from (2.14) or from (3.1):

$$(\mathcal{V}_A^{(3)})^2 = \frac{4}{9}(1 + \cos\theta + \cos^2\theta). \tag{6.25}$$

Note, that this result differs from the result of $\tilde{\mathcal{V}}$ in Bimonte and Musto [Eq. (8) of Ref. [17]] by a factor of 4/3 which follows from the different scaling of the generalized visibilities, see Eq. (5.8). For the generalized distinguishability we obtain from (6.15),

$$(\mathcal{D}_A^{(3)})^2 = \frac{4}{9}(1 + \sin^2\theta - \cos\theta), \tag{6.26}$$

where we take into account that the generalized predictability and the concurrence are given as

$$(\mathcal{P}_A^{(3)})^2 = 0, \tag{6.27}$$

$$(\mathcal{C}_{AB}^{(2)})^2 = \frac{4}{9}(1 + \sin^2\theta - \cos\theta), \tag{6.28}$$

for the system under consideration. Note that the generalized concurrence (6.28) covers the range

$$0 \le (\mathcal{C}_{AB}^{(n=2)})^2 \le \frac{2(n-1)}{n} = 1, \tag{6.29}$$

which explains the superscript $n=2$. This follows from the fact that the dimensionality of the detector's Hilbert space is two and, thus, smaller than the Hilbert space dimension of the interfering quantum object, as discussed earlier. The properly defined generalized distinguishability (6.26) is significantly different from the distinguishability $\tilde{\mathcal{D}}$ as derived from the definition (6.10). In the test case of Bimonte and Musto the following expression for $\tilde{\mathcal{D}}$ was found [see Eq. (17) in [17]]:

$$\tilde{\mathcal{D}}^2 = \begin{cases} \frac{1}{3}\sin^2\theta & \text{for } 0 \le \theta < \frac{2\pi}{3}, \\ \frac{4}{9}\sin^2\left(\frac{\theta}{2}\right) & \text{for } \frac{2\pi}{3} < \theta \le \pi. \end{cases} \tag{6.30}$$

In contrast to the visibility, this expression differs not only by a scaling factor of 4/3 from the properly defined distinguishability (6.26), but also in its analytical form. Moreover, this function exhibits a discontinuity at $\theta=2\pi/3$, a property which is highly unsatisfactory on physical grounds.

Although the state of the quantum system is pure the complementarity between $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{V}}=\frac{3}{4}\mathcal{V}_A^{(3)}$ is not saturated over the complete range of θ. In contrast, the complementarity between the properly defined distinguishability (6.26) and the generalized visibility (6.25) is saturated

$$(\mathcal{D}_A^{(3)})^2 + (\mathcal{V}_A^{(3)})^2 = \frac{4}{9}(2 + \sin^2\theta + \cos^2\theta) = \frac{4}{3}, \tag{6.31}$$

for all θ. Therefore, we claim that the properly defined distinguishability cannot always be related to a quantity that can be measured by a detector observable that maximizes the which-way knowledge $\mathcal{K}(W)$. This happens whenever the dimensionality of the Hilbert space of the detector is smaller than that of the Hilbert space of the interfering quantum particle, except for the case of a double-slits interferometer where the trivial case of a one-dimensional Hilbert space of the detector states corresponds to the situation with no which-path measurement. Consequently, the composite quantum system will contain nonorthogonal detector states that cannot be resolved perfectly and one cannot extract the total amount of the properly defined distinguishability by a measurement at the detector.

We have emphasized throughout that the properly defined distinguishability contains a genuine bipartite quantum property in the form of the concurrence. It is the concurrence that enforces complementarity in bipartite systems and mutually excludes single-partite properties of the individuals. Consequently, the interpretation of the distinguishability as the maximum possible which-path information that one can extract by a proper measurement at the detector is not only misleading but wrong. The quantum correlations among the detector and the interfering quantum object in form of the entanglement measure concurrence, inherently present in the distinguishability, are responsible for the decrease of single-partite properties. These quantum correlations can be a resource for *possible* which-path information if one measures suitable observables at the detector but they cannot necessarily be extracted completely from such a measurement as shown in the above example. We note again that this happens whenever the Hilbert space dimensionality of the detector states is smaller than that of the interfering quantum object.

With this in mind let us consider the case where we interchange the role of the interfering quantum object and the detector. In other words, we assume subsystem B in (6.20) to be the interfering particle and A to be the detector. In this case the dimensionality of the detector's Hilbert space is larger than that of the interfering quantum object and we expect that the distinguishability can be interpreted as the maximum possible which-path information that one can extract from a suitable measurement at the detector. Consequently, all of the genuine quantum correlations of the composite system AB can be extracted from such a measurement as it was declared above. We can rewrite the state (6.20) as

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{3}}\left[\left(|1_A\rangle + \sqrt{\frac{1+\cos\theta}{2}}(|2\rangle_A + |3\rangle_A)\right)^{(+)} \otimes |2\rangle_B + \left(\sqrt{\frac{1-\cos\theta}{2}}(|2\rangle_A - |3\rangle_A)\right)^{(-)} \otimes |1\rangle_B\right]. \tag{6.32}$$

It is not difficult to recognize that the detector states $(+)$ and $(-)$ which are correlated with the states $|2\rangle_B$ and $|1\rangle_B$, respectively, are orthogonal. In other words, by performing a projective measurement on the detector states $(+)$ and $(-)$ we gain complete which-path information about the interfering quantum object. Consequently, the distinguishability $\tilde{\mathcal{D}}$, as defined in (6.10), is given by

$$\tilde{\mathcal{D}} = 1, \tag{6.33}$$

independent from the angle θ and the visibility $\tilde{\mathcal{V}}$ is zero. This is completely equivalent to the generalized distinguishability derived from Eq. (6.15) which gives

$$(\mathcal{D}_B^{(2)})^2 = (\mathcal{C}_{AB}^{(2)})^2 + (\mathcal{P}_B^{(2)})^2 = \frac{4(1+\sin^2\theta - \cos\theta)}{9} + \frac{1+4\cos^2\theta + 4\cos\theta}{9} = 1. \tag{6.34}$$

Thus, in this case, the distinguishability can be interpreted as the maximum possible which-path information that one can extract from the system by a suitable measurement at the detector. This is possible because the Hilbert space dimensionality of the detector states is larger than that of the interfering quantum object. Whenever this is not the case, we cannot interpret the generalized distinguishability as the maximum possible which-path information because we cannot extract, in general, all the genuine quantum correlations inherent in the generalized distinguishability. Thus, the optimum detector should have the same dimensionality as the qudit, i.e., the quanton exposed to the d-dimensional interferometer, it is aimed at measuring. Fewer dimensions are insufficient and more will not lead to better results.

VII. CONCLUSIONS

In this paper we have derived a complementarity relation for a composite bipartite quantum system of arbitrary dimensions $n \otimes m$. The complementarity relation contains single-partite properties in the form of properly defined predictability and visibility and a bipartite property in the form of the generalized I concurrence [19]. This latter quantifies entanglement, a genuine bipartite quantum property, that does not have any classical counterpart. We have shown that the single-partite properties can be derived from the generators of SU(n) and their upper bound can be related to the length of the Bloch vector which, in turn, represents the information content present in the system. Accordingly, the upper bound of the complementarity relation is dependent on the dimensionality of the system and, in general, it can exceed unity. The generalized predictability and visibility derived from the generators of SU(n) can be related to the predictability and visibility which have been suggested from interferometrical considerations [16]. They are equivalent up to a dimension-dependent scaling factor which is, however, crucially important when composite systems are considered. This scaling factor ensures that the definition of the generalized I concurrence is independent of the dimensions of the system. This situation is very satisfactory since the amount of entanglement should not change if one adds an unused dimension to one of the subsystems [19].

In the case when nonclassical correlations between the subsystems emerge, i.e., entanglement is present, the amount of entanglement quantified by the n-dimensional generalization of the concurrence, i.e., the I concurrence, mutually excludes the single-partite properties of the subsystems. In other words, the properties of the individual systems cease to exist and the information is transferred to genuine bipartite quantum correlations. Thus, complementarity is enforced by quantum correlations which force the information of single-partite properties to decrease. Genuine bipartite quantum properties and single-partite properties can only coexist according to a complementarity relation. If one of these properties increases the other necessarily must decrease. This also sheds light on questions concerning nonlocality in quantum mechanics. The complementarity relations clearly state that there exist bipartite realities which mutually exclude local or single-partite realities. In other words, the element of local reality ceases to exist when genuine bipartite quantum correlations are present. On the other hand, the complementarity relation is equally important for the definition of entanglement measures. In particular, the upper bound of the complementarity relation, for a pure composite quantum system, reduced by the amount of single-partite information present in the subsystems, defines and quantifies the proper entanglement measure which has been shown here to be identical to the I concurrence.

We have discussed the relation of the I concurrence to the entropy of entanglement and have shown that the entropy of entanglement is a strictly monotonic function of the I concurrence which ensures that the I concurrence forms a measure of entanglement on its own. We have further discussed the effect of the mixing of bipartite quantum systems on the complementarity relation. In this case the complementarity relation becomes an inequality but, more importantly, based on recent progress in the quantification of the generalized I concurrence for mixed quantum systems [39], a lower bound for the complementarity relation has been found.

Finally, we have studied some implications of the complementarity relation on n-port interferometers with which-path detection schemes and derived a proper definition of distinguishability. The concept of distinguishability in an n-port interferometer has been a controversial subject [16–18] since the complementarity relation between distinguishability and visibility, in general, was not saturated even in pure quantum systems [17,18]. This, however, makes the concept of complementarity between distinguishability and visibility useless since both quantities can increase or decrease at the same time and are thus not mutually exclusive.

We have shown that this unsatisfactory behavior has its origin in a misconception of distinguishability and derived a proper definition of the squared distinguishability which is the sum of the squared I concurrence and the squared predictability. Thus, distinguishability is a quantity which combines two mutually exclusive properties in one and, consequently, there are cases where a definition based on measurements cannot give satisfactory results. This is the case, in particular, when the Hilbert space dimension of the detector is smaller than that of the interfering quantum object for which the measurement-based definition of distinguishability can be smaller than that of the properly defined distinguishability. With the properly defined distinguishability the complementarity relation between distinguishability and visibility is, of course, saturated for pure systems. In summary, the concept of complementarity goes far beyond wave-particle duality and can be effectively applied for composite systems of arbitrary dimensions.

The authors gratefully acknowledge discussions with M. Freyberger, N. Lütkenhaus, M. Nölle, M. Peev, W. P. Schleich, S. Stenholm, M. Suda, and N. Vitanov. This work has been supported by a PSC-CUNY research grant [to one of the authors (J.B.)], and by the EU ToK project CAMEL (Grant No. MTKD-CT-2004-014427) and the EU RTN project EMALI (Grant No. MRTN-CT-2006-035369) [to one of the authors (M.J.)]. One of the authors (M.J.) appreciates the support and warm hospitality of Norbert Lütkenhaus during his stay at the Quantum Information Theory Group of the Universität Erlangen-Nürnberg in Germany where parts of this work began.

[1] N. Bohr, Naturwissenschaften **16**, 245 (1928); Nature (London) **121**, 580 (1928).
[2] W. K. Wootters and W. H. Zurek, Phys. Rev. D **19**, 473 (1979).
[3] D. M. Greenberger and A. Yasin, Phys. Lett. A **128**, 391 (1988).
[4] G. Jaeger, M. A. Horne, and A. Shimony, Phys. Rev. A **48**, 1023 (1993).
[5] G. Jaeger, A. Shimony, and L. Vaidman, Phys. Rev. A **51**, 54 (1995).
[6] B.-G. Englert, Phys. Rev. Lett. **77**, 2154 (1996).
[7] G. Björk and A. Karlsson, Phys. Rev. A **58**, 3477 (1998).
[8] B.-G. Englert and J. A. Bergou, Opt. Commun. **179**, 337 (2000).
[9] M. Jakob and J. A. Bergou, e-print arXiv:quant-ph/0302075.
[10] M. Jakob and J. A. Bergou (unpublished).
[11] T. E. Tessier, Found. Phys. Lett. **18**, 107 (2005).
[12] A. Hosoya, A. Carlini, and S. Okano, e-print arXiv:quant-ph/0405116.
[13] C. Brukner, M. Aspelmeyer, and A. Zeilinger, Found. Phys. **35**, 1909 (2005).
[14] M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen, U. Sen, and B. Synak-Radtke, Phys. Rev. A **71**, 062307 (2005); M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen, U. Sen, and B. Synak-Radtke, e-print arXiv:quant-ph/0410090.
[15] M. Jakob and J. A. Bergou, Int. J. Mod. Phys. B **20**, 1371 (2006).
[16] S. Dürr, Phys. Rev. A **64**, 042113 (2001).
[17] G. Bimonte and R. Musto, Phys. Rev. A **67**, 066101 (2003).
[18] G. Bimonte and R. Musto, J. Phys. A **36**, 11481 (2003).
[19] P. Rungta, V. Buzek, C. M. Caves, M. Hillery, and G. J. Milburn, Phys. Rev. A **64**, 042315 (2001).
[20] A. Luis, J. Phys. A **34**, 8597 (2001).
[21] F. Bloch, Phys. Rev. **70**, 460 (1946).
[22] F. T. Hioe and J. H. Eberly, Phys. Rev. Lett. **47**, 838 (1981).
[23] J. Pöttinger and K. Lendi, Phys. Rev. A **31**, 1299 (1985).
[24] K. Lendi, Phys. Rev. A **34**, 662 (1986).
[25] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Application*, Lecture Notes in Physics, Vol. 286 (Springer-Verlag, Berlin, 1987).
[26] G. Mahler and V. A. Weberruß, *Quantum Networks* (Springer-Verlag, Berlin, 1995).
[27] G. Kimura, Phys. Lett. A **314**, 339 (2003).
[28] M. S. Byrd and N. Khaneja, Phys. Rev. A **68**, 062322 (2003).
[29] M. Freyberger, Phys. Lett. A **242**, 193 (1998).
[30] C. Brukner and A. Zeilinger, Phys. Rev. Lett. **83**, 3354 (1999).
[31] C. Brukner and A. Zeilinger, Phys. Rev. A **63**, 022113 (2001).
[32] C. Brukner and A. Zeilinger, Philos. Trans. R. Soc. London, Ser. A **360**, 1061 (2002).
[33] C. Brukner and A. Zeilinger, in *Time, Quantum and Information*, edited by L. Castell and O. Ischebeck (Springer-Verlag, Berlin, 2003).
[34] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[35] W. K. Wootters, Quantum Inf. Comput. **1**, 27 (2001).
[36] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
[37] E. Gerjuoy, Phys. Rev. A **67**, 052308 (2003).

[38] A. Loziński, A. Buchleitner, K. Zyczkowski, and T. Wellens, Europhys. Lett. **62**, 168 (2003).
[39] F. Mintert, M. Kuś, and A. Buchleitner, Phys. Rev. Lett. **92**, 167902 (2004).
[40] K. Chen, S. Albeverio, and S. M. Fei, Phys. Rev. Lett. **95**, 040504 (2005).
[41] K. Chen, S. Albeverio, and S. M. Fei, Phys. Rev. Lett. **95**, 210501 (2005).
[42] C. S. Yu and H. S. Song, Phys. Rev. A **71**, 042331 (2005).
[43] G. Vidal, J. Mod. Opt. **47**, 355 (2000).
[44] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[45] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[46] U. Fano, Rev. Mod. Phys. **29**, 74 (1957).
[47] A. Rényi, *Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics, and Probability* (University of California Press, Berkeley, 1961), Vol. 1.
[48] A. Rényi, *Probability Theory* (North-Holland, Amsterdam, 1970).
[49] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **210**, 377 (1996).
[50] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A **58**, 883 (1998).
[51] K. Furuya, M. C. Nemes, and G. Q. Pellegrino, Phys. Rev. Lett. **80**, 5524 (1998).
[52] P. Zanardi, C. Zalka, and L. Faoro, Phys. Rev. A **62**, 030301(R) (2000).
[53] J. Gemmer and G. Mahler, Eur. Phys. J. D **17**, 385 (2001).
[54] A. S. M. de Castro and V. V. Dodonov, J. Opt. B: Quantum Semiclassical Opt. **5**, S593 (2003).
[55] O. Gühne and M. Lewenstein, Phys. Rev. A **70**, 022316 (2004).
[56] S. Stenholm, Ann. Phys. (N.Y.) **302**, 142 (2002).
[57] M. Jakob and S. Stenholm, Phys. Rev. A **67**, 032111 (2003).
[58] M. Jakob and S. Stenholm, Phys. Rev. A **69**, 042105 (2004).
[59] M. Jakob and S. Stenholm, Phys. Rev. A **70**, 012104 (2004).
[60] S. Stenholm and M. Jakob, Ann. Phys. (N.Y.) **310**, 106 (2004).
[61] S. Stenholm and M. Jakob, J. Mod. Opt. **51**, 841 (2004).
[62] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
[63] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **89**, 180402 (2002).
[64] L. Mandel, Opt. Lett. **16**, 1882 (1991).
[65] M. O. Scully and K. Drühl, Phys. Rev. A **25**, 2208 (1982).
[66] M. Jakob and J. Bergou, Phys. Rev. A **66**, 062107 (2002).
[67] A. C. Elitzur and S. Dolev, Phys. Rev. A **63**, 062109 (2001).