Einstein-Podolsky-Rosen correlations of photons: Quantum-field-theory approach

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We formulate a description of Einstein-Podolsky-Rosen-type experiments with photons which is especially convenient in the discussion of questions concerning Lorentz covariance. We classify all Lorentz-covariant two-photon states with sharp momenta and define observables corresponding to measurements of the linear polarization of photons. We also calculate explicitly the Einstein-Podolsky-Rosen correlation function and coincidence rate in the scalar two-photon state.

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I. INTRODUCTION

In the last decade the relativistic aspects of quantum information theory and Einstein-Podolsky-Rosen (EPR) correlations have been widely discussed [1–42], mostly for massive particles. Photons have been discussed in such a context only in a few papers [4–6,14,25,28–30,35,38]. In a recent paper [8] the correlation function in the EPR type of experiments with massive particles has been calculated in the covariant framework of the quantum field theory formalism. However, a lot of subtle experiments testing the basics of quantum mechanics (violation of Bell inequalities, teleportation, etc.) were performed with photons which are massless particles [43–45]. Therefore it is very interesting to consider EPR correlations and other aspects of quantum information theory for photons in the proper Lorentz-covariant framework. For a discussion of relativistic covariance the most appropriate is the quantum field theory approach. Nevertheless, quantum information problems are usually formulated in the language of spin degrees of freedom of nonrelativistic particles. Therefore in this paper we adopt a description of photon polarization which on the one hand is based on quantum-electrodynamical Fock space and on the other resembles the nonrelativistic spin. Then we classify all Lorentz-covariant two-photon states with sharp momenta and define, in the language of quantum electrodynamics, the proper observable which can describe the linear polarization measurements in EPR-type experiments. Next we use these tools to calculate explicitly the EPR correlation function and decay rates in the covariant two-photon states. We discuss also the freedom of choice of the explicit form of Lorentz group action on basis vectors of the carrier space of the irreducible unitary representation of Poincaré group. This freedom corresponds to the freedom of choice of the vector $\mathbf{a_k}$ defining the basis states (3) via the rotation (5). The form of polarization vectors $e^{\mu}(k)$ [Eq. (26)] and the explicit form of the linearly polarized photon state [Eqs. (38) and (39)] depend also on the choice of vector $\mathbf{a_k}$. Appendix A is devoted to a detailed discussion of these questions, and we take them into account throughout the paper. The well-defined behavior of all the states and observables discussed here under Lorentz transformations is the main advantage of our approach.

In Sec. II we establish the notation and briefly recall basic facts concerning massless representations of the Poincaré group and free quantum electromagnetic field. In Sec. III we adopt the description of linear polarization of light which is especially convenient in the discussion of quantum information issues. In Sec. IV we classify all two-photon states which transform covariantly under the Lorentz group action. The next section is devoted to a discussion of the proper observables used by Alice and Bob in the description of EPR-type experiments. In Sec. VI we calculate correlation function in EPR experiment in the covariantly transforming two-photon states. Section VII is devoted to a discussion of coincidence rates which are usually measured in the EPR experiments with photons. The last section contains concluding remarks.

II. PRELIMINARIES

A. Massless representations of the Poincaré group

Let us denote by $\mathcal{H}$ the carrier space of the irreducible massless representation of the Poincaré group. This space is spanned by the eigenvectors of the four-momentum operators $|k,\lambda\rangle$,

$$\hat{P}^{\mu}|k,\lambda\rangle = k^{\mu}|k,\lambda\rangle, \tag{1}$$

with $k^2=0$ and λ denoting helicity. We assume that vectors $|k,\lambda\rangle$ span the space $\mathcal{H}$. We use the following Lorentz-covariant normalization:

$$\langle k',\lambda'|k,\lambda\rangle = 2k^0\delta^3(\mathbf{k}'-\mathbf{k})\delta_{\lambda'\lambda}. \tag{2}$$

The vectors $|k,\lambda\rangle$ can be generated from standard vector $|\tilde{k},\lambda\rangle$, where $\tilde{k}=(1,0,0,1)$. We have

$$|k,\lambda\rangle = U(L_k)|\tilde{k},\lambda\rangle, \tag{3}$$

where

$$k = L_k\tilde{k} = R_{\mathbf{n_k}}B(k^0)\tilde{k}. \tag{4}$$

Here $B(k^0)$ denotes pure Lorentz boost along z axis taking vector $\tilde{k}$ to $k^0\tilde{k}$ and $R_{\mathbf{n_k}}$ denotes the rotation which acting on the vector (1, 0, 0, 1) gives the vector $(1,\mathbf{n_k})$, where $\mathbf{n_k}=\mathbf{k}/|\mathbf{k}|$. The most general form of $R_{\mathbf{n_k}}$ fulfilling Eq. (4) and with determinant equal to 1 is the following:

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$$R_{\mathbf{n_k}} = \left(\begin{array}{c|c} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{a_k}|\mathbf{n_k}\times\mathbf{a_k}|\mathbf{n_k} \end{array}\right), \tag{5}$$

where $|\mathbf{a_k}|=1$, $\mathbf{a_k}\perp\mathbf{n_k}$, and we treat the vectors in Eq. (5) as column matrices. The explicit form of $B(k^0)$ reads

$$B(k^0) = \begin{pmatrix} \frac{k^{02}+1}{2k^0} & 0 & 0 & \frac{k^{02}-1}{2k^0} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{k^{02}-1}{2k^0} & 0 & 0 & \frac{k^{02}+1}{2k^0} \end{pmatrix}. \tag{6}$$

It should be satisfied that

$$R_{\tilde{\mathbf{n}}} = I \quad \text{where } \tilde{\mathbf{n}} = (0,0,1). \tag{7}$$

Note that conditions (4) and (7) do not determine the rotation $R_{\mathbf{n_k}}$ uniquely—for the details see Appendix A.

Now, by means of the Wigner procedure we get

$$U(\Lambda)|k,\lambda\rangle = e^{i\lambda\psi(\Lambda,k)}|\Lambda k,\lambda\rangle, \tag{8}$$

where

$$e^{i\lambda\psi(\Lambda,k)} = U(R(\Lambda,k)), \tag{9}$$

with the Wigner rotation $R(\Lambda,k)$ given by

$$R(\Lambda,k) = L_{\Lambda k}^{-1}\Lambda L_k. \tag{10}$$

In the next subsection we will focus on the representations with $\lambda=\pm 1$ corresponding to photons.

B. Electromagnetic field

The electromagnetic four-potential operator is defined as

$$\hat{A}^\mu(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3\mathbf{k}}{2k^0}\sum_{\lambda=\pm1}[e^{ikx}e^{\mu\lambda}(k)a_\lambda^\dagger(k) + e^{-ikx}e^{*\mu\lambda}(k)a_\lambda(k)], \tag{11}$$

where the asterisk denotes complex conjugation and creation and annihilation operators $a_\lambda^\dagger(k)$ and $a_{\lambda'}(k')$ fulfill the following commutation relations:

$$[a_\lambda^\dagger(k),a_{\lambda'}^\dagger(k')] = [a_\lambda(k),a_{\lambda'}(k')] = 0, \tag{12}$$

$$[a_\lambda(k),a_{\lambda'}^\dagger(k')] = 2k^0\delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}. \tag{13}$$

Furthermore, we introduce a Poincaré-invariant vacuum $|0\rangle$ defined by

$$\langle 0|0\rangle = 1, \quad a_\lambda(k)|0\rangle = 0. \tag{14}$$

Thus the one-particle states

$$a_\lambda^\dagger(k)|0\rangle \tag{15}$$

are the basis vectors $|k,\lambda\rangle$ defined by Eq. (3) of the space $\mathcal{H}$ iff

$$U(\Lambda)a_\lambda^\dagger(k)U^\dagger(\Lambda) = e^{i\lambda\psi(\Lambda,k)}a_\lambda^\dagger(\Lambda k), \tag{16}$$

$$U(\Lambda)a_\lambda(k)U^\dagger(\Lambda) = e^{-i\lambda\psi(\Lambda,k)}a_\lambda(\Lambda k). \tag{17}$$

The electromagnetic field operator

$$\hat{F}^{\mu\nu}(x) = \partial^\mu\hat{A}^\nu(x) - \partial^\nu\hat{A}^\mu(x) \tag{18}$$

transforms like a tensor

$$U(\Lambda)\hat{F}^{\mu\nu}(x)U^\dagger(\Lambda) = \Lambda^{-1\mu}{}_\alpha\Lambda^{-1\nu}{}_\beta\hat{F}^{\alpha\beta}(\Lambda x). \tag{19}$$

We assume the Coulomb gauge

$$\hat{A}^0(x) = 0 \tag{20}$$

and the transversality condition

$$\partial_\mu\hat{A}^\mu(x) = 0. \tag{21}$$

The first of these equations implies $e^{0\lambda}(k)=0$ and the second $k_\mu e^{\mu\lambda}(k)=0$. As we know $e^{\mu\lambda}(k)$ are not four-vectors [46] but satisfy the following Weinberg condition:

$$e^{\mu\lambda}(k')e^{-i\lambda\psi(\Lambda,k)} = \left(\Lambda^\mu{}_\nu - \frac{k'^\mu}{k'^0}\Lambda^0{}_\nu\right)e^{\nu\lambda}(k), \tag{22}$$

where $k'=\Lambda k$. Therefore of course also $\hat{A}^\mu(x)$ is not a four-vector. To find an explicit form of $e^{\mu\lambda}(k)$ let us first determine $e^{\mu\lambda}(\tilde{k})$. We arrive at (choosing arbitrarily the normalization)

$$e^{\mu\lambda}(\tilde{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i\lambda \\ 0 \end{pmatrix}. \tag{23}$$

Now, using Eqs. (3), (4), (8), and (22) we find

$$e^{\mu\lambda}(k) = R_{\mathbf{n_k}}{}^\nu{}_\mu e^{\nu\lambda}(\tilde{k}), \tag{24}$$

which means that

$$e^{0\lambda}(k) = 0, \quad \mathbf{e}^\lambda(k) = R_{\mathbf{n_k}}\mathbf{e}^\lambda(\tilde{k}). \tag{25}$$

Inserting the explicit form of $R_{\mathbf{n_k}}$ given by Eq. (5) we obtain

$$\mathbf{e}^\lambda(k) = \frac{1}{\sqrt{2}}[\mathbf{a_k} - i\lambda(\mathbf{n_k}\times\mathbf{a_k})]. \tag{26}$$

Equations (4), (23), and (25) imply the following conditions:

$$\mathbf{e}^{*\lambda'}(k)\mathbf{e}^\lambda(k) = \delta^{\lambda'\lambda}, \tag{27}$$

$$\sum_\lambda e^{*\lambda i}(k)e^{\lambda j}(k) = \delta^{ij} - \frac{k^ik^j}{|\mathbf{k}|^2}, \tag{28}$$

$$\mathbf{k}\cdot\mathbf{e}^\lambda(k) = 0. \tag{29}$$

Finally, the charge conjugation and parity act on the field operator as follows:

$$\mathsf{C}\hat{\mathbf{A}}(x)\mathsf{C}^{-1} = -\hat{\mathbf{A}}(x), \tag{30}$$

$$\mathsf{P}\hat{\mathbf{A}}(x)\mathsf{P}^{-1} = -\hat{\mathbf{A}}(x^{\pi}), \tag{31}$$

where $x^{\pi}=(x^0,-\mathbf{x})$. These conditions lead to the following transformation rule for creation operators:

$$\mathsf{C}a^{\dagger}_{\lambda}(k)\mathsf{C}^{-1} = -a^{\dagger}_{\lambda}(k), \tag{32}$$

$$\mathsf{P}a^{\dagger}_{\lambda}(k)\mathsf{P}^{-1} = \chi_{\lambda}(k)a^{\dagger}_{-\lambda}(k^{\pi}). \tag{33}$$

From Eq. (33) and the condition $\mathsf{P}^2=\pm I$ we get

$$\chi_{\lambda}(k)\chi_{-\lambda}(k^{\pi}) = 1. \tag{34}$$

Note that the explicit form of the phase factor $\chi_{\lambda}(k)$ depends on the choice of the vector $\mathbf{a}_{\mathbf{k}}$ in Eq. (5)—a detailed discussion of this point can be found in Appendix A after Eq. (A5).

III. POLARIZATION

In this section we will remind the description of polarization of photons which is especially convenient in the discussion of correlations in EPR experiments.

In the space of state vectors with definite momentum we can introduce the following basis:

$$\sum_{\lambda} \mathbf{e}^{\lambda}(k)|k,\lambda\rangle \tag{35}$$

[there are only two independent vectors because $\mathbf{k}\cdot\mathbf{e}^{\lambda}(k)=0$]. Therefore the most general one-photon state with four-momentum k can be written as

$$|\boldsymbol{\varepsilon},k\rangle \equiv \sum_{\lambda} \boldsymbol{\varepsilon}\cdot\mathbf{e}^{\lambda}(k)|k,\lambda\rangle, \tag{36}$$

where $\boldsymbol{\varepsilon}$ is an arbitrary complex vector fulfilling $|\boldsymbol{\varepsilon}|^2=1$. It holds that

$$\langle\boldsymbol{\varepsilon},k|\boldsymbol{\varepsilon}',p\rangle = 2k^0\delta^3(\mathbf{k}-\mathbf{p})\left(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon} - \frac{(\boldsymbol{\varepsilon}^*\cdot\mathbf{k})(\boldsymbol{\varepsilon}'\cdot\mathbf{k})}{|\mathbf{k}|^2}\right). \tag{37}$$

The state of the linearly polarized photon with four-momentum k reads [46]

$$|\boldsymbol{\varepsilon}_{\theta},k\rangle = \frac{1}{\sqrt{2}}(e^{i\theta}|k,+1\rangle + e^{-i\theta}|k,-1\rangle), \tag{38}$$

where the real vector $\boldsymbol{\varepsilon}_{\theta}$ has the form

$$\boldsymbol{\varepsilon}_{\theta} = \frac{1}{\sqrt{2}}\sum_{\lambda}(\mathbf{e}^{\lambda}e^{-i\lambda\theta}) = \mathbf{a}_{\mathbf{k}}\cos\theta - (\mathbf{n}_{\mathbf{k}}\times\mathbf{a}_{\mathbf{k}})\sin\theta. \tag{39}$$

Equation (37) implies

$$\langle\boldsymbol{\varepsilon}_{\theta},k|\boldsymbol{\varepsilon}_{\theta'},p\rangle = 2k^0\delta(\mathbf{k}-\mathbf{p})\cos(\theta-\theta'). \tag{40}$$

The angle of polarization, θ, is measured with respect to the vector

$$\boldsymbol{\varepsilon}_{\theta=0} = \frac{1}{\sqrt{2}}\sum_{\lambda}\mathbf{e}^{\lambda} = \mathbf{a}_{\mathbf{k}}. \tag{41}$$

There is one point which should be clarified here. As was mentioned before, the choice of vector $\mathbf{a}_{\mathbf{k}}$ is a matter of convention—see Appendix A for the details. It is obvious that in an experiment the linear polarization can be measured with respect to arbitrarily chosen axis—say, $\mathbf{l}$. However, if $\mathbf{l}$ is not parallel to $\mathbf{a}_{\mathbf{k}}$, but an angle between these vectors is equal to α, then the photon polarized linearly under the angle β with respect to $\mathbf{l}$ is described by the state vector $|\boldsymbol{\varepsilon}_{\alpha+\beta},k\rangle$ or $|\boldsymbol{\varepsilon}_{\alpha-\beta},k\rangle$, depending on the space configuration of vectors $\mathbf{l}$, $\mathbf{a}_{\mathbf{k}}$, and $\mathbf{n}_{\mathbf{k}}\times\mathbf{a}_{\mathbf{k}}$. Of course the most comfortable situation we have when in the experiment the angle of polarization is measured with respect to $\mathbf{a}_{\mathbf{k}}$. For example, we can choose the convention denoted as convention I in Appendix A, in which for every momentum vector $\mathbf{k}$ parallel to xy plane the vector $\mathbf{a}_{\mathbf{k}}=(0,0,1)$.

So instead of $\{|k,\lambda\rangle\}_{\lambda=\pm1}$ we can use basis consisting of two vectors describing states polarized linearly in two orthogonal directions

$$|\boldsymbol{\varepsilon}_{\theta},k\rangle \quad \text{and} \quad |\boldsymbol{\varepsilon}_{\theta_{\perp}},k\rangle = |\boldsymbol{\varepsilon}_{\theta+\pi/2},k\rangle, \tag{42}$$

where $\boldsymbol{\varepsilon}_{\theta}$ is given by Eq. (39) and the explicit form of $\mathbf{e}_{\theta_{\perp}}$ reads

$$\boldsymbol{\varepsilon}_{\theta_{\perp}} = \frac{-i}{\sqrt{2}}\sum_{\lambda}(\lambda\mathbf{e}^{\lambda}e^{-i\lambda\theta}) = -\mathbf{a}_{\mathbf{k}}\sin\theta - (\mathbf{n}_{\mathbf{k}}\times\mathbf{a}_{\mathbf{k}})\cos\theta. \tag{43}$$

Moreover, for arbitrary state vector $|\varepsilon,k\rangle$ defined in Eq. (36) we have

$$|\boldsymbol{\varepsilon},k\rangle = (\boldsymbol{\varepsilon}_{\theta}\cdot\boldsymbol{\varepsilon})|\boldsymbol{\varepsilon}_{\theta},k\rangle + (\boldsymbol{\varepsilon}_{\theta_{\perp}}\cdot\boldsymbol{\varepsilon})|\boldsymbol{\varepsilon}_{\theta_{\perp}},k\rangle, \tag{44}$$

where we have used Eq. (37) and the condition $\mathbf{k}\cdot\boldsymbol{\varepsilon}_{\theta}=\mathbf{k}\cdot\boldsymbol{\varepsilon}_{\theta_{\perp}}=0$, which follows from Eq. (29).

We can introduce creation operators $a^{\dagger}_{\theta}(k)$ which acting on the vacuum create photons with four-momentum k, polarized linearly under the angle θ. From Eq. (38) we get

$$a^{\dagger}_{\theta}(k) = \frac{1}{\sqrt{2}}\sum_{\lambda}e^{i\lambda\theta}a^{\dagger}_{\lambda}(k) = \sum_{\lambda}\boldsymbol{\varepsilon}_{\theta}\cdot\mathbf{e}^{\lambda}(k)a^{\dagger}_{\lambda}(k). \tag{45}$$

Therefore

$$a^{\dagger}_{\lambda}(k) = \frac{e^{-i\lambda\theta}}{\sqrt{2}}[a^{\dagger}_{\theta}(k) - i\lambda a^{\dagger}_{\theta_{\perp}}(k)]. \tag{46}$$

IV. COVARIANT STATES

To discuss correlations of photons in the relativistic context it is very convenient to introduce vectors which transform covariantly under Lorentz transformations.

A. One-particle states

Now we want to define one-particle states with definite transformation properties. To construct covariant states we can use vectors $\mathbf{k}$ and $\mathbf{e}^{\lambda}(k)$. Unfortunately vector $\mathbf{e}^{\lambda}(k)$ is not space part of any four-vector. However, using the above vectors we can construct, in analogy with the electromagnetic field tensor, the following antisymmetric, gauge-independent, and mutually dual quantities:

$$f^{\mu\nu}(k)^{\lambda} = k^{\mu}e^{\nu\lambda}(k) - k^{\nu}e^{\mu\lambda}(k), \tag{47}$$

$$\tilde{f}^{\mu\nu}(k)^{\lambda} = \varepsilon^{\mu\nu\alpha\beta}[k_{\alpha}e^{\lambda}_{\beta}(k) - k_{\beta}e^{\lambda}_{\alpha}(k)]. \tag{48}$$

Note that it holds that

$$\hat{F}^{\mu\nu}(x)|0\rangle = \frac{i}{(2\pi)^{3/2}}\int \frac{d^3\mathbf{k}}{2k^0}\sum_{\lambda=-1,1} f^{\mu\nu}(k)^{\lambda}e^{ikx}|k,\lambda\rangle. \tag{49}$$

Therefore, from Eqs. (8) and (19) we have

$$f^{\mu\nu}(k)^{\lambda}e^{i\lambda\psi(\Lambda,k)} = \Lambda^{-1\mu}{}_{\alpha}\Lambda^{-1\nu}{}_{\beta}f^{\alpha\beta}(\Lambda k)^{\lambda}, \tag{50}$$

which implies that the state $\sum_{\lambda=\pm1}f^{\mu\nu}(k)^{\lambda}|k,\lambda\rangle$ transforms like a tensor,

$$U(\Lambda)\sum_{\lambda=\pm1} f^{\mu\nu}(k)^{\lambda}|k,\lambda\rangle = \Lambda^{-1\mu}{}_{\alpha}\Lambda^{-1\nu}{}_{\beta}\sum_{\lambda=\pm1} f^{\alpha\beta}(\Lambda k)^{\lambda}|\Lambda k,\lambda\rangle, \tag{51}$$

while the state $\sum_{\lambda=\pm1}\tilde{f}^{\mu\nu}(k)^{\lambda}|k,\lambda\rangle$ transforms like a tensor dual to $\sum_{\lambda=\pm1}f^{\mu\nu}(k)^{\lambda}|k,\lambda\rangle$.

The most general one-photon state can be written as

$$|\varphi\rangle = \int \sum_{\lambda=\pm1}\frac{d^3\mathbf{k}}{2k^0}\varphi^{\lambda}(k)|k,\lambda\rangle. \tag{52}$$

The state (52) can be expressed as a combination of covariantly transforming one-particle states,

$$|\varphi\rangle = \int \frac{d^3\mathbf{k}}{2k^0}\varphi^{\mu\nu}(k)\left(\sum_{\lambda=\pm1} f_{\mu\nu}(k)^{\lambda}|k,\lambda\rangle\right), \tag{53}$$

where

$$\varphi^{\lambda}(k) = \varphi^{\mu\nu}(k)f_{\mu\nu}(k)^{\lambda}. \tag{54}$$

Note that for the state (52) it holds that

$$\langle\varphi|\varphi\rangle = \int \sum_{\lambda=\pm1}\frac{d^3\mathbf{k}}{2k^0}\varphi_{\lambda}(k)\varphi^{\lambda}(k). \tag{55}$$

Similar formulas can be obtained with the help of $\tilde{f}_{\mu\nu}(k)^{\lambda}$.

B. Two-particle states

The most general two-photon state has the form

$$|\Phi\rangle = \int \frac{d^3\mathbf{k}}{2k^0}\frac{d^3\mathbf{p}}{2p^0}\sum_{\lambda\sigma}\Phi_{\lambda\sigma}(k,p)|(k,\lambda),(p,\sigma)\rangle, \tag{56}$$

where

$$|(k,\lambda),(p,\sigma)\rangle = a^{\dagger}_{\lambda}(k)a^{\dagger}_{\sigma}(p)|0\rangle. \tag{57}$$

From Eq. (12) we have

$$|(k,\lambda),(p,\sigma)\rangle = |(p,\sigma),(k,\lambda)\rangle, \tag{58}$$

which implies, taking into account that antisymmetric part of $\Phi(k,p)_{\lambda\sigma}$ vanishes in the integral (56), that we can assume

$$\Phi_{\lambda\sigma}(k,p) = \Phi_{\sigma\lambda}(p,k). \tag{59}$$

Equations (13) and (57) imply the following normalization of the two-photon states:

$$\begin{aligned}\langle(k,\lambda),(p,\sigma)|(k',\lambda'),(p',\sigma')\rangle &= 4k^0p^0[\delta(\mathbf{k}-\mathbf{k}')\delta(\mathbf{p}-\mathbf{p}')\delta^{\lambda\lambda'}\delta^{\sigma\sigma'} \\ &\quad + \delta(\mathbf{k}-\mathbf{p}')\delta(\mathbf{p}-\mathbf{k}')\delta^{\lambda\sigma'}\delta^{\sigma\lambda'}].\end{aligned} \tag{60}$$

Note that it holds

$$\langle\Phi|\Phi\rangle = 2\int \frac{d^3\mathbf{k}}{2k^0}\frac{d^3\mathbf{p}}{2p^0}\sum_{\lambda\sigma}\Phi^{*}_{\lambda\sigma}(k,p)\Phi_{\lambda\sigma}(k,p). \tag{61}$$

Now, we construct covariantly transforming, two-photon states.

1. Scalar states

The only candidates which can be used to construct scalar states are the following quantities (we are interested in construction of two-particle covariant states, so we consider quantities with two free indices λ,σ):

$$f^{\mu\nu}(k)^{\lambda}f_{\mu\nu}(p)^{\sigma}, \quad p_{\mu}f^{\mu\alpha}(k)^{\lambda}f_{\alpha\nu}(p)^{\sigma}k^{\nu}, \tag{62}$$

$$f^{\mu\nu}(k)^{\lambda}\tilde{f}_{\mu\nu}(p)^{\sigma}, \quad p_{\mu}f^{\mu\alpha}(k)^{\lambda}\tilde{f}_{\alpha\nu}(p)^{\sigma}k^{\nu}, \tag{63}$$

$$\tilde{f}^{\mu\nu}(k)^{\lambda}\tilde{f}_{\mu\nu}(p)^{\sigma}, \quad p_{\mu}\tilde{f}^{\mu\alpha}(k)^{\lambda}\tilde{f}_{\alpha\nu}(p)^{\sigma}k^{\nu}. \tag{64}$$

Let us introduce the following notation:

$$\phi_{\lambda\sigma}(k,p) \equiv (kp)[\mathbf{e}^{\lambda}(k)\cdot\mathbf{e}^{\sigma}(p)] + [\mathbf{p}\cdot\mathbf{e}^{\lambda}(k)][\mathbf{k}\cdot\mathbf{e}^{\sigma}(p)], \tag{65}$$

$$\psi_{\lambda\sigma}(k,p) \equiv (k^0\mathbf{p} - p^0\mathbf{k})\cdot[\mathbf{e}^{\lambda}(k)\times\mathbf{e}^{\sigma}(p)]. \tag{66}$$

Note that it holds that

$$\sum_{\lambda\sigma}\phi^{*}_{\lambda\sigma}(k,p)\phi_{\lambda\sigma}(k,p) = \sum_{\lambda\sigma}\psi^{*}_{\lambda\sigma}(k,p)\psi_{\lambda\sigma}(k,p) = 2(kp)^2. \tag{67}$$

Moreover, using Eq. (26) one can find that

$$\psi_{\lambda\sigma}(k,p) = -i\frac{\lambda+\sigma}{2}\phi_{\lambda\sigma}(k,p). \tag{68}$$

Using formulas (65) and (66) we find

$$f^{\mu\nu}(k)^{\lambda}f_{\mu\nu}(p)^{\sigma} = -\frac{2}{(kp)}p_{\mu}f^{\mu\alpha}(k)^{\lambda}f_{\alpha\nu}(p)^{\sigma}k^{\nu} = -2\phi_{\lambda\sigma}(k,p), \tag{69}$$

$$\tilde{f}^{\mu\nu}(k)^{\lambda}\tilde{f}_{\mu\nu}(p)^{\sigma} = -\frac{2}{(kp)}p_{\mu}\tilde{f}^{\mu\alpha}(k)^{\lambda}\tilde{f}_{\alpha\nu}(p)^{\sigma}k^{\nu} = 8\phi_{\lambda\sigma}(k,p), \tag{70}$$

$$f^{\mu\nu}(k)^{\lambda}\tilde{f}_{\mu\nu}(p)^{\sigma} = -\frac{2}{(kp)}p_{\mu}f^{\mu\alpha}(k)^{\lambda}\tilde{f}_{\alpha\nu}(p)^{\sigma}k^{\nu} = 4\psi_{\lambda\sigma}(k,p). \tag{71}$$

So finally we find only two states which are Lorentz scalars:

$$|\phi(k,p)\rangle = \sum_{\lambda\sigma}\phi_{\lambda\sigma}(k,p)|(k,\lambda),(p,\sigma)\rangle, \tag{72}$$

$$|\psi(k,p)\rangle = \sum_{\lambda\sigma} \psi_{\lambda\sigma}(k,p)|(k,\lambda),(p,\sigma)\rangle. \tag{73}$$

In the center-of-mass frame (for example, when photons result from the decay of a massive particle) we have

$$|\phi(k,k^{\pi})\rangle = 2|\mathbf{k}|^2\sum_{\lambda\sigma} \mathbf{e}^{\lambda}(k)\cdot\mathbf{e}^{\sigma}(k^{\pi})|(k,\lambda),(k^{\pi},\sigma)\rangle, \tag{74}$$

$$|\psi(k,k^{\pi})\rangle = -2|\mathbf{k}|\sum_{\lambda\sigma} \mathbf{k}\cdot[\mathbf{e}^{\lambda}(k)\times\mathbf{e}^{\sigma}(k^{\pi})]|(k,\lambda),(k^{\pi},\sigma)\rangle. \tag{75}$$

The action of the parity operator on the above vectors has the form

$$\hat{\mathsf{P}}|\phi(k,k^{\pi})\rangle = +|\phi(k,k^{\pi})\rangle, \tag{76}$$

$$\hat{\mathsf{P}}|\psi(k,k^{\pi})\rangle = -|\psi(k,k^{\pi})\rangle. \tag{77}$$

So we conclude that the state (72) is a scalar state while the state (73) is a pseudoscalar one.

2. Four-vector states

Similarly, to construct four-vector states we consider the quantities

$$f^{\mu\nu}(k)^{\lambda}f_{\nu\alpha}(p)^{\sigma}k^{\alpha}, \quad f^{\mu\nu}(k)^{\lambda}\tilde{f}_{\nu\alpha}(p)^{\sigma}k^{\alpha}, \tag{78}$$

$$\tilde{f}^{\mu\nu}(k)^{\lambda}\tilde{f}_{\nu\alpha}(p)^{\sigma}k^{\alpha}. \tag{79}$$

Four-vectors similar to (79) can be obtained by exchange $k\leftrightarrow p$, $\lambda\leftrightarrow\sigma$. After some calculations we get

$$f^{\mu\nu}(k)^{\lambda}f_{\nu\alpha}(p)^{\sigma}k^{\alpha} = -\frac{1}{4}\tilde{f}^{\mu\nu}(k)^{\lambda}\tilde{f}_{\nu\alpha}(p)^{\sigma}k^{\alpha} = k^{\mu}\phi_{\lambda\sigma}(k,p), \tag{80}$$

$$f^{\mu\nu}(k)^{\lambda}\tilde{f}_{\nu\alpha}(p)^{\sigma}k^{\alpha} = -2k^{\mu}\psi_{\lambda\sigma}(k,p). \tag{81}$$

Therefore the case of four-vector states reduces to the case of scalar states. Indeed, for example, the most general state constructed from Eq. (80) has the form

$$\int \frac{d^3\mathbf{k}}{2k^0}\frac{d^3\mathbf{p}}{2p^0}\sum_{\lambda\sigma} g^{\mu}(k,p)k_{\mu}\phi_{\lambda\sigma}(k,p)|(k,\lambda),(p,\sigma)\rangle, \tag{82}$$

which can be written as

$$\int \frac{d^3\mathbf{k}}{2k^0}\frac{d^3\mathbf{p}}{2p^0}\sum_{\lambda\sigma} g(k,p)\phi_{\lambda\sigma}(k,p)|(k,\lambda),(p,\sigma)\rangle, \tag{83}$$

where $g(k,p)=g^{\mu}(k,p)k_{\mu}$. But (83) can be constructed directly using the state (72). We have a similar situation for the state (81).

3. Tensor states

Finally, second-rank tensor states can be constructed using the following quantities:

$$f^{\mu\alpha}(k)^{\lambda}f_{\alpha\nu}(p)^{\sigma}, \quad f^{\mu\alpha}(k)^{\lambda}\tilde{f}_{\alpha\nu}(p)^{\sigma}, \quad \tilde{f}^{\mu\alpha}(k)^{\lambda}\tilde{f}_{\alpha\nu}(p)^{\sigma}. \tag{84}$$

Other covariant quantities vanish, because $k_{\mu}f^{\mu\nu}(k)^{\lambda}=0$, $p_{\mu}p_{\nu}f^{\mu\nu}(k)^{\lambda}=0$ (and similarly when we change $p\leftrightarrow k$). After some calculation we get

$$f^{\mu\alpha}(k)^{\lambda}f_{\alpha\nu}(p)^{\sigma} = -\{e^{\mu\lambda}(k)e^{\sigma}_{\nu}(p)(kp) - k^{\mu}p_{\nu}[\mathbf{e}^{\lambda}(k)\cdot\mathbf{e}^{\sigma}(p)] + e^{\mu\lambda}(k)p_{\nu}[\mathbf{k}\cdot\mathbf{e}^{\sigma}(p)] + k^{\mu}e^{\sigma}_{\nu}(p)[\mathbf{p}\cdot\mathbf{e}^{\lambda}(k)]\}, \tag{85}$$

$$f^{\mu\alpha}(k)^{\lambda}\tilde{f}_{\alpha\nu}(p)^{\sigma} = 2[k^{\mu}e^{\alpha\lambda}(k) - k^{\alpha}e^{\mu\lambda}(k)]\varepsilon_{\alpha\nu\beta j}p^{\beta}e^{j\sigma}(p), \tag{86}$$

$$\tilde{f}^{\mu\alpha}(k)^{\lambda}\tilde{f}_{\alpha\nu}(p)^{\sigma} = -4\delta^{\mu}_{\nu}\phi_{\lambda\sigma}(k,p) + 4f^{\mu\alpha}(k)^{\lambda}f_{\alpha\nu}(p)^{\sigma}. \tag{87}$$

We see that states constructed from Eq. (87) can be obtained from Eqs. (85) and (72). Therefore we have only two independent two-particle tensor states

$$\sum_{\lambda\sigma} f^{\mu\alpha}(k)^{\lambda}f_{\alpha\nu}(p)^{\sigma}|(k,\lambda),(p,\sigma)\rangle \tag{88}$$

and

$$\sum_{\lambda\sigma} f^{\mu\alpha}(k)^{\lambda}\tilde{f}_{\alpha\nu}(p)^{\sigma}|(k,\lambda),(p,\sigma)\rangle. \tag{89}$$

Summarizing, we found the following covariantly transforming two-particle states: (72), (73), (88), and (89). Of course, among the variety of states (88) and (89) corresponding to fixed values of μ, ν only two are independent since for fixed k and p two-particle states $|(k,\lambda),(p,\sigma)\rangle$ span the four-dimensional subspace.

V. OBSERVABLES

In this section we will construct observables used by Alice and Bob in EPR-type experiments with photons. In such experiments Alice and Bob usually measure the linear polarization of photons flying in a well-defined direction—say, $\mathbf{n_k}$ for Alice and $\mathbf{n_p}$ for Bob. Therefore we need an observable which gives $+1$ when acting on photons with momentum $\mathbf{k}=|\mathbf{k}|\mathbf{n_k}$, polarized linearly under the angle θ, and -1 for the similar photon polarized under the angle $\theta_{\perp}=\theta+\pi/2$. Of course, we cannot use the operator

$$|\boldsymbol{\varepsilon}_{\theta},k\rangle\langle\boldsymbol{\varepsilon}_{\theta},k| - |\boldsymbol{\varepsilon}_{\theta_{\perp}},k\rangle\langle\boldsymbol{\varepsilon}_{\theta_{\perp}},k| \tag{90}$$

as our observable. The reason is that the above operator is nonzero only in one-particle subspace of Fock space but Alice and Bob perform measurements on two-photon states. Thus we need an analog of the quantum mechanical observables $(|\boldsymbol{\varepsilon}_{\theta},k\rangle\langle\boldsymbol{\varepsilon}_{\theta},k|-|\boldsymbol{\varepsilon}_{\theta_{\perp}},k\rangle\langle\boldsymbol{\varepsilon}_{\theta_{\perp}},k|)\otimes I$ for Alice and $I\otimes(|\boldsymbol{\varepsilon}_{\theta},k\rangle\langle\boldsymbol{\varepsilon}_{\theta},k|-|\boldsymbol{\varepsilon}_{\theta_{\perp}},k\rangle\langle\boldsymbol{\varepsilon}_{\theta_{\perp}},k|)$ for Bob.

Now we construct the proper observable. Recall that particle number operator has the form

$$\hat{N} = \int \frac{d^3\mathbf{k}}{2k^0} \sum_{\lambda} a_{\lambda}^{\dagger}(k) a_{\lambda}(k). \tag{91}$$

With the help of Eqs. (45) and (46) we can write

$$\hat{N} = \int \frac{d^3\mathbf{k}}{2k^0} [a_{\theta}^{\dagger}(k) a_{\theta}(k) + a_{\theta_{\perp}}^{\dagger}(k) a_{\theta_{\perp}}(k)]. \tag{92}$$

It suggests that $a_{\theta}^{\dagger}(k) a_{\theta}(k)$ can be interpreted as the density of the number of photons with four-momentum k and polarized linearly under the angle θ. Therefore we can expect that integrating $a_{\theta}^{\dagger}(k) a_{\theta}(k)$ with the Dirac delta $\delta(\mathbf{n} - \frac{\mathbf{k}}{|\mathbf{k}|})$ projecting on the fixed direction $\mathbf{n}$ (see Appendix B) we will obtain a projector on states with momentum parallel to $\mathbf{n}$ and polarized linearly under the angle θ. So we define the operator

$$\Pi_{\mathbf{n}}^{\theta} = \int \frac{d^3\mathbf{k}}{2k^0} \delta\left(\mathbf{n} - \frac{\mathbf{k}}{|\mathbf{k}|}\right) a_{\theta}^{\dagger}(k) a_{\theta}(k) = \int \frac{1}{2} \omega d\omega a_{\theta}^{\dagger}(\omega, \omega\mathbf{n}) a_{\theta}(\omega, \omega\mathbf{n}), \tag{93}$$

where $a_{\theta}^{\dagger}(\omega, \omega\mathbf{n}) = a_{\theta}^{\dagger}(k)$ for $k = (\omega, \omega\mathbf{n})$ and properties of the delta function $\delta(\mathbf{n} - \frac{\mathbf{k}}{|\mathbf{k}|})$ are summarized in Appendix B. It holds

$$\Pi_{\mathbf{n}}^{\theta} | \boldsymbol{\varepsilon}_{\theta_1}, k\rangle = \delta(\mathbf{n} - \mathbf{n}_{\mathbf{k}}) \cos(\theta - \theta_1) | \boldsymbol{\varepsilon}_{\theta}, k\rangle. \tag{94}$$

Note that from Eq. (44) we have

$$| \boldsymbol{\varepsilon}_{\theta_1}, k\rangle = \cos(\theta - \theta_1) | \boldsymbol{\varepsilon}_{\theta}, k\rangle + \cos(\theta_{\perp} - \theta_1) | \boldsymbol{\varepsilon}_{\theta_{\perp}}, k\rangle. \tag{95}$$

This formula corresponds to the classical Malus's law stating that the intensity of linearly polarized beam of light transmitted through the polarizer is equal to $I_0 \cos^2 \alpha$, where I_0 is the intensity of the incident beam and α is an angle between the transmission axis of the polarizer and the plane of polarization of the incident beam. The smeared operator

$$\Pi_{\Omega}^{\theta} = \int_{\Omega} \sin \beta d\alpha d\beta \Pi_{\mathbf{n}}^{\theta}, \tag{96}$$

where Ω is a solid angle and $\mathbf{n} = (\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta)$, is a proper projector, i.e.,

$$(\Pi_{\Omega}^{\theta})^2 = \Pi_{\Omega}^{\theta}. \tag{97}$$

For two-particle states we have

$$\Pi_{\mathbf{n}}^{\theta} | (\boldsymbol{\varepsilon}_{\theta_1}, k), (\boldsymbol{\varepsilon}_{\theta_2}, p)\rangle = \delta(\mathbf{n} - \mathbf{n}_{\mathbf{k}}) \cos(\theta - \theta_1) | (\boldsymbol{\varepsilon}_{\theta}, k), (\boldsymbol{\varepsilon}_{\theta_2}, p)\rangle + \delta(\mathbf{n} - \mathbf{n}_{\mathbf{p}}) \cos(\theta - \theta_2) | (\boldsymbol{\varepsilon}_{\theta_1}, k), (\boldsymbol{\varepsilon}_{\theta}, p)\rangle, \tag{98}$$

where $|(\boldsymbol{\varepsilon}_{\theta_1}, k), (\boldsymbol{\varepsilon}_{\theta_2}, p)\rangle = a_{\theta_1}^{\dagger}(k) a_{\theta_2}^{\dagger}(p) |0\rangle$ describes the state of two photons with four-momenta k and p and polarized linearly under the angles θ_1 and θ_2, respectively. For two-photon states produced in decays of elementary particles conservation laws imply that $\mathbf{n}_{\mathbf{k}} \neq \mathbf{n}_{\mathbf{p}}$. Therefore, when we choose the solid angle Ω in such a way that $\mathbf{n}_{\mathbf{k}} \in \Omega$ and $\mathbf{n}_{\mathbf{p}} \notin \Omega$ we get

$$\Pi_{\Omega}^{\theta} | (\boldsymbol{\varepsilon}_{\theta_1}, k), (\boldsymbol{\varepsilon}_{\theta_2}, p)\rangle = \cos(\theta - \theta_1) | (\boldsymbol{\varepsilon}_{\theta}, k), (\boldsymbol{\varepsilon}_{\theta_2}, p)\rangle, \tag{99}$$

and similarly in the case when $\mathbf{n}_{\mathbf{k}} \notin \Omega$ and $\mathbf{n}_{\mathbf{p}} \in \Omega$.

So finally we define the following observable:

$$\hat{S}_{\Omega}^{\theta} = \Pi_{\Omega}^{\theta} - \Pi_{\Omega}^{\theta_{\perp}}, \tag{100}$$

which describes measurements performed by Alice and Bob in EPR experiments with photons.

VI. CORRELATION FUNCTION

In this section we will calculate correlation function in EPR experiments with photons. Let Alice measure the observable $\hat{S}_{\Omega_A}^{\theta}$ and Bob $\hat{S}_{\Omega_B}^{\tilde{\theta}}$, and we assume that $\Omega_A \cap \Omega_B = \emptyset$ [compare Eq. (100)]. This assumption corresponds to the experimental situation in which Alice's and Bob's detectors catch photons coming from directions contained in the solid angles Ω_A and Ω_B, respectively. The correlation function in the arbitrary pure state $|\Phi\rangle$ has the form

$$\mathcal{C}_{\Omega_A, \Omega_B}^{\Phi}(\theta, \tilde{\theta}) = \frac{\langle \Phi | \hat{S}_{\Omega_A}^{\theta} \hat{S}_{\Omega_B}^{\tilde{\theta}} | \Phi\rangle}{\langle \Phi | \Phi\rangle}. \tag{101}$$

Now let us take as $|\Phi\rangle$ an arbitrary state with sharp four-momenta k and p,

$$|\Phi(k,p)\rangle = \sum_{\lambda\sigma} \Phi_{\lambda\sigma}(k,p) |(k,\lambda),(p,\sigma)\rangle, \tag{102}$$

and let us assume that $\mathbf{n}_{\mathbf{k}} \in \Omega_A$ and $\mathbf{n}_{\mathbf{p}} \in \Omega_B$ ($\Omega_A \cap \Omega_B = \emptyset$).

Taking into account the explicit form of the observables $\hat{S}_{\Omega_A}^{\theta}$, $\hat{S}_{\Omega_B}^{\tilde{\theta}}$ [Eqs. (100), (96), and (93)] we find in this case

$$\begin{aligned}
\mathcal{C}_{\Omega_A, \Omega_B}^{\Phi(k,p)}(\theta, \tilde{\theta}) &= \frac{4(\delta^3(\mathbf{0}))^2 k^0 p^0}{\langle \Phi(k,p) | \Phi(k,p)\rangle} \\
&\times \Bigg\{ \left| \sum_{\lambda\sigma} \Phi_{\lambda\sigma}^{*}(k,p) [\boldsymbol{\varepsilon}_{\theta}(k) \cdot \mathbf{e}^{\lambda}(k)][\boldsymbol{\varepsilon}_{\tilde{\theta}}(p) \cdot \mathbf{e}^{\sigma}(p)] \right|^2 \\
&+ \left| \sum_{\lambda\sigma} \Phi_{\lambda\sigma}^{*}(k,p) [\boldsymbol{\varepsilon}_{\theta_{\perp}}(k) \cdot \mathbf{e}^{\lambda}(k)][\boldsymbol{\varepsilon}_{\tilde{\theta}_{\perp}}(p) \cdot \mathbf{e}^{\sigma}(p)] \right|^2 \\
&- \left| \sum_{\lambda\sigma} \Phi_{\lambda\sigma}^{*}(k,p) [\boldsymbol{\varepsilon}_{\theta_{\perp}}(k) \cdot \mathbf{e}^{\lambda}(k)][\boldsymbol{\varepsilon}_{\tilde{\theta}}(p) \mathbf{e}^{\sigma} \cdot (p)] \right|^2 \\
&- \left| \sum_{\lambda\sigma} \Phi_{\lambda\sigma}^{*}(k,p) [\boldsymbol{\varepsilon}_{\theta}(k) \cdot \mathbf{e}^{\lambda}(k)][\boldsymbol{\varepsilon}_{\tilde{\theta}_{\perp}}(p) \cdot \mathbf{e}^{\sigma}(p)] \right|^2 \Bigg\}
\end{aligned} \tag{103}$$

and

$$\langle \Phi(k,p) | \Phi(k,p)\rangle = 4[\delta^3(\mathbf{0})]^2 k^0 p^0 \sum_{\lambda\sigma} |\Phi_{\lambda\sigma}(k,p)|^2. \tag{104}$$

Of course, the factors $[\delta^3(\mathbf{0})]^2$ in the above equations should be understood as a result of the normalization procedure. Moreover, it holds that

$$\boldsymbol{\varepsilon}_{\theta_\perp}(k) = -\mathbf{n_k} \times \boldsymbol{\varepsilon}_\theta(k), \quad \boldsymbol{\varepsilon}_{\tilde{\theta}_\perp}(p) = -\mathbf{n_p} \times \boldsymbol{\varepsilon}_{\tilde{\theta}}(p). \tag{105}$$

To calculate correlation function explicitly we have to specify the two-photon state $|\Phi(k,p)\rangle$. We will choose as $|\Phi(k,p)\rangle$ some of the covariant transforming states discussed in Sec. IV.

A. Correlations in the scalar states

Now let us take as $|\Phi(k,p)\rangle$ the scalar state $|\phi(k,p)\rangle$ [Eq. (72)]. We have

$$\sum_{\lambda\sigma} \phi^*_{\lambda\sigma}(k,p)[\boldsymbol{\varepsilon}_\theta(k)\cdot \mathbf{e}^\lambda(k)][\boldsymbol{\varepsilon}_{\tilde{\theta}}(p)\cdot \mathbf{e}^\sigma(p)] = (kp)[\boldsymbol{\varepsilon}_\theta(k)\cdot\boldsymbol{\varepsilon}_{\tilde{\theta}}(p)] + [\mathbf{p}\cdot\boldsymbol{\varepsilon}_\theta(k)][\mathbf{k}\cdot\boldsymbol{\varepsilon}_{\tilde{\theta}}(p)]. \tag{106}$$

To proceed further we can assume (without loss of generality) that the momentum vectors **k** and **p** are in one plane. We can therefore choose the coordinate frame in such a way that **k** and **p** lie in the xy plane. Now we choose the convention mentioned earlier of choice of vector **a**, denoted as convention I in Appendix A. In this convention from Eq. (A1) we have $\mathbf{a_k}=\mathbf{a_p}=(0,0,1)$; therefore, both angles θ and $\tilde{\theta}$ are measured from the same vector (0,0,1). Under this choice we have explicitly

$$\mathbf{n_k} \times \mathbf{a_k} = (n^2_{\mathbf{k}}, -n^1_{\mathbf{k}}, 0), \tag{107}$$

$$\mathbf{e}^\lambda(\mathbf{k}) = \frac{1}{\sqrt{2}}(-i\lambda n^2_{\mathbf{k}}, i\lambda n^1_{\mathbf{k}}, 1), \tag{108}$$

$$\boldsymbol{\varepsilon}_\theta(\mathbf{k}) = (-n^2_{\mathbf{k}}\sin\theta, n^1_{\mathbf{k}}\sin\theta, \cos\theta). \tag{109}$$

Using these explicit formulas we have

$$\sum_{\lambda\sigma} \phi^*_{\lambda\sigma}(k,p)[\boldsymbol{\varepsilon}_\theta(k)\cdot \mathbf{e}^\lambda(k)][\boldsymbol{\varepsilon}_{\tilde{\theta}}(p)\cdot \mathbf{e}^\sigma(p)] = (kp)\cos(\theta+\tilde{\theta}). \tag{110}$$

And now, inserting Eqs. (110) and (67) into Eqs. (103) and (104) we find

$$C^{\phi(k,p)}_{\Omega_A,\Omega_B}(\theta,\tilde{\theta}) = \cos 2(\theta+\tilde{\theta}). \tag{111}$$

One can check that Eq. (68) implies that for the pseudoscalar state (73) we receive the same correlation function

$$C^{\psi(k,p)}_{\Omega_A,\Omega_B}(\theta,\tilde{\theta}) = \cos 2(\theta+\tilde{\theta}). \tag{112}$$

It should be noted that in the center-of-mass frame, due to opposite direction of vectors $\mathbf{n_k}$ and $\mathbf{n}_{\mathbf{k}^\pi}=-\mathbf{n_k}$, the angle $\theta+\tilde{\theta}$ corresponds to the angle between polarizers used by Alice and Bob. It should be also noted that, assuming convention I, scalar and pseudoscalar states in the helicity basis have the form

$$|\phi(k,p)\rangle = 2(kp)[|(k,+1),(p,+1)\rangle + |(k,-1),(p,-1)\rangle], \tag{113}$$

$$|\psi(k,p)\rangle = -2i(kp)[|(k,+1),(p,+1)\rangle - |(k,-1),(p,-1)\rangle]. \tag{114}$$

VII. COINCIDENCE RATE

The observable (100) and the correlation function (101) are defined in a way resembling the nonrelativistic case of spin measurements. However, in many experiments with photons what is really measured is the coincidence rate—the number of photons registered by Alice and Bob divided by the number of emitted photons [47,48]. To calculate the coincidence rate we need observables which give 1 when the photon passes the analyzer (is registered) and 0 when the photon is not registered. Such an observable is simply the projector Π^θ_Ω [Eq. (96)]. Therefore, in the configuration considered in the previous section, the coincidence rate in the arbitrary pure state $|\Phi\rangle$ has the form

$$R^\Phi_{\Omega_A\Omega_B}(\theta,\tilde{\theta}) = \frac{\langle\Phi|\Pi^\theta_{\Omega_A}\Pi^{\tilde{\theta}}_{\Omega_B}|\Phi\rangle}{\langle\Phi|\Phi\rangle}. \tag{115}$$

For the state with sharp momenta (102) we get

$$\langle\Phi(k,p)|\Pi^\theta_{\Omega_A}\Pi^{\tilde{\theta}}_{\Omega_B}|\Phi(k,p)\rangle = 4[\delta^3(\mathbf{0})]^2 k^0 p^0 \times \left|\sum_{\lambda\sigma}\Phi^*_{\lambda\sigma}(k,p)[\boldsymbol{\varepsilon}_\theta(k)\cdot\mathbf{e}^\lambda(k)][\boldsymbol{\varepsilon}_{\tilde{\theta}}(p)\cdot\mathbf{e}^\sigma(p)]\right|^2. \tag{116}$$

Now, for the scalar state (72) in convention I, taking into account Eqs. (104), (110), and (67) we get

$$R^{\phi(k,p)}_{\Omega_A\Omega_B}(\theta,\tilde{\theta}) = \frac{1}{4}[1+\cos 2(\theta+\tilde{\theta})]. \tag{117}$$

It should be noted here that the two-photon state obtained in the calcium-cascade experiments can be described by scalar state (113) with appropriately chosen momenta k and p [47,49]. Our result (117) coincides with the coincidence rates obtained for the calcium-cascade experiments (see, e.g., [47]). Of course, coincidence rates for other two-photon states $|\Phi\rangle$, like states of photons produced in the decays of positronium or π^0, can be also easily calculated with the help of Eqs. (115) and (116). However, in these decays there are produced high-energy photons for which it is difficult to measure the linear polarization [48].

VIII. CONCLUSIONS

We have described the formalism appropriate for the discussion of Lorentz covariance of quantum information protocols with photons. It is based on the description of photon polarization resembling the nonrelativistic spin degrees of freedom. We have also found all two-photon states with sharp momenta transforming covariantly under Lorentz group action. Using the language of quantum electrodynamics we have also constructed observables corresponding to the linear polarization measurements in EPR experiments. Finally, we have calculated a correlation function in the arbitrary two-photon state with sharp momenta. As an example we have given an explicit form of the correlation function for the scalar state.

We have shown also that in our framework one can easily calculate the coincidence rates measured usually in EPR experiments with photons. As an example we have calculated the coincidence rate in the scalar state corresponding to the two-photon state obtained in the calcium-cascade experiments.

The main advantage of the present approach lies in the fact that the behavior of all of the states and observables discussed here under the Lorentz transformations is well defined. Also the classification of all two-photon states which are covariant under Lorentz group action is given explicitly. Using these results we plan to discuss the Lorentz covariance of the quantum information protocols in a forthcoming paper.

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APPENDIX A: CONVENTIONS

In this appendix we will discuss briefly the freedom of choice of explicit form of Poincaré group action on basis vectors and in the resulting freedom of choice of polarization vectors which is left even after the gauge is explicitly fixed. All of the facts discussed here are well known but scattered in the literature.

As we have noted before, conditions (4) and (7) do not determine the rotation $R_{\mathbf{n_k}}$ uniquely. Indeed, we can write $R'_{\mathbf{n_k}}=R_{\mathbf{n_k}}R_3(\mathbf{n})$, where $R_3(\mathbf{n})$ denotes arbitrary rotation around z axis satisfying $R_3(\tilde{\mathbf{n}})=I$. Rotation $R'_{\mathbf{n_k}}$ defined in such a way fulfills conditions (4) and (7). Now we give explicitly few possible forms of $R_{\mathbf{n_k}}$ fulfilling Eqs. (4) and (7).

Convention I. One of the possible forms of $R_{\mathbf{n_k}}$ is the following:

$$R_{\mathbf{n_k}}=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -n_{\mathbf{k}}^3\left[\dfrac{n_{\mathbf{k}}^2}{\sqrt{1-(n_{\mathbf{k}}^3)^2}}+n_{\mathbf{k}}^1\right] & -\dfrac{n_{\mathbf{k}}^1(n_{\mathbf{k}}^3)^2}{\sqrt{1-(n_{\mathbf{k}}^3)^2}}+n_{\mathbf{k}}^2 & n_{\mathbf{k}}^1\\ 0 & n_{\mathbf{k}}^3\left[\dfrac{n_{\mathbf{k}}^1}{\sqrt{1-(n_{\mathbf{k}}^3)^2}}-n_{\mathbf{k}}^2\right] & -\dfrac{n_{\mathbf{k}}^2(n_{\mathbf{k}}^3)^2}{\sqrt{1-(n_{\mathbf{k}}^3)^2}}-n_{\mathbf{k}}^1 & n_{\mathbf{k}}^2\\ 0 & 1-(n_{\mathbf{k}}^3)^2 & n_{\mathbf{k}}^3\sqrt{1-(n_{\mathbf{k}}^3)^2} & n_{\mathbf{k}}^3 \end{pmatrix}. \tag{A1}$$

This form corresponds to the choice

$$\mathbf{a}_{\mathbf{k}}^T=\left(-n_{\mathbf{k}}^3\left(\frac{n_{\mathbf{k}}^2}{\sqrt{1-(n_{\mathbf{k}}^3)^2}}+n_{\mathbf{k}}^1\right),\; n_{\mathbf{k}}^3\left(\frac{n_{\mathbf{k}}^1}{\sqrt{1-(n_{\mathbf{k}}^3)^2}}-n_{\mathbf{k}}^2\right),1-(n_{\mathbf{k}}^3)^2\right). \tag{A2}$$

Convention II. Another possible form of $R_{\mathbf{n_k}}$ can be obtained taking

$$\mathbf{a}_{\mathbf{k}}^T=\left(\frac{n_{\mathbf{k}}^1(n_{\mathbf{k}}^3)^2}{\sqrt{1-(n_{\mathbf{k}}^3)^2}}-n_{\mathbf{k}}^2,\frac{n_{\mathbf{k}}^2(n_{\mathbf{k}}^3)^2}{\sqrt{1-(n_{\mathbf{k}}^3)^2}}+n_{\mathbf{k}}^1,-n_{\mathbf{k}}^3\sqrt{1-(n_{\mathbf{k}}^3)^2}\right). \tag{A3}$$

Convention III. Yet another form of $R_{\mathbf{n_k}}$ can be received from

$$\mathbf{a}_{\mathbf{k}}^T=\left(1-\frac{(n_{\mathbf{k}}^1)^2}{n_{\mathbf{k}}^3+1},-\frac{n_{\mathbf{k}}^1n_{\mathbf{k}}^2}{n_{\mathbf{k}}^3+1},-n_{\mathbf{k}}^1\right). \tag{A4}$$

It should be noted that the explicit form of the phase factor $\chi_\lambda(k)$ in Eq. (33) depends on the explicit form of $\mathbf{a_k}$. Indeed, Eq. (33) is fulfilled provided that

$$\mathbf{e}^\lambda(k)=-\chi_{-\lambda}(k^\pi)\mathbf{e}^{-\lambda}(k^\pi). \tag{A5}$$

But $\mathbf{e}^\lambda(k)$ depends on the choice of the rotation $R_{\mathbf{n_k}}$.

One can show that when we choose $\mathbf{a_k}$ in Eq. (5) such that $\mathbf{a}_{-\mathbf{k}}=\mathbf{a_k}$, then $\chi_\lambda(k)=-1$. Therefore for the $R_{\mathbf{n_k}}$ given in Eq. (A1) (convention I) we have $\chi_\lambda(k)=-1$.

One can show that when we choose $\mathbf{a_k}$ in Eq. (5) such that $\mathbf{a}_{-\mathbf{k}}=-\mathbf{a_k}$, then $\chi_\lambda(k)=1$. Therefore for the $R_{\mathbf{n_k}}$ given in Eq. (A3) (convention II) we have $\chi_\lambda(k)=1$.

On the other hand, for the explicit form of $R_{n_{\mathbf{k}}}$ given in Eq. (A4) (convention III) we have

$$\chi_\lambda(k)=\frac{k^1+i\lambda k^2}{k^1-i\lambda k^2}. \tag{A6}$$

The four-potential $\hat{A}^\mu(x)$ should not depend on the conventional choice of $R_{\mathbf{n_k}}$. Let us discuss this point now. According to Eq. (11),

$$\hat{A}^\mu(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3\mathbf{k}}{2k^0}\sum_{\lambda=-1,1}[e^{ikx}e_0^{\mu\lambda}(k)a_{0\lambda}^\dagger(k)+e^{-ikx}e_0^{*\mu\lambda}(k)a_{0\lambda}(k)], \tag{A7}$$

where we have added subscript 0 for the convenience of the further discussion. We have, from Eq. (25), $\mathbf{e}_0^\lambda(k)=R_{\mathbf{n_k}}^0\mathbf{e}^\lambda(\tilde{k})$. But we can write $R_{\mathbf{n_k}}=R_{\mathbf{n_k}}^0R_3(\mathbf{n})$, where $R_3(\mathbf{n})$ denotes rotation around z axis, so

$$\mathbf{e}^\lambda(k)=R_{\mathbf{n_k}}\mathbf{e}^\lambda(\tilde{k})=R_{\mathbf{n_k}}^0R_3(\mathbf{n})\mathbf{e}^\lambda(\tilde{k})=R_{\mathbf{n_k}}^0\mathbf{e}^\lambda(\tilde{k})\xi^{*\lambda}(k)=\mathbf{e}_0^\lambda(k)\xi^{*\lambda}(k), \tag{A8}$$

where $\xi^{*\lambda}(k)$ denotes the phase factor [using the explicit form of $\mathbf{e}^\lambda(\tilde{k})$ and rotation R_3 we have $R_3(\mathbf{n})\mathbf{e}^\lambda(\tilde{k})=\xi^{*\lambda}\mathbf{e}^\lambda(\tilde{k})$]. Now, to leave the left-hand side of Eq. (A7) unchanged we have to redefine creation and annihilation operators in the following way:

$$a_\lambda=\xi^\lambda(k)a_{0\lambda}(k). \tag{A9}$$

But if according to Eq. (16) we have

$$U(\Lambda)a_{0\lambda}^\dagger(k)U^\dagger(\Lambda)=e^{i\lambda\psi_0(\Lambda,k)}a_{0\lambda}^\dagger(\Lambda k), \tag{A10}$$

then from Eq. (A9)

$$U(\Lambda)a_{\lambda}^{\dagger}(k)U^{\dagger}(\Lambda) = \frac{\xi^{\lambda}(k)}{\xi^{\lambda}(\Lambda k)}e^{i\lambda\psi_0(\Lambda,k)}a_{\lambda}^{\dagger}(\Lambda k). \quad \text{(A11)}$$

Therefore we can write

$$e^{i\lambda\psi(\Lambda,k)} = \frac{\xi^{\lambda}(k)}{\xi^{\lambda}(\Lambda k)}e^{i\lambda\psi_0(\Lambda,k)}. \quad \text{(A12)}$$

Analogously for the action of the parity operator we have [compare Eq. (33)]

$$\mathsf{P}a_{0\lambda}^{\dagger}(k)\mathsf{P}^{-1} = \chi_{0\lambda}(k)a_{0-\lambda}^{\dagger}(k^{\pi}), \quad \text{(A13)}$$

which implies

$$\mathsf{P}a_{\lambda}^{\dagger}(k)\mathsf{P}^{-1} = \frac{\xi^{\lambda}(k)}{\xi^{-\lambda}(k^{\pi})}\chi_{0\lambda}(k)a_{-\lambda}^{\dagger}(k^{\pi}). \quad \text{(A14)}$$

Therefore we can write

$$\chi_{\lambda}(k) = \frac{\xi^{\lambda}(k)}{\xi^{-\lambda}(k^{\pi})}\chi_{0\lambda}(k). \quad \text{(A15)}$$

APPENDIX B: DIRAC DELTA IN SPHERICAL COORDINATES

We have by definition

$$f(\mathbf{p}) = \int d^3\mathbf{k}\,\delta^3(\mathbf{k}-\mathbf{p})f(\mathbf{k}). \quad \text{(B1)}$$

When we change variables to spherical coordinates we have

$$d^3k \to |\mathbf{k}|^2 d|\mathbf{k}|d\Omega, \quad \text{(B2)}$$

$$\delta^3(\mathbf{k}-\mathbf{p}) \to \frac{\delta(|\mathbf{k}|-|\mathbf{p}|)}{|\mathbf{k}|^2}\delta(\mathbf{n}_k-\mathbf{n}_p), \quad \text{(B3)}$$

$$\mathbf{n}_k = \frac{\mathbf{k}}{|\mathbf{k}|}, \quad \mathbf{n}_p = \frac{\mathbf{p}}{|\mathbf{p}|}, \quad \text{(B4)}$$

where

$$\int d\Omega\,\delta(\mathbf{n}_k-\mathbf{n})g(\mathbf{n}_k) = g(\mathbf{n}), \quad \text{(B5)}$$

and $d\Omega$ denotes a differential solid angle.

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