Quantum tunneling in the adiabatic Dicke model

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The Dicke model describes N two-level atoms interacting with a single-mode bosonic field and exhibits a second-order phase transition from the normal to the superradiant phase. The energy levels are not degenerate in the normal phase but have degeneracy in the superradiant phase, where quantum tunneling occurs. By means of the Born-Oppenheimer approximation and the instanton method in quantum field theory, the tunneling splitting, inversely proportional to the tunneling rate for the adiabatic Dicke model, in the superradiant phase can be evaluated explicitly. It is shown that the tunneling splitting vanishes as $\exp(-N)$ for large N, whereas for small N it disappears as $\sqrt{N}/\exp(N)$. The dependence of the tunneling splitting on the relevant parameters, especially on the atom-field coupling strength, is also discussed.

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The Dicke model, which describes N two-level atoms coupled by a single-mode bosonic field in quantum optics, illustrates the importance of collective and coherent radiation for many atoms [1,2] and has attracted much attention in modern physics, as in theoretical quantum optics [3] and nuclear physics [4]. It has been demonstrated that this model can exhibit a second-order phase transition from the normal to the superradiant phase as a function of the atom-field interaction strength, where the atomic ensemble (or photon) in the normal phase is collectively unexcited while is macroscopically excited with coherent radiation in the superradiant phase [5–9]. Furthermore, the critical exponent for the order parameter vanishes as $N^{-2/3}$ at the transition point [10]. As an important development in quantum information and quantum computing, this superradiant phase transition has some interesting relations, to quantum entanglement [11–13] and to the Berry phase [14,15] as well as to quantum chaos [16,17]. In the transition from quantum mechanics to classical mechanics for the Dicke model, decoherence can also be well described with the standard framework of quantum mechanics [18]. It should be emphasized that the energy levels of the Dicke model in the normal phase are nondegenerate, whereas in the superradiant phase they have degeneracy, where quantum tunneling occurs. However, quantitative estimation of the quantum tunneling between these two degenerate eigenstates has not been discussed systematically.

The phenomenon of tunneling, which has no counterpart in classical physics, describes a process in which a system penetrates into a classically forbidden region and is an intrinsic quantum effect in many fields of modern physics such as condensed matter physics, nuclear physics, and theoretical chemistry, etc. [19–26]. A well-known consequence of the tunneling between two degenerate states is the lifting of their degeneracy. The two new eigenstates are a symmetric and antisymmetric superposition of the original states, which can be characterized by tunneling splitting inversely proportional to the tunneling rate. Therefore, the quantity of interest to determine the occurrence of tunneling is the tunneling splitting between the two lowest eigenstates for a given quantum system. A well-studied case for quantum tunneling is that of a two-level system or a spin-1/2 atom coupled to the environment, which gives rise to the famous spin-boson model [27]. Moreover, in systems with multiple degrees of freedom, quantum tunneling has another important property, known as chaos-assisted tunneling [28,29]. However, for a nonintegral system with multiple degrees of freedom associated with a many-body collective excitation as in the Dicke model, the tunneling process remains an interesting and open problem. The main difficulty may be that sufficient constants of motion cannot be obtained and the tunneling splitting cannot therefore be evaluated directly by the conventional path-integral approach.

In this paper, we consider only the Dicke model in the adiabatic approximation where the frequency between the two levels of the atom is larger than that of the electromagnetic wave. Based on the Born-Oppenheimer approximation, the many-body Dicke model can be mapped into a single-particle Hamiltonian with a symmetric double-well potential in the superradiant phase. Therefore, the tunneling splitting can be evaluated explicitly with the help of the instanton method in quantum field theory. It is shown that the tunneling splitting vanishes as $\exp(-N)$ for large N, whereas for small N it disappears as $\sqrt{N}/\exp(N)$. The dependence of the tunneling splitting on the relevant parameters, especially on the atom-field coupling strength, is also discussed.

The Hamiltonian for the Dicke model without the rotating-wave approximation is given by

$$H = \omega a^\dagger a + \sum_{j=1}^{N}\left(\omega_0\sigma_z^j + \frac{\lambda}{\sqrt{N}}(\sigma_+^j + \sigma_-^j)(a^\dagger + a)\right), \qquad (1)$$

where a and $a^\dagger$ are the photon annihilation and creation operators; σ_+ and σ_- are the spin operators for the jth atom defined as $\sigma_\pm=\sigma_x \pm i\sigma_y$, where σ_l $(l=x,y,z)$ is the lth component of the Pauli matrices; ω is the frequency of the electromagnetic wave; $\hbar\omega_0$ is the energy difference between the two levels of the atom; λ is the atom-field interaction strength; and N is the total number of the atoms. The prefactor $1/\sqrt{N}$ gives a finite free energy per atom in the thermodynamic limit $(N\rightarrow\infty)$. It has previously been shown that the Dicke model exhibits a second-order phase transition at the critical point $\lambda_c=\sqrt{\omega_0\omega/2}$; in the normal phase $(\lambda<\lambda_c)$ the energy levels are nondegenerate, whereas above λ_c the energy levels have degeneracy, where quantum tunneling occurs. Therefore, in the rest of this paper we consider only the case for $\lambda > \lambda_c$ and also take the adiabatic approximation, that is, $\omega_0 \gg \omega$, where the field is regarded as a slow oscillator whereas the atom changes fast [30,31]. Throughout this paper, natural units ($\hbar=1$) are used.

By introducing the collective spin operators defined as $J_l = \Sigma_{j=1}^N \sigma_l$ and then using a rotation $\exp(i\pi J_y/4)$, Hamiltonian (1) can be written as

$$H = \frac{\omega}{2}\left(p^2 + q^2 + \beta J_x + \frac{\gamma q}{\sqrt{N}} J_z\right), \qquad (2)$$

where $q=(a^\dagger + a)/\sqrt{2}$, $p=i(a^\dagger - a)/\sqrt{2}$, $\beta = 2\omega_0/\omega$, and $\gamma = 2\sqrt{2}\lambda/\omega$. It is known that in the adiabatic regime the total wave function of a composite system with one quickly and one slowly changing part can be derived from the Born-Oppenheimer approximation as follows: $|\psi_{tot}\rangle = \int dq\, \phi(q)|q\rangle \otimes |\chi(q)\rangle$, where $|\chi(q)\rangle$ is the eigenstate of the adiabatic equation of the atomic part for each fixed value of the slow variable q and satisfies

$$(\beta J_x + \gamma q J_z/\sqrt{N})|\chi(q)\rangle = E(q)|\chi(q)\rangle. \qquad (3)$$

Since the atoms are identical, $|\chi(q)\rangle$ can be written as $|\chi(q)\rangle = |\chi(q)\rangle_1 \otimes |\chi(q)\rangle_2 \otimes \cdots \otimes |\chi(q)\rangle_N$. The lowest eigenstate corresponding to the Hamiltonian H^-, which is interesting in investigating quantum tunneling, is therefore obtained via $|\chi^-(q)\rangle = \{[A_-(q)|\uparrow\rangle - A_+(q)|\downarrow\rangle]/\sqrt{2}\}^{\otimes N}$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of σ_z with the eigenvalues ± 1 and $A_\pm(q) = \sqrt{1 \pm \gamma q/\sqrt{N}\tau(q)}$ with $\tau(q) = \sqrt{\beta^2 + \gamma^2 q^2/N}$. The eigenvalue corresponding to $|\chi^-(q)\rangle$ can also be given by $E^-(q) = -N\sqrt{\beta^2 + \gamma^2 q^2/N}$, which contributes an effective adiabatic potential felt by the slow bosonic mode.

Substituting the lowest total wave function $|\psi_{tot}^-\rangle = \int dq\, \phi^-(q)|q\rangle \otimes |\chi^{(-)}(q)\rangle$, together with $E^-(q)$, into Hamiltonian (3) yields

$$\left(-\frac{d^2}{dq^2} + U^-(q)\right)\phi^-(q) = \varepsilon^- \phi^-(q), \qquad (4)$$

where $\varepsilon^- = 2H^-/\omega + \beta N(1-\eta)^2/2\eta^2$ and $U^-(q) = q^2 - N\sqrt{\beta^2 + \gamma^2 q^2/N} + \beta N(1-\eta)^2/2\eta^2$. The corresponding Hamiltonian for eigenvalue ε^- is written as H_L^-. Since in the adiabatic approximation $\beta = 2\omega_0/\omega \to \infty$, the variable $\beta\sqrt{1+\gamma^2 q^2/\beta^2 N}$ can be expanded to the order of $O(q^4)$, and $U^-(q)$ can therefore be rewritten as

$$U^-(q) = \frac{\eta^2}{2\beta N}\left(q^2 - \frac{\beta N(\eta-1)}{\eta^2}\right)^2, \qquad (5)$$

where $\eta = \gamma^2/2\beta = 2\lambda^2/\omega\omega_0$. It is straightforward to find that, for $\eta < 1$, $U^-(q)$ can be viewed as a broadened harmonic oscillator potential well with the minimum 0 at $q_m = 0$; whereas for $\eta > 1$, $U^-(q)$ turns into a symmetric double-well potential with the same minimum at $\pm q_m$, where $q_m = \sqrt{2\beta N(\eta-1)/\eta^2}$. It is shown that this quantum system undergoes a second-order phase transition crossing the critical point $\eta = 1$, that is, $\lambda_c = \sqrt{\omega_0\omega/2}$. Furthermore, this symmetric double-well potential gives rise to interesting quantum tunneling.

To evaluate the tunneling splitting $\Delta\varepsilon^-$ for $\lambda > \lambda_c$ we begin with the Feynman propagator in configuration space through the potential barrier [32–39]

$$\langle -q_m, \beta_T | q_m, -\beta_T\rangle = \langle -q_m | e^{-2\beta_T H_L^-}|q_m\rangle = \int \mathcal{D}\{q\} e^{-S}, \qquad (6)$$

where $\beta_T = (k_B T)^{-1}$ with k_B being the Boltzmann constant and T the temperature. $\mathcal{D}\{q\} = \Pi_{i=1}^{N-1} dq_i$,

$$S = \lim_{\beta_T \to \infty} \int_{-\beta_T}^{\beta_T} \mathcal{L}_e d\tau \qquad (7)$$

is the Euclidean action evaluated along the tunneling trajectory of a pseudoparticle in the barrier region (called an instanton), and the Euclidean Lagrangian is given by $\mathcal{L}_e = \dot{q}^2 + U^-(q)$. $\dot{q} = dq/d\tau$ denotes the imaginary-time derivative where $\tau = it$ is the imaginary time.

The quantum tunneling removes the degeneracy of the ground states $|\pm q_m\rangle$ in two potential wells located at $\pm q_m$, respectively. $|\pm q_m\rangle$ describe the two equilibrium orientations of the giant spin and thus may be called the Schrödinger cat states. In the two-level approximation we have $H_L^-|1\rangle = \varepsilon_1^-|1\rangle$ and $H_L^-|0\rangle = \varepsilon_0^-|0\rangle$, where $|0\rangle = (|q_m\rangle - |-q_m\rangle)/\sqrt{2}$ and $|1\rangle = (|q_m\rangle + |-q_m\rangle)/\sqrt{2}$ are the ground state and the first excited state, respectively. The low-lying energy spectra can be evaluated as $\varepsilon_{1,0}^- = \overline{\varepsilon^-} \pm \Delta\varepsilon^-/2$, where $\overline{\varepsilon^-} = \langle \pm q_m | H_L^- | \pm q_m\rangle$ and $\Delta\varepsilon^- = \varepsilon_1^- - \varepsilon_0^- = -(\langle q_m | H_L^- | -q_m\rangle + \langle -q_m | H_L^- | q_m\rangle)$. Inserting the complete set $|\pm q_m\rangle$ in the transition amplitude given by Eq. (6), the tunneling splitting can be derived from the path-integral approach by

$$\Delta\varepsilon^- \sim \frac{e^{2\beta_T \overline{\varepsilon^-}}}{2\beta_T \psi(q_m)\psi^*(-q_m)} \int \mathcal{D}\{q\} e^{-S}, \qquad (8)$$

where $\psi(\pm q_m)$ are the approximate wave functions for the harmonic oscillator. The functional integral $\int \mathcal{D}\{q\} e^{-S}$ can be evaluated in terms of the stationary-phase approximation such that $\int \mathcal{D}\{q\} e^{-S} \sim I e^{-S_c}$, where S_c is the action along the classical trajectory of instanton $q_c(\tau)$ which the solution of classical equation of motion: $\delta S = 0$, and $I = \int \mathcal{D}\{\eta\} e^{-(1/2)\eta^2(\delta^2 S/\delta q \delta q)|_{q_c}}$ denotes the contribution of quantum fluctuations around the classical trajectory such that $q(\tau) = q_c(\tau) + \eta_1(\tau)$ with η_1 being the small fluctuation.

The classical trajectory of instanton can be derived from $\dot{q}^2 - U^-(q) = 0$ by

$$q_c(\tau) = \left[\frac{\beta N(\eta-1)}{\eta^2}\right]^{1/2} \tanh(\sqrt{(\eta-1)}\tau), \qquad (9)$$

and the corresponding classical action is given by

$$S_c = -\frac{\beta N(\eta-1)^{3/2}}{\eta^2}\left(-\frac{1}{3}\tanh^3[\sqrt{(\eta-1)}\beta_T] + \tanh(\sqrt{(\eta-1)}\beta_T)\right). \qquad (10)$$

If the contributions of the infinite number of instanton and

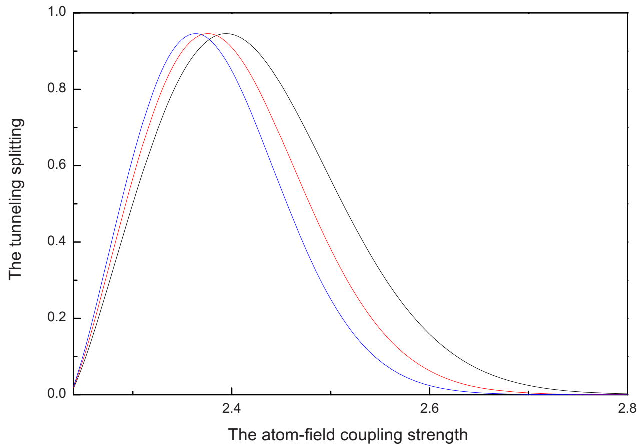

FIG. 1. (Color online) Tunneling splitting ΔE^- versus the atom-field coupling strength λ for $N=6$ [rightmost (black) line], 8 [center (red) line], and 10 [leftmost (blue) line] with $\omega_0=10$ and $\omega=1$.

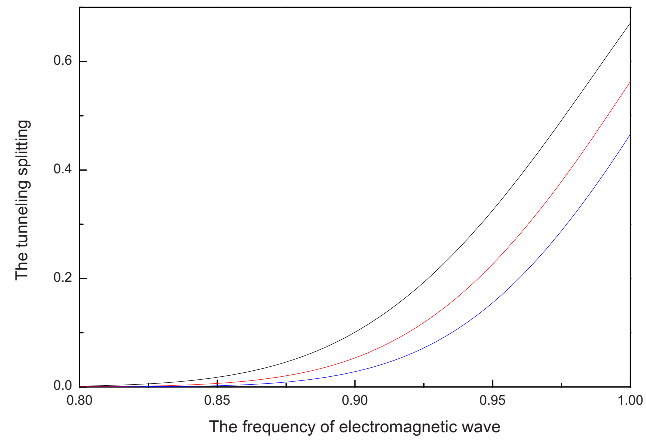

FIG. 2. (Color online) Tunneling splitting ΔE^- versus frequency ω of the electromagnetic wave for $N=5$ [leftmost (black) line], 6 [center (red) line], and 7 [rightmost (blue) line] with $\lambda=2.5$ and $\omega_0=10$.

instanton–anti-instanton pairs to the one-instanton contribution are taken into account and the interactions among instantons and anti-instantons are omitted in the dilute-instanton-gas approximation, we can obtain the tunneling splitting [32–39]

$$\Delta\varepsilon^- = \frac{8\sqrt{2}}{\pi}\left(\frac{\beta N(\eta-1)^{5/2}}{\eta^2}\right)^{1/2}\exp\left(-\frac{2\beta N(\eta-1)^{5/2}}{3\eta^2}\right). \tag{11}$$

Finally, the tunneling splitting for the Dicke model in the adiabatic approximation can be obtained from

$$\Delta E^- = \frac{4\sqrt{2}}{\pi}\omega\sigma^{1/2}\exp\left(-\frac{2\sigma}{3}\right), \tag{12}$$

where $\sigma=\sqrt{\omega_0}N(2\lambda^2-\omega_0\omega)^{5/2}/2\lambda^4\omega^{3/2}$.

Equation (12) basically contains the main results of the present paper. The first observation is that for large N the tunneling splitting ΔE^- vanishes as $\exp(-N)$, whereas for small N it disappears as $\sqrt{N}/\exp(N)$. The well-known mean-field approximation gives only the result that the tunneling splitting vanishes as $\exp(-N)$. In fact, it can be easily understood that the mean-field approximation is valid for the large-N limit. It should be pointed out that a rigorous phase transition can be realized only in the thermodynamic limit $N\to\infty$, and the corresponding tunneling splitting ΔE^- is zero. However, recent interest is mainly focused on a finite particle number, where the basic features of the phase transition can still be demonstrated [40,41]. Moreover, the superradiant phase transition in the Dicke model for finite particle number can also occur [10]. Therefore, we argue that for finite particle number the tunneling splitting given in Eq. (12) does not disappear; it is valid and can be connected with the critical coupling strength λ_c.

Figure 1 shows the tunneling splitting ΔE^- as a function of the atom-field coupling strength λ for different N with $\omega_0=10$ and $\omega=1$. It can be seen clearly that when $\lambda\le\sqrt{\omega_0\omega/2}$ no quantum tunneling occurs, and the corresponding phase is the normal phase, whereas for $\lambda \ge\sqrt{\omega_0\omega/2}$ quantum tunneling governed by the tunneling splitting ΔE^- occurs, and the corresponding phase is the superradiant phase. It is interesting that for $\lambda<\lambda'$, with λ' being the value at which the tunneling splitting ΔE^- reaches maximum in the superradiant phase, the tunneling splitting ΔE^- increases and the corresponding tunneling rate (inversely proportional to the tunneling splitting ΔE^-) decreases, whereas above λ' the tunneling rate increases monotonically. The physics may be understood as follows. The Dicke-like Hamiltonian can be rewritten as $H=H_0+\lambda V$ with $H_0=\omega a^\dagger a+\omega_0 J_z$ and $V=\lambda(J_+ + J_-)(a^\dagger+a)/\sqrt{N}$ being an effective potential. Whereas H_0 is integrable, the full Hamiltonian H is not for any $\lambda\neq 0$. If the parameter λ is increased upward from zero gradually, this quantum system is driven away from integrability and toward chaos. Therefore, for $\lambda<\lambda'$ the tunneling mechanism maybe arises from competition between pure quantum tunneling and chaos-assisted quantum tunneling, whereas above λ' the tunneling mechanism is mainly chaos-assisted quantum tunneling [28]. The tunneling splitting ΔE^- as a function of the frequency ω of the electromagnetic wave for different N is plotted in Fig. 2. It is shown that, with increasing frequency ω, the tunneling splitting ΔE^- is increased and the corresponding tunneling rate is decreased. The reason is that, when the frequency ω is increased, the contribution of the effective potential V governing the quantum tunneling is cut back in the full Hamiltonian H and therefore the tunneling rate is decreased. Moreover, the dependence of the tunneling splitting ΔE^- on N is also attributed to the coefficient $\lambda/\sqrt{N}$ of the effective potential V.

In conclusion, we first discussed the tunneling splitting for the Dicke model in the adiabatic approximation based on the Born-Oppenheimer approximation and instanton method in quantum field theory. It was shown that for large N the tunneling splitting ΔE^- given in Eq. (12) vanishes as $\exp(-N)$, whereas for small N it disappears as $\sqrt{N}/\exp(N)$.

Moreover, this tunneling splitting might enable one to investigate the transition from pure quantum tunneling to chaos-assisted quantum tunneling. Recently, it has been shown that quantum tunneling has some important connections to quantum information and quantum computing [39]. We believe that our considerations open up further opportunities for studies of chaos, quantum tunneling, and information theory in the Dicke model or others. This awaits further validation in both theory and experiment.

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