Complex weak values in quantum measurement

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In the weak value formalism of Aharonov *et al.*, the weak value A_w of any observable A is generally a complex number. We derive a physical interpretation of its value in terms of the shift in the measurement pointer's mean position and mean momentum. In particular, we show that the mean position shift contains a term jointly proportional to the imaginary part of the weak value and the rate at which the pointer is spreading in space as it enters the measurement interaction.

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I. INTRODUCTION

In quantum mechanics the essential connection between theory and experimental outcomes may be thought of as being embodied in the formula

$$\langle A\rangle = \langle\psi_i|A|\psi_i\rangle \tag{1}$$

for the measured mean value of an observable A upon (strong) measurement of a quantum system prepared in state $|\psi_i\rangle$. The formalism of weak values introduced by Aharonov and co-workers [2] (see [1], Chaps. 16 and 17, for a review and further references therein) provides an alternative foundation for quantum measurement theory [3]. In this formalism the above formula becomes replaced [2] by a more general expression:

$$A_w = \frac{\langle\psi_f|A|\psi_i\rangle}{\langle\phi_f|\phi_i\rangle}. \tag{2}$$

A_w is called the *weak value* of observable A for a quantum system preselected in state $|\psi_i\rangle$ and postselected in state $|\psi_f\rangle$, and it characterizes the observed outcomes of weak measurements. If $|\psi_i\rangle$ (or $|\psi_f\rangle$) is an eigenstate of A then $\langle A\rangle$ and A_w agree (both equaling the corresponding eigenvalue) but more generally A_w need not lie within the range of eigenvalues [4–10] and may even be complex. Thus its significance is more subtle than the straightforward interpretation of $\langle A\rangle$ as a measured mean value. In this Brief Report, we establish a physical interpretation for A_w in its most general context.

The formalism of weak values has two ingredients that differ from the usual approach that leads to Eq. (1): first, in addition to preparation of quantum systems in a given initial state, we also impose *postselection* into a given final state; second, we consider a scenario in which the measurement interaction is suitably weak so that after measurement the system state is left largely undisturbed. As a framework for our main results, we begin by briefly reviewing the weak value formalism and the origin of the expression A_w in Eq. (2).

Consider a quantum system prepared in state $|\psi_i\rangle$ upon which we wish to measure A. For measurement process we use the standard von Neumann paradigm [11] introducing a pointer in initial state $|\phi\rangle$ with wave function $\phi(q)$, and interaction Hamiltonian

$$H_{\text{int}} = g(t)Ap, \quad g(t) = g\,\delta(t-t_0), \tag{3}$$

where g is a coupling constant and p is the pointer momentum conjugate to the position coordinate q. Here we have taken the interaction to be impulsive at time $t=t_0$ (and the expression Ap is shorthand for $(A\otimes I)(I\otimes p)$, where the first and second slots refer to the system and pointer, respectively.)

After interaction the system and pointer are in joint state $e^{-igAp}|\psi_i\rangle|\phi\rangle$, and we postselect the system on state $|\psi_f\rangle$, resulting in the (subnormalized) pointer state

$$|\alpha\rangle = \langle\psi_f|e^{-igAp}|\psi_i\rangle|\phi\rangle. \tag{4}$$

(Here and hereafter we adopt units making $\hbar=1$.) In practice, the postselection is achieved by running the process many times with initial state $|\psi_i\rangle$, and after all tasks are completed we perform a further final measurement of the projector Π_f onto $|\psi_f\rangle$ in each run. Then for statistical analysis of measurement outcomes or any other considerations, we retain only those runs for which Π_f yielded 1. (The subnormalization of $|\alpha\rangle$ reflects the probability of success in this Π_f measurement.) It is a remarkable fact that quantum theory allows both pre- and postselection of systems, whereas classical physics allows imposition of only either initial or final boundary conditions, but not both (cf. [1], Sec. 16.3).

It is a standard tenet of quantum theory that measurement irrevocably disturbs a quantum system. The measurement interaction Eq. (3) is said to be strong if the translated wave functions $\phi(q-ga_i)$ for eigenvalues a_i of A, correspond to states that have negligible overlap. In that case, after the measurement interaction the pointer position will be observed, on average, to have shifted by $g\langle A\rangle$. In contrast to this standard scenario, the second basic ingredient in the formalism of weak values is the requirement that the measurement interaction Eq. (3) be suitably weak so that we may obtain information about A while the system state is left largely intact. To restrict the strength of interaction we consider the limit of small g, retaining only terms to first order in g. Alternatively, weakness may be imposed by requiring p to remain small which, by the $\Delta p\Delta q$ uncertainty relation, corresponds to a limit of increasingly broad initial wave functions of the pointer in the q representation. In the following, we will work only with the limit of small g. In both cases the translates $\phi(q-ga_i)$ will retain a large overlap (of size 1

$-O(g)$ or $1-O(p)$). Expanding Eq. (4) to terms of $O(g)$ yields

$$|\alpha\rangle \approx \langle\psi_f|I - igAp|\psi_i\rangle|\phi\rangle = \langle\psi_f|\psi_i\rangle(I - igA_w p)|\phi\rangle \quad (5)$$

$$\approx \langle\psi_f|\psi_i\rangle e^{-igA_w p}|\phi\rangle. \quad (6)$$

Thus it is clear that all subsequent measurement properties of the pointer depend on the ingredients A, $|\psi_i\rangle$ and $|\psi_f\rangle$ only through the single c number A_w.

From Eq. (2) we see that A_w can generally be a complex number and its effect on mean values of pointer variables, such as the mean position and mean momentum, is not immediately clear from Eqs. (5) and (6). Mathematically, Eq. (6) simply represents a translation $\phi(q-gA_w)$ of the wave function by gA_w. However, as the latter is generally complex and we use the resulting translated function only along the real q axis, its quantum mean properties are now not simply characterizable in terms of translates of those of $\phi(q)$. In the literature, only some special restricted cases have been considered. Introduce the initial and final pointer means

$$\langle q\rangle_i = \langle\phi|q|\phi\rangle, \quad \langle q\rangle_f = \frac{\langle\alpha|q|\alpha\rangle}{\langle\alpha|\alpha\rangle}, \quad (7)$$

and similarly the momentum means $\langle p\rangle_i$ and $\langle p\rangle_f$ with p replacing q in the above. Also, introduce the variances of position and of momentum in the initial pointer state:

$$\mathrm{Var}_q = \langle\phi|q^2|\phi\rangle - \langle\phi|q|\phi\rangle^2, \quad \mathrm{Var}_p = \langle\phi|p^2|\phi\rangle - \langle\phi|p|\phi\rangle^2. \quad (8)$$

Then the following cases have been noted [1,2,10,12]. (i) if A_w is *real* then $\langle q\rangle_f=\langle q\rangle_i+gA_w$; (ii) if A_w is complex but the pointer wave function $\phi(q)$ is *real valued* then $\langle q\rangle_f=\langle q\rangle_i+g\,\mathrm{Re}(A_w)$ and $\langle p\rangle_f=\langle p\rangle_i+2g\,\mathrm{Im}(A_w)\mathrm{Var}_p$; (iii) it has also been noted ([1], p. 237) that in the expression Eq. (6) the imaginary part of A_w contributes a nonunitary operation, which can thus be thought of as increasing or decreasing the size $\langle\alpha|\alpha\rangle$ of the pre- and postselected ensemble of runs.

II. COMPLEX WEAK VALUES

We now consider the most general case of complex A_w and complex-valued wave function $\phi(q)$. Our resulting general formulas will display the role of the imaginary part of A_w in the shift of pointer mean position. We will demonstrate the following.

Theorem. Let $A_w=a+ib$. Then after a weak von Neumann measurement interaction on a system with pre- and postselected states $|\psi_i\rangle$ and $|\psi_f\rangle$, the mean pointer position and momentum satisfy

$$\langle q\rangle_f = \langle q\rangle_i + ga + gb\left(m\frac{d}{dt}\mathrm{Var}_q\right), \quad (9)$$

$$\langle p\rangle_f = \langle p\rangle_i + 2gb(\mathrm{Var}_p). \quad (10)$$

Here m is the mass of the pointer and $(d/dt)\mathrm{Var}_q$ is the time derivative of its position variance as $t\to t_0$, the time of the impulsive measurement interaction.

Thus in particular there is a contribution to the pointer's mean position shift that is proportional to the imaginary part of A_w and the rate at which the pointer is spreading in space as it enters the interaction.

To derive Eq. (9) we begin by substituting $p=-i\partial/\partial q$ into Eq. (5). Retaining only terms to $O(g)$, we get

$$\alpha(q)\bar{\alpha}(q) = |\langle\psi_f|\psi_i\rangle|^2[\phi\bar{\phi} - ga(\phi'\bar{\phi} + \bar{\phi}'\phi) - igb(\phi'\bar{\phi} - \bar{\phi}'\phi)], \quad (11)$$

where ϕ' denotes the space derivative and $\bar{\phi}$ denotes the complex conjugate. The coefficient of ga is the space derivative of the probability density $\phi\bar{\phi}$, whereas the coefficient of gb is recognized as the spatial part of the conserved probability current for $|\phi\rangle$. To exploit these features we introduce

$$\phi = Re^{iS}, \quad \rho = R^2. \quad (12)$$

Then

$$\alpha\bar{\alpha} = |\langle\psi_f|\psi_i\rangle|^2[\rho - ga\rho' + gb(2\rho S')],$$

and a straightforward calculation to $O(g)$ gives (writing $\mu=\langle q\rangle_i$)

$$\langle q\rangle_f = \frac{\int \bar{\alpha}q\alpha}{\int \bar{\alpha}\alpha} = \mu - ga\int q\rho' + gb\int 2\rho S'(q-\mu), \quad (13)$$

where the integration is over all space. Integration by parts gives

$$\langle q\rangle_f = \langle q\rangle_i + ga - gb\int (q-\mu)^2(\rho S')'. \quad (14)$$

Now consider the Schrödinger equation of the pointer up to the time t_0 of interaction:

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2m}\phi'' + V(q)\phi.$$

Substituting Eq. (12) and taking the imaginary part of the resulting equation gives the continuity equation for probability conservation:

$$\frac{\partial\rho}{\partial t} + \left(\rho\frac{S'}{m}\right)' = 0. \quad (15)$$

Hence $(\rho S')'=-m\rho_t$ and Eq. (14) finally gives

$$\langle q\rangle_f = \langle q\rangle_i + ga + gb\left(m\frac{d}{dt}\mathrm{Var}_q\right), \quad (16)$$

as claimed. We note that a wave function is (instantaneously) real valued (up to an overall constant phase) if and only if $S'=0$ and then $d\,\mathrm{Var}_q/dt$ is zero [via Eq. (15), giving $\rho_t=0$] so we regain the previously quoted results (i) and (ii) for the change in $\langle q\rangle$ in these restricted cases.

Next, we present an alternative Heisenberg representation derivation of Eq. (16) which generalizes immediately to other pointer observables (such as p) replacing q. Let M be any pointer observable. From Eq. (5) we get to $O(g)$

$$\begin{aligned}\langle M\rangle_f &= \frac{\langle\alpha|M|\alpha\rangle}{\langle\alpha|\alpha\rangle}\\ &= \frac{\langle\alpha|M|\alpha\rangle - igA_w\langle\phi|Mp|\phi\rangle + ig\bar{A}_w\langle\phi|pM|\phi\rangle}{\langle\phi|\phi\rangle - igA_w\langle\phi|p|\phi\rangle + ig\bar{A}_w\langle\phi|p|\phi\rangle}\\ &= \langle M\rangle_i + iga\langle pM - Mp\rangle_i + gb(\langle pM + Mp\rangle_i - 2\langle p\rangle_i\langle M\rangle_i)\end{aligned} \tag{17}$$

(where, for any observable N, $\langle N\rangle_i=\langle\phi|N|\phi\rangle$ is its mean value in state $|\phi\rangle$).

For $M=q$ we have the commutation relations

$$[p,q] = -i$$

and the Heisenberg equations of motion (with $H=p^2/2m+V(q)$)

$$i\frac{d}{dt}\langle q\rangle = \langle[q,H]\rangle = \frac{i\langle p\rangle}{m},$$

$$i\frac{d}{dt}\langle q^2\rangle = \langle[q^2,H]\rangle = \frac{i\langle pq + qp\rangle}{m}.$$

Substitution of these into Eq. (17) immediately gives Eq. (16).

If instead we set $M=p$ then $pM-Mp$ in the coefficient of ga in Eq. (17) becomes zero and the coefficient of gb becomes $2\langle p^2\rangle_i - 2\langle p\rangle_i = 2\,\mathrm{Var}_p$, giving

$$\langle p\rangle_f = \langle p\rangle_i + 2gb\,\mathrm{Var}_p,$$

as claimed in the theorem. Note that the pointer observable p commutes with the measurement interaction Hamiltonian gAp, so this shift in $\langle p\rangle$ is an artifact of postselection rather than a quantum dynamical effect, in contrast to the more interesting case of the shift in $\langle q\rangle$.

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