Analytical theory for the propagation of laser beams in nonlinear media

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The propagation of a laser beam of intensity I in a nonlinear medium with a refractive index $n(I)$ of arbitrary form is studied. In particular, the influence of the functional form $n=n(I)$ on self-focusing and self-trapping is investigated. Starting from the propagation equations and using symmetry considerations and the Bogoliubov renormalization group approach, we derive a general equation relating the self-focusing distance, the intensity, and $n(I)$. For different polynomial dependences of $n(I)$ on I, we construct analytical solutions for the spatial intensity profile $I(\mathbf{r})$ for an initially collimated Gaussian beam inside the medium. We also explicitly analyze the case of nonlinear self-focusing accompanied by multiphoton ionization. For particular (already studied) cases, we considerably improve the accuracy of the results with respect to previous semianalytical studies and obtain very good agreement with recent numerical simulations.

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I. INTRODUCTION

The problem of intense light propagation in media exhibiting nonlinear response has been investigated in detail since the early 1960s [1–4]. Already in the case of linear intensity dependence of the refractive index, numerous self-action effects have been observed [5,6], like solitary waves, self-focusing, self-phase-modulation, etc. Great success in mathematical modeling of these processes was achieved due to this simple dependence of the index of refraction on the intensity. In particular, analytical soliton solutions to the problem were obtained by Zakharov and Shabat in the framework of the inverse scattering problem [7]. Self-similar solutions for parabolic and hyperbolic-secant beam shapes were found by Akhmanov *et al.* [3] under the geometrical optics approximation. Approximate solutions for several initial beam intensity distributions were obtained by Kovalev based on the Lie symmetry group analysis and the renormalization group approach [8,9].

However, modern experimental techniques allow one to obtain strong time compression of laser pulses and electric field intensities above the ionization threshold. For a theoretical description of such experiments, the linear approximation becomes insufficient, and more complicated forms of the refractive index should be considered [10,11]. The theoretical results obtained so far were mainly based on extensive numerical simulations. In particular, self-guiding versus collapse in air [12,13], filament formation in fused silica [14–16], splitting of one filament into several [17], and their stability and features of their interactions [18,19] have been investigated. In general, those numerical simulations are very time consuming and require a preliminary analytical investigation.

The most widely used analytical method for this goal is known as the variational approach [20] with its different corrections, e.g., [10,13,21–23]. Its basic assumption is an ansatz on the fixed functional form of the beam, in which the average beam radius, the amplitude, and the phase are assumed to be slowly varying functions of the propagation distance. Substitution of this trial function into the corresponding Lagrangian of the problem followed by its averaging over the radial variable and application of the variational procedure allows the original nonlinear wave equation to be reduced to a system of rather complicated ordinary differential equations. These equations are usually solved numerically for given initial conditions and the model parameters.

The main shortcoming of the variational method is related to the restriction that the form of the trial solution remains unchanged upon propagation. Usually, the spatial profile of the beam is taken to be of Gaussian or super-Gaussian form. However, it was demonstrated [9] that the Gaussian beam rapidly loses its initial form when passing through a nonlinear medium.

In the present paper, we search for approximate analytical solutions of the problem of light propagation in nonlinear media, avoiding any assumption on the form of the beam during propagation and any restriction on the functional dependence of the refractive index on the light intensity. For this goal we used a method based on the Lie symmetry group analysis, which is currently being applied to a large number of problems in physics and engineering [24]. The method has been developed to find analytical solutions for boundary value problems by Shirkov and Kovalev [8]. They pointed out that this method is closely related to the Bogoliubov renormalization group, well known in quantum field theory and the theory of condensed mater. This method has already been used to construct analytical solutions of the Vlasov equation [25], and also to describe self-focusing wave collapse in optical media where the refractive index is a linear function of intensity [9].

Here, we apply the method to find analytical solutions to the problem of the propagation of intense laser pulses in a medium with arbitrary nonlinearities. Starting from the propagation equation for the electric field and the equation for the temporal evolution of the photoexcited carrier density, and using the semiclassical approximation, we first formulate the boundary value problem. Then, we make use of the Bogoliubov renormalization group method and find the approximate symmetry group admitted by the equations. This allows us to construct the desired analytical solutions.

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For the general case we find an expression to determine the self-focusing distance for an arbitrary functional dependence of the refractive index $n(I)$ on the light intensity I [Eq. (24)]. Then, we calculate the intensity profile of the light inside the sample for different polynomial dependences of n on I.

The paper is organized as follows. In Sec. II, starting from the nonlinear wave equation, we derive the corresponding generalized nonlinear Schrödinger and eikonal equations (semiclassical approximation). In Sec. III we search for approximate analytical solutions for eikonal equations and discuss the results obtained for different nonlinear forms of the refractive index. Particular attention is given to the case when $n(I)$ reflects both nonlinear Kerr self-focusing and multiphoton plasma generation. Section IV is devoted to the application of the obtained results to the problem of ultrashort laser pulse propagation in air and a comparison with results of numerical simulations. In Sec. V the results are summarized and further applications are discussed.

II. MODEL EQUATIONS

We consider the propagation of a linearly polarized laser beam in a medium. The scalar wave equation that governs the electric field E distribution reads

$$\nabla_{\perp}^2 E - \frac{1}{c^2}\partial_{tt}E = \frac{4\pi}{c^2}\partial_{tt}P\frac{4\pi}{c^2} + \partial_t J, \tag{1}$$

where c is the light velocity, $\nabla_{\perp} \equiv \partial_x + \partial_y + \partial_z$, and ∂_{tt} stands for the second derivative with respect to time. $P = P_L + P_{\mathrm{NL}}$ is the polarization of the medium, where $P_L = \varepsilon_0 E$ denotes the linear contribution, ε_0 refers to the dielectric constant, and P_{NL} is a nonlinear function of the electric field. Usually the nonlinear polarization P_{NL} is presented in the form of a Taylor series expansion in odd (for homogeneous media) powers of the incident field amplitude: $P_{\mathrm{NL}} = \chi^{2j+1}E^{2j+1}$, $j = 1, \ldots, \infty$. The nonlinear susceptibility coefficients of higher than the third order ($j = 2, \ldots, \infty$) are usually neglected [5,6].

Strong interaction of an electric field with the medium can produce very large carrier densities through multiphoton (tunnel and avalanche) ionization processes. This is described in the equations by the term proportional to the current J and expressed as $J = \sigma E$. Here, σ is the conductance, which can be calculated using methods of statistical mechanics [26] for a cw beam or long pulses. For intense and short pulses it is in accordance with Drude's model [5] and reads $\sigma = \rho\tau_{\mathrm{col}}e^2/m(1+\omega^2\tau_{\mathrm{col}}^2)$, where ω is the light frequency, τ_{col} is the electron collision time, and ρ is the density of electrons with mass m and charge e.

Direct integration of Eq. (1) with the terms discussed above requires enormous computational effort and in many cases does not provide an insight into basic physical understanding of the various linear and nonlinear effects involved. However, significant simplification of the mathematical model can be achieved if we consider the propagation of laser beam along the z axis with fixed carrier frequency ω_0, and neglect dispersion of the medium. Then, using the slowly varying envelope approximation $E(\mathbf{r},t) = \mathcal{E}(\mathbf{r},t)e^{i(k_0 z - \omega_0 t)} + \mathrm{c.c.}$, where $k_0 = n_0\omega_0/c$, and transforming into the retarded coordinate system, where t is the retarded time $t - z/v_g$ with the group velocity v_g, we obtain the generalized nonlinear Schrödinger equation

$$i\partial_z\mathcal{E} + \frac{1}{2k_0}\nabla_{\perp}^2\mathcal{E} + k_0 n(|\mathcal{E}|^2)\mathcal{E} = 0, \tag{2}$$

where z is the propagation length. The Laplacian $\nabla_{\perp}^2$ describes wave diffraction in the transverse plane. For the sake of simplicity we consider the $(1+1)$-dimensional problem, i.e., $\nabla_{\perp}^2 = \partial_{xx}$, where x is the axis transverse to the beam direction. $n = n(|\mathcal{E}|^2)$ is the index of nonlinear refraction. In general, $n(|\mathcal{E}|^2)$ is quite a complicated function of the electric field intensity. However, in many problems the main contribution of the response of a nonlinear medium is related to the third-order Kerr nonlinearity and multiphoton ionization (MPI). For such cases, the refractive index in (2) reads

$$n = n_2|\mathcal{E}|^2 - \rho(|\mathcal{E}|^2)/2\rho_c, \tag{3}$$

where ρ is the density of electrons created by the photoinduced ionization process. ρ_c denotes the critical density above which the plasma becomes opaque. This value is related to the Drude model as [27] $\rho_c = (n_0 c\sigma\tau_c)^{-1}$. The spatial and temporal evolution of the plasma density is usually determined as

$$\partial_t\rho = W_{\mathrm{MPI}}(|\mathcal{E}|)(\rho_{\mathrm{at}} - \rho) + \nu\rho|\mathcal{E}|^2 - \rho/\tau_{\mathrm{rec}}, \tag{4}$$

Here, ρ_{at} is the initial atomic density. ν in Eq. (4) is the rate of tunneling ionization, and τ_{rec} is the plasma recombination rate. Both of these processes can be neglected for the propagation of a femtosecond pulse with intensity smaller than 10^{14} W/cm^2.

It has been demonstrated [14,28] that, in many cases, the rate of multiphoton ionization can be approximated by a power function of the intensity as $W_{\mathrm{MPI}}(|\mathcal{E}|) = \sigma_K|\mathcal{E}|^{2K}$. Here $K = \mathrm{mod}(U_i/\hbar\omega_0 + 1)$ corresponds to the number of photons required for ionization of an atom having an ionization potential (or gap) equal to U_i. The quantity σ_K is proportional to the multiphoton adsorption cross section. Moreover, from a direct numerical simulation [27], it was proven that $\rho \simeq \sigma_K|\mathcal{E}|^K\rho_{\mathrm{at}}t_p$ is a sufficiently good approximation to the solution of Eq. (4), were t_p is the pulse duration time.

All the simplifications discussed above allow us to investigate the beam propagation Eq. (2) in a closed form. We will consider that the refractive index is a polynomial function of the electric field.

Equation (2) was investigated analytically in the frame of the inverse scattering method [7], under the semiclassical approximation and the renormalization group [9], only for the simplest nonlinearity $n = n_2|\mathcal{E}|^2$.

In the present paper, we construct an approximate analytical solution of the light propagation Eq. (2) for a more general form of the refractive index, and particularly for the case of multiphoton plasma generation.

Let us represent the electric field $\mathcal{E}$ in the eikonal form $\mathcal{E} = \sqrt{I}\exp(ik_0 S)$. Then, starting from (2), after some algebraic manipulations, we obtain

$$\partial_z S = -\frac{1}{2}(\partial_x S)^2 + n(I) + \theta\left(\frac{\partial_{xx} I}{I} - \frac{(\partial_x I)^2}{2I^2}\right),$$

$$\partial_z I = -\partial_x(I\partial_x S), \tag{5}$$

where both coordinates z and x are normalized to the initial beam radius w_0. I is the intensity normalized with respect to its value at the entry plane, I_0, and $\theta=(2k_0^2 w_0^2)^{-1}$.

Typical values of the parameters in modern experiments are $\lambda=800$ nm and initial beam radius $w_0=1$ mm. Using these values, the dimensionless prefactor of the diffraction term in Eq. (5) becomes $\theta \simeq 8\times 10^{-8}$. This means that the diffraction term is negligible and that the geometrical optics approximation should give a reasonable description of the problem of intense laser beam propagation in nonlinear media for several experimental conditions.

Thus, we consider the following system of equations:

$$\partial_z S = -\frac{1}{2}(\partial_x S)^2 + n(I),$$

$$\partial_z I = -\partial_x(I\partial_x S).$$

These equations are canonical, i.e., they can be written as $\partial_z I = \delta H/\delta S$, $\partial_z S = -\delta H/\delta I$, where H is a Hamiltonian of the form

$$H = \int\left(\frac{I}{2}(\partial_x S)^2 - \Phi(I)\right)dx \tag{6}$$

with $\Phi(I)=\int_0^I n(I')dI'$.

We introduce now the dimensionless variable $v=\partial_x S$, and differentiate the eikonal Eq. (5) with respect to x. Then a refractive index of the form (3) gives the term $n_2 I_0(1-\partial_I \rho/2\rho_c)\partial_x I \equiv a\varphi\partial_x I$. For an intensity I_0 of the order 10^{13} W/cm^2 and light propagation in air ($n_2=3.2\times 10^{-19}$ cm^2/W), $a\equiv n_2 I_0 \simeq 10^{-5}$. Therefore, a turns out to be a small parameter. In what follows we construct an approximate analytical solution making use of the fact that the parameter a is small.

Finally, we obtain a boundary value problem

$$\partial_z v + v\partial_x v - a\varphi(I)\partial_x I = 0,$$

$$\partial_z I + v\partial_x I + I\partial_x v = 0,$$

$$v(0,x) = 0, \quad I(0,x) = \exp(-x^2), \tag{7}$$

which describes propagation of an initially collimated Gaussian beam in a nonlinear medium.

The system (7) is linear with respect to first-order derivatives. Therefore, it is convenient to use the hodograph transformation [29] in order to transform it into a linear system:

$$\partial_w \tau - \frac{I}{\varphi(I)}\partial_I \chi = 0, \quad \partial_w \chi + a\partial_I \tau = 0, \tag{8}$$

where $\tau=Iz$, $\chi=x-vz$, $w=v/a$. The boundary conditions are transformed as follows:

$$w = 0, \quad \chi = \sqrt{\ln(1/I)}. \tag{9}$$

In spite of their simple form, Eqs. (8) can be exactly solved analytically only for a few particular functions $\varphi(I)$ [30].

III. ANALYTICAL SOLUTIONS

In order to solve Eqs. (8) we use an approach based on the Lie symmetry group analysis of differential equations. The regular scheme to make use of this method in the case of boundary value problems was developed by Shirkov and Kovalev [8,30–32]. In the following we briefly discuss those aspects important for our calculations. A detailed description of the method is, however, out of the scope of this paper. For more details we address the interested reader to mathematical textbooks (see, e.g., [33,34]).

The main steps of the method consist in the construction of the renormalization group manifold, the calculation of the symmetry group admitted by this manifold, and the subsequent restriction of this group on the particular solution of the boundary value problem. The constructed exact or approximate symmetry is used to find the desired solution.

In the problem under consideration the renormalization group manifold is given by Eqs. (8). Now, we have to construct the Lie symmetry admitted by Eqs. (8), namely, we have to find the corresponding infinitesimal transformation operator X. For this goal we have to solve the equation

$$XF|_{F=0} = 0, \tag{10}$$

where F is the frame of Eqs. (8). Equation (10) is known as the determining equation [24,33]; its solution is one of the central problems in the Lie symmetry analysis of differential equations.

We search for X in the canonical Lie-Bäcklund form

$$X = f\partial_\tau + g\partial_\chi, \tag{11}$$

where and g are unknown functions of n, w, χ, τ, and their derivatives: $\partial\chi/\partial n$, $\partial\tau/\partial n$, etc.

Applying the operator of Eq. (11) to the equations of motion (8), we obtain the determining equation [35]

$$D_w(g) + aD_I(f) = 0,$$

$$D_w(f) - \frac{I}{\varphi(I)}D_I(g) = 0, \tag{12}$$

where

$$D_I \equiv \partial_I + \sum_{s=0}^{\infty}(\tau_{I^{s+1}}\partial_{\tau_{I^s}} + \chi_{I^{s+1}}\partial_{\chi_I^s}), \tag{13}$$

$$D_w \equiv \partial_w + \sum_{s=0}^{\infty}(\tau_{w^{s+1}}\partial_{\tau_{w^s}} + \chi_{w^{s+1}}\partial_{\chi_w^s}) \tag{14}$$

are total derivatives. Here, the index s stands for the order of the derivatives: $\tau_I^s \equiv \partial^s\tau/\partial I^s$, etc. Derivatives with respect to w in the parentheses in Eq. (14) should be excluded in accordance with Eqs. (8): $\tau_w^1 = I/\varphi\chi_I^1$, $\tau_w^2 = I/\varphi\chi_{Iw}^1 = -aI/\varphi\tau_I^2$, and so on. As a result, the operator of Eq. (14) takes the form of a series in powers of the small parameter a.

To determine f and g we propose formal series in powers of a as well:

$$f=\sum_{i=0}^{\infty}a^i f^i, \quad g=\sum_{i=0}^{\infty}a^i g^i, \tag{15}$$

Substituting Eqs. (15) into Eqs. (12) and collecting terms with the same power of a, we obtain up to the first order in a

$$\partial_w f^0 - \frac{I}{\varphi}D_I(g^0)=0, \tag{16}$$

$$\partial_w g^1 - D'_w(g^0)+D_I(f^0)=0, \tag{17}$$

$$\partial_w f^1 - D'_w(f^0) - \frac{I}{\varphi}D_I(g^1)=0, \tag{18}$$

where $D'_w=\Sigma_{s=0}^{\infty}\tau_{I^{s+1}}\partial_{\chi_{I^s}}$. The system of differential equations (16)–(18) can be solved sequentially starting from a given g^0. The requirement that the symmetry group satisfies the boundary conditions means that f and g vanish under substitution of Eqs. (9) in each order of approximation. Therefore, the natural choice for the first step in solving equations (16)–(18) is to put $g^0=0$. However, in Ref. [8], it was demonstrated that, for an initial Gaussian beam, the choice

$$g^0 = 1+2I\chi\chi_I \tag{19}$$

provides the better result, since it corresponds to a further iteration in the renormalization procedure.

Substituting Eq. (19) into Eq. (16), and solving Eqs. (16)–(18) step by step, after some calculations, we obtain

$$g^1 = 1+2I\chi\chi_I + 2a\left(1+I\frac{\varphi_I}{\varphi}\right)\chi\chi_a - 2a\varphi\tau\tau_I - 2a\varphi_I\tau^2 + G^1, \tag{20}$$

$$f^1 = 2I\left(\chi_I\tau+\chi\tau_I+\frac{\varphi_I}{\varphi}\chi\tau\right)+2a\chi\tau_a\left(1+I\frac{\varphi_I}{\varphi}\right)+F^1. \tag{21}$$

Here, $F^i(I,\chi_{I^s},\tilde{\tau}_{I^s})$ and $G^i(I,\chi_{I^s},\tilde{\tau}_{I^s})$ $(i\geq 0)$ are, in principle, arbitrary functions of their arguments and

$$\tilde{\tau}_{I^s}=\tau_{I^s}-w\sum_{p=0}^{s}\frac{s!}{p!\,(s-p)!}\frac{\partial^p[I/\varphi(I)]}{\partial I^p}\chi_{I^{s-p+1}}.$$

The first-order partial differential equations $g^1=0$ and $f^1=0$ serve as a tool for constructing approximate solutions for the boundary value problem Eq. (8). The arbitrary functions F^1 and G^1 contained in these equations should be chosen to satisfy the requirements of the renormalization group analysis: $f^1=0$, $g^1=0$ at the boundary [8]. We imposed additional restrictions from the requirement that solutions should correspond, in their limit for $z\to\infty$, to the equilibrium solutions of the Hamiltonian Eq. (6). Then, based on Eq. (20) and the restrictions mentioned above, we construct the desired solution

$$-\chi^2=\ln\{I[1-\tau^2 n(I)/I^2]\},$$

$$w=-2\tau\chi n(I)/I, \tag{22}$$

where $n(I)$ is the refractive index as a function of the light intensity.

Returning to the original variables, we obtain

$$\frac{-x^2}{[1-2z^2n(I)]^2}=\ln\{I[1-z^2n(I)]\},$$

$$v=-\frac{2zxn(I)}{1-2z^2n(I)}. \tag{23}$$

These expressions constitute an approximate solution of the nonlinear geometrical optical equations (8) for an initially collimated Gaussian beam and arbitrary nonlinear refractive index $n(I)$. By direct substitution of these functions into Eqs. (8) one can verify that the remaining terms at the beam axis are of order $O(a^2)$.

Note that expressions (23) exhibit singularities, which correspond to the self-focusing of the beam. The self-focusing point z_{sf} can be found from the condition

$$I[1-z^2n(I)]=1. \tag{24}$$

This is one of the central equations of this work and provides a transparent relationship between the functional form of the index of refraction and the self-focusing distance. This equation might be useful for experimentalists, in order to determine *a priori* the self-focusing distance for materials where the function $n(I)$ is known. And vice versa, from the experimental value of the self-focusing distance, one would be able to draw conclusions on the microscopic properties of material.

Equation (24) can even be solved analytically for $n(I)=n_2I$. In this case, the self-focusing distance of a Gaussian beam turns out to be $z_{sf}=1/2\sqrt{n_2I_0}$, which is the same expression as the one obtained in Ref. [8] for this particular case of nonlinearity. We notice that this value coincides with the exact Khokhlov solution [3] for a beam with an initial intensity distribution defined as $I(x,0)=\cosh^{-2}(x)$, and the same form of nonlinearity.

The spatial intensity distribution for the self-focusing case is presented in Fig. 1. A different example is shown in Fig. 2, where we plot the intensity distribution corresponding to a material with refractive index of negative sign $(-n_2)$, which leads to defocusing. One can observe the broadening of the beam for increasing propagation distance. In both cases the change in the shape of the intensity profile upon propagation is clearly seen. Note that the variational method is not able to describe this effect.

If $n(I)$ has a more complicated dependence on I, Eq. (24) cannot be solved analytically. Nevertheless, its numerical solution is significantly simpler than that of the differential equations provided by other methods; e.g., [10,13].

We now apply the result of Eqs. (23) to the case where $n(I)$ shows nonlinearities in the form of a power law $n(I)=n_2I^K$, with $K>1$. We find that the intensity distribution does not qualitatively differ from that of the previously con-

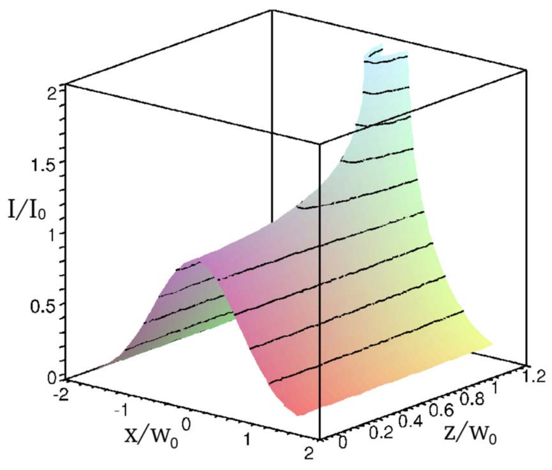

FIG. 1. (Color online) Self-focusing of a laser beam in a nonlinear medium with positive refractive index $n=n_2I(n_2I_0=0.3)$. Intensity distribution I/I_0 versus propagation distance z/w_0 and radius x_0/x_0. Intensity is normalized with respect to its peak value at the entrance plane of the nonlinear medium, I_0. w_0 is the initial beam radius. The peak intensity tends to infinity at the self-focusing distance $z_{\mathrm{sf}}=1/(2\sqrt{0.3})$.

sidered case. In Fig. 3 we plot the on-axial intensity as a function of the propagation distance for a fixed value of intensity at the border of the medium and various power dependences of the refractive index on intensity. In all cases one can observe the collapsing behavior. An increase of the power K leads to a decrease of the self-focusing distance.

In Fig. 4 we plot the on-axial intensity distribution for a fixed form of refractive index $n=n_2I^2$ and various values of intensity at the border of the medium.

Clearly, and as expected, an increase of the incident intensity of the beam leads to a decrease of the self-focusing distance.

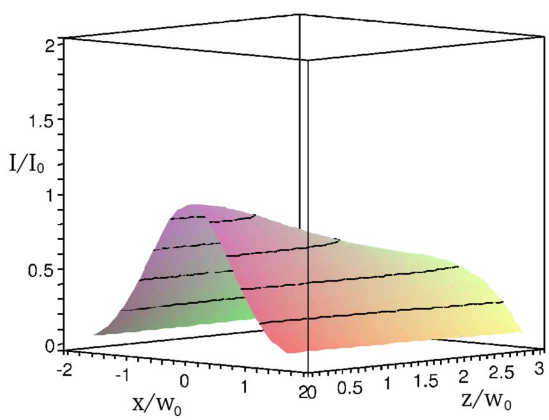

FIG. 2. (Color online) Self-defocusing of a laser beam in a nonlinear medium with negative refractive index $n=n_2I$ $(n_2I_0=-0.3)$. Intensity distribution I/I_0 versus propagation distance z/w_0 and radius x/w_0. The peak intensity of the beam (of Gaussian form at $z=0$) decreases during the propagation. I_0 is the peak value of the intensity at the entrance plane of the nonlinear medium. w_0 is the initial beam radius.

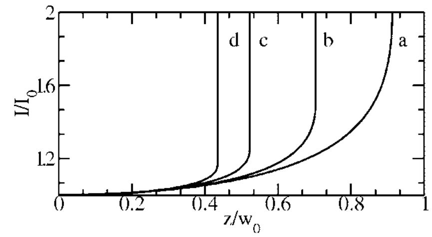

FIG. 3. Self-focusing of a laser beam in a nonlinear medium with positive refractive index. On-axial intensity distribution for different dependences of the refractive index on the beam intensity: $n=$ (a) $0.3I$; (b) $0.3I^2$, (c) $0.3I^4$, (d) $0.3I^6$. I_0 is the peak value of the intensity at the entrance plane of the nonlinear medium. w_0 is the initial beam radius.

IV. MULTIPHOTON IONIZATION IN AIR

In this section, we study the situation when the index of refraction has competing contributions, i.e., it consists of terms of different signs. We consider the form $n=n_2(I-\beta/n_2I^K)$, which includes both self-focusing and -defocusing terms. This form of nonlinearity describes the case of multiphoton plasma formation discussed above. One can notice that in realistic optical media the coefficient β related to multiphoton ionization is sufficiently smaller than the Kerr nonlinearity n_2 that our perturbative calculations are still valid.

Substituting $n=n_2(I-\beta/n_2I^K)$ into Eqs. (23), we get the spatial intensity distribution for the process of nonlinear self-focusing accompanied by the multiphoton ionization. The corresponding on-axial intensity distribution for this case is presented in Fig. 5. We analyze the effect by assuming different types of defocusing terms.

In contrast to the collapsing or broadening behavior shown in the previous section, we see in Fig. 5 that the intensity increases with the propagation distance and achieves saturation at a given value, which depends on the form of the defocusing term. This saturation value corresponds to an equilibrium of the Hamiltonian Eq. (6), and remains stationary during the beam propagation as long as the energy loss is not taken into account in Eqs. (2). Such a spatial intensity distribution is referred to as the self-guiding

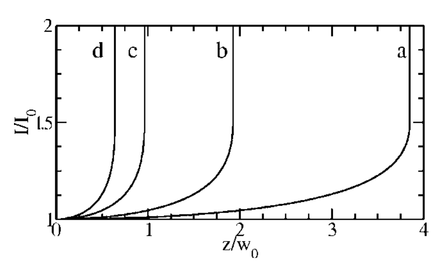

FIG. 4. Self-focusing of a laser beam in a nonlinear medium with positive refractive index of the form $n=n_2I^2$. On-axial intensity distribution for various initial beam intensities: $I=$ (a) I_0; (b) $2I_0$; (c) $4I_0$; (d) $6I_0$. I_0 is the peak value of the intensity at the entrance plane of the nonlinear medium. w_0 is the initial beam radius.

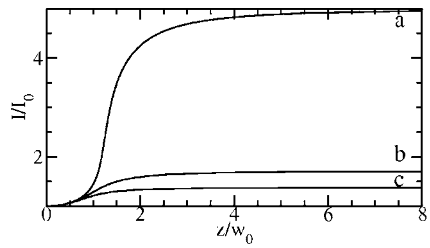

FIG. 5. Filamentation in a medium exhibiting Kerr nonlinearity and multiphoton ionization. A saturated value of on-axial intensity is achieved due to competition between linear and power terms. n = (a) $0.3(I-0.2I^2)$; (b) $0.3(I-0.2I^4)$; (c) $0.3(I-0.2I^6)$. I_0 is the peak value of the intensity at the entrance plane of the nonlinear medium. w_0 is the initial beam radius.

propagation or process of filament formation in the literature: upon propagation in a nonlinear medium the laser beam reaches a point, which we denote again as z_{sf}, where the intensity sharply increases until it achieves saturation at the stationary value.

The rate of the intensity increase at z_{sf} depends on the ratio between the Kerr nonlinearity and the rate of multiphoton absorption. Very small values of $\beta I_0^{(K-1)}$ lead to a strong increase of the derivative $\partial I/\partial z$. We must point out that for very large values of the derivative $\partial I/\partial z$ the result of Eqs. (23) is no longer rigorous, since it was based on the geometrical optics approximation. However, this does not affect the overall physical picture since a strong increase of $I(z)$ occurs only on a small interval.

Based on our analytical results given by Eqs. (23) there are different scenarios for the propagation of a laser beam in a medium with competing contributions to the index of refraction. (i) For a system with a multiphoton contribution comparable to or greater than the Kerr nonlinearity, a smooth on-axial intensity increase takes place, followed by a saturation at the stationary value, which is equal to $I=(n_2/\beta I_0^{(K-1)})^{(1/(K-1))}$. This value is obtained from the condition that for $z\to\infty$ the intensity tends to the value which provides a minimum of the functional (6) [36]. (ii) If, on the other hand, the rate of multiphoton ionization is significantly smaller than the Kerr nonlinearity, its contribution is insufficient to prevent self-focusing followed by a strong intensity increase at the point z_{sf}.

Although in those regions where $\partial I/\partial z$ is big our analytical solutions for $I=I(z)$ will differ from the exact solution, since they were obtained under the geometrical optics approximation, z_{sf} can still be rigorously obtained from the analytical expressions (23). In fact, one could describe the intensity profile $I(z)$ as a steplike function: it increases monotonically along the propagation axis up to the point z_{sf}, and for $z>z_{sf}$ it remains stationary at the value defined from the extremum of (6).

Finally, we use the analytical method presented in this paper to study intense laser pulse propagation in air. This problem has many aspects from the pure theoretical point of view (e.g., the question of existence and stability of solitary wave solutions) and also numerous applications. The problem has been analyzed theoretically in a number of papers (see, e.g., [12,17,27]).

TABLE I. Two sets of laser beam parameters are taken for comparison between numerical and analytical results.

Parameter	Reference [27]	Reference [19]
t_p	120 fs	30 fs
w_0	14 mm	1 mm
E_{in}	50 mJ	0.2 mJ
K	8	10
$\beta^{(K)}$	3.7×10^{-95} cm^{13}/W^7	1.27×10^{-126} cm^{17}/W^9

We apply now the analytical expressions (23) to this problem. We take the beam parameters typically used in current experiments and a model for the air response used in numerical simulations. For the comparison, we consider the two cases treated in Refs. [27] and [19]. In both cases, a beam with central wavelength $\lambda=800$ nm is considered. The value of the Kerr nonlinear refractive index is $n_2=3.2\times10^{-19}$ cm^2/W, and the air atomic density at the normal pressure is $\rho_{at}=2.7\times10^{19}$ cm^{-3}. The other beam parameters considered are shown in Table I. From the input energy E_{in} used in the table one can obtain the input power of the pulse as $P_{in}=E_{in}/t_p\sqrt{\pi/2}$. The input intensity I_0 is computed from the input power, the transverse waist of the beam at the border of the medium, w_0, and the shape of the beam. For a Gaussian beam one gets $I_0=2P_{in}/\pi w_0^2$.

In addition to the different input laser pulse parameters, the numerical simulations of Refs. [27] and [19] were performed using slightly different response models for air. In particular, in Ref. [19] a multiphoton ionization model based on values $U_i=14.6$ eV [37] and $K=10$ for the mean ionization potential and the required number of photons was used. The multiphoton absorption cross section $\beta^{(K)}$ was taken in the form $\sigma_K\times K\hbar\omega_0\rho_{at}\simeq1.27\times10^{-126}$ cm^{17}/W^9. More recent parameter estimates [28] were used in Ref. [27]: $U_i=12.06$ eV, $K=8$, and the multiphoton absorption cross section was set to $\beta^{(K)}\simeq3.7\times10^{-95}$ cm^{13}/W^7.

We performed analytical calculations of the self-focusing distance and saturation intensity using the parameters of Table I in order to compare our analytical predictions with the results of numerical simulations for these two cases. The results are summarized in Table II. In addition, we show the predictions of the variational approach for the same sets of parameters for comparison.

TABLE II. Comparison of results from extensive numerical simulations from Refs. [27] (first and second rows) and [19] (third and fourth rows) with those provided by our analytical solutions [Eq. (23)] for the corresponding parameters. The last column refers to results given by the variational approach.

Quantity	Numerical simulations	This work Eq. (23)	Variational approach
I_{sat}	2×10^{13} W/cm^2	1.8×10^{13} W/cm^2	
z_{sf}	38.7 m	37.7 m	100 m
I_{sat}	5.75×10^{13} W/cm^2	5.29×10^{13} W/cm^2	
z_{sf}	1.9 m	1.2 m	4 m

The table shows a good agreement between the analytical results obtained in Sec. III and results of numerical simulations. For both cases analyzed above, the maximal intensity in the filament is very close to the values obtained in the numerical experiments. In our calculations, the saturated intensity is given by

$$I_{\mathrm{sat}} = (2n_2\rho_c/\sigma_K\rho_{\mathrm{at}}t_p)^{1/(K-1)}. \quad (25)$$

The self-focusing distance obtained by us is, however, shorter than the one obtained numerically. This discrepancy is due to several effects that were included in the numerical simulations and which could not be taken into account in the Eqs. (8). The most important of these effects is that our calculations were carried out in the frame of the geometrical optics approximation and did not take diffraction into account. For the same reason our analytical results of Eqs. (23) give a better agreement with the numerical results of Ref. [27], where the initial beam radius is 14 times greater than in Ref. [19]. Moreover, in contrast to the models solved numerically, we did not consider the energy loss in multiphoton absorption, group velocity dispersion, and delayed Raman contribution in the Kerr effect. Taking into account all these effects should increase the self-focusing distance. However, in the case of air all these terms are small and will not significantly alter the result [37].

In Table II we compare our resulting values with those provided by the variational approach. We use the variational approach equations from Ref. [38] and the set of parameters from Refs. [27] and [19], respectively. Clearly, the variational approach gives significant discrepancies with the numerical results, in contrast to our analytical solution (23). We note that the same order of discrepancy between numerical simulations and the variational approach is usual [13]. Moreover, the variational approach does not predict the saturated value of the intensity, but infinite-intensity oscillations along the propagation axis [10,13].

Summarizing this part, we can conclude that the analytical result Eqs. (23) allows us to obtain the most important quantities for the problem within quite good precision. This opportunity is very useful for preliminary estimation preceding both complicated numerical simulations and experiments. In comparison with the variational approach, which is traditionally used for these goals, the expressions (23) and (25) are more accurate and significantly easier to obtain.

V. CONCLUSIONS AND SUMMARY

In the present paper, the problem of light propagation in nonlinear media was considered. The electric field intensity was assumed to be strong enough to give rise to both Kerr nonlinearity and multiphoton ionization processes.

Up to now this problem was investigated analytically only under the assumption of fixed (e.g., Gaussian) shape of the beam profile. However, this ansatz has no deep physical or mathematical ground. Consequently, the semianalytical results obtained with this restriction lead to discrepancies with respect to results of numerical simulation and experiments, especially for high intensities.

In the present paper we obtain approximate analytical solutions without any *a priori* ansatz.

Based on typical estimates for parameters in current experiments, we demonstrate that for several experimental conditions the geometrical optics approximation yields satisfactory results. Moreover, for laser beam intensity up to 10^{15} W/cm^2, the nonlinearity parameter $a \equiv n_2 I$ is still small enough to allow a perturbative calculation in powers of a. Thus, the mathematical model used in this paper to describe intense laser pulse propagation in nonlinear media is based on the eikonal equations with a small nonlinearity parameter.

An approximate analytical solution for these equations was constructed using the renormalization group method for the boundary value problem. This method of mathematical physics was formulated and discussed in [8,32,39]. It is based on the Lie symmetry analysis of differential equations. An introduction to this field of mathematics can be found in the textbook [34], or in more more detail in Refs. [24,33].

In accordance with the renormalization group method, we started from the renormalization group manifold, Eqs. (8), and constructed, to the first order of a, the Lie-Bäcklund operator Eq. (11) with coordinates Eqs. (17) and (18). As a next step, we had to restrict the obtained symmetry group to the solutions that satisfied the boundary conditions (possible ways to do this were discussed in [8,32,39]). This restriction yields first-order partial differential equations $g^1=0$, $f^1=0$, whose solution was constructed in such a way as to provide the correct limit for $z \to \infty$.

We applied the obtained solutions Eqs. (23) to various polynomial forms of the nonlinear index of refraction. As expected, we observed self-induced defocusing for negative a, and self-focusing collapse of the beam for positive a. The position of the collapse point z_{sf} is a function of both the initial intensity of the beam I_0 and the order of nonlinearity K.

Special attention was paid to the propagation of a laser beam of intensity I above the ionization threshold. This problem has been studied intensively in recent years, and most of these works were devoted to pulse propagation in air. We considered this problem in the frame of the results Eqs. (23). This allowed us to describe the process of self-guiding of collimated Gaussian laser beams. The predicted values of the self-focusing position and saturated beam intensity were demonstrated to be in good agreement with numerical simulations.

In comparison to the widely used ansatz-based methods, the analytical result of Eqs. (23) might be more suitable for the following reasons: (i) the spatial intensity distribution is easier to find numerically from the implicit analytical expressions (23) than from the system of differential equations provided by the variational approach; (ii) as demonstrated, it yields more precise values of z_{sf} and the filament intensity.

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