Maximally polarized states for quantum light fields

Luis L. Sánchez-Soto,[1] Eulogio C. Yustas,[1] Gunnar Björk,[2] and Andrei B. Klimov[3]
[1]*Departamento de Óptica, Facultad de Física, Universidad Complutense, 28040 Madrid, Spain*
[2]*School of Information and Communication Technology, Royal Institute of Technology (KTH), Electrum 229, SE-164 40 Kista, Sweden*
[3]*Departamento de Física, Universidad de Guadalajara, 44420 Guadalajara, Jalisco, Mexico*
(Received 13 October 2006; revised manuscript received 16 July 2007; published 15 October 2007)

The degree of polarization of a quantum field can be defined as its distance to an appropriate set of states. When we take unpolarized states as this reference set, the states optimizing this degree for a fixed average number of photons $\bar{N}$ present a fairly symmetric, parabolic photon statistic, with a variance scaling as $\bar{N}^2$. Although no standard optical process yields such a statistic, we show that, to an excellent approximation, a highly squeezed vacuum can be taken as maximally polarized. We also consider the distance of a field to the set of its SU(2) transformed, finding that certain linear superpositions of SU(2) coherent states make this degree to be unity.

DOI: 10.1103/PhysRevA.76.043820 PACS number(s): 42.50.Dv, 03.65.Ca, 03.65.Yz, 42.25.Ja

I. INTRODUCTION

In classical optics, the polarization of a light beam can be elegantly visualized on the Poincaré sphere and is determined by the Stokes parameters [1,2]: indeed, the corresponding degree of polarization is simply the modulus of the Stokes vector. While this affords a very intuitive image, it also has serious drawbacks that can be traced back to the fact that the Stokes parameters are proportional to the second-order correlations of the field amplitudes. This may be sufficient for most classical problems, but for quantum fields higher-order correlations are crucial. This has prompted some novel definitions and generalizations of the degree of polarization, both in the classical [3–6] and the quantum [7–10] domains.

Recently, we have related the idea of distance measure with the problem of ascertaining the polarization characteristics of a quantum field, exploring suitable definitions that avoid the aforementioned difficulties that previous approaches based on Stokes parameters encounter [11]. This concept of distance measure has been successfully used in assessing a number of key concepts in quantum optics. The notions of nonclassicality [12], entanglement [13], information [14], and localization [15], to cite only a few relevant examples, have been systematically formulated within this framework. The rationale behind this is quite direct: once we have identified a convex set with the desired physical properties (classicality, separability, etc.), the distance determines the distinguishability of a state with respect to that set [16].

Irrespective of our particular choice for the distance, a natural question emerges: what states maximize the corresponding measure? A good deal of effort has been devoted to characterizing maximally nonclassical or entangled states. However, as far as we know, maximally polarized states have not been considered thus far, except for some trivial cases. It is precisely our purpose here to fill this gap, providing a complete description of such states, as well as feasible experimental schemes for their generation.

The plan of this paper is as follows. In Sec. II we briefly recall the basic ingredients needed to define a degree of polarization in terms of a distance measure. Two very different families of polarization degrees can be introduced, depending on the states chosen for determining the distance. The first class employs unpolarized states as a reference set: the corresponding measure is fully analyzed in Sec. III, using a quadratic program to find the associated optimal states. We show that, to an excellent approximation, a highly squeezed vacuum can be considered as maximally polarized. The second approach is analyzed in Sec. IV and uses the distance from a field to the set of all its SU(2)-transformed counterparts. The degree of polarization in this case can be made unity by taking certain superpositions of SU(2) coherent states. Nevertheless, the same scaling for the squeezed vacuum is also recovered in this approach. Finally, we summarize our conclusions in Sec. V.

II. POLARIZATION FLUCTUATIONS AND DISTANCE MEASURES

Let us start by briefly discussing some basic concepts about quantum polarization. We assume a monochromatic plane wave propagating in the z direction, whose electric field lies in the xy plane. Under these conditions, the field can be represented by two complex amplitude operators, denoted by $\hat{a}_H$ and $\hat{a}_V$ when using the basis of linear (horizontal and vertical) polarizations. They obey the bosonic commutation relations

$$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}, \quad j,k \in \{H,V\}. \tag{2.1}$$

The Stokes operators are subsequently introduced as the quantum counterparts of the classical variables [17], namely,

$$\hat{S}_0 = \hat{a}_H^\dagger \hat{a}_H + \hat{a}_V^\dagger \hat{a}_V, \quad \hat{S}_1 = \hat{a}_H^\dagger \hat{a}_V + \hat{a}_V^\dagger \hat{a}_H,$$

$$\hat{S}_2 = i(\hat{a}_H \hat{a}_V^\dagger - \hat{a}_H^\dagger \hat{a}_V), \quad \hat{S}_3 = \hat{a}_H^\dagger \hat{a}_H - \hat{a}_V^\dagger \hat{a}_V, \tag{2.2}$$

and their mean values are precisely the Stokes parameters $(\langle \hat{S}_0 \rangle, \langle \hat{\mathbf{S}} \rangle)$, where $\hat{\mathbf{S}} = (\hat{S}_1, \hat{S}_2, \hat{S}_3)^T$ and the superscript T indicates the transpose. The Stokes operators satisfy the commutation relations of the algebra su(2):

$$[\hat{S}_1,\hat{S}_2] = 2i\hat{S}_3, \tag{2.3}$$

and cyclic permutations. The noncommutability of these operators precludes the simultaneous exact measurement of their physical quantities, which is expressed by the uncertainty relation

$$\langle(\Delta\hat{\mathbf{S}})^2\rangle = \langle(\Delta\hat{S}_1)^2\rangle + \langle(\Delta\hat{S}_2)^2\rangle + \langle(\Delta\hat{S}_3)^2\rangle \geqslant 2\langle\hat{S}_0\rangle. \tag{2.4}$$

The standard definition of the degree of polarization reads as [18]

$$\mathbb{P}_{\mathrm{sc}} = \frac{|\langle\hat{\mathbf{S}}\rangle|}{\langle\hat{S}_0\rangle} = \frac{\sqrt{\langle\hat{S}_1\rangle^2 + \langle\hat{S}_2\rangle^2 + \langle\hat{S}_3\rangle^2}}{\langle\hat{S}_0\rangle}, \tag{2.5}$$

where the subscript "sc" indicates that this is a semiclassical definition, mimicking the form of the classical one. Alternatively, this can be recast as

$$\mathbb{P}_{\mathrm{sc}} = \frac{\sqrt{\langle\hat{S}_0(\hat{S}_0+2)\rangle - \langle(\Delta\hat{\mathbf{S}})^2\rangle}}{\langle\hat{S}_0\rangle}, \tag{2.6}$$

which clearly shows that $\mathbb{P}_{\mathrm{sc}}$ takes into account both mean polarization and its fluctuations. However, $\mathbb{P}_{\mathrm{sc}}$ depends exclusively on the first moments of the Stokes operators. Higher-order moments turn out to be crucial for a full understanding of quantum phenomena as, e.g., polarization squeezing [19]. We also note that for any factorized state of the form $|\psi\rangle_H|0\rangle_V$, i.e., an arbitrary state in the horizontal mode and the vacuum in the vertical one, we get $\mathbb{P}_{\mathrm{sc}}=1$, which seems unphysical from a variety of reasons. A number of additional flaws, such as that of quantum states with hidden polarization [20,21], have also been put forward before.

To bypass all these problems it has been suggested to resort to distance measures [11]. According to the discussion made in the Introduction, we propose to quantify the degree of polarization of a state described by the density matrix $\hat{\varrho}$ as

$$\mathbb{P}(\hat{\varrho}) \propto \inf_{\hat{\sigma}\in\mathcal{S}} D(\hat{\varrho},\hat{\sigma}), \tag{2.7}$$

where $\mathcal{S}$ denotes a convex set of states with physical outstanding properties as polarization is concerned and $D(\hat{\varrho},\hat{\sigma})$ is any measure of distance between the density matrices $\hat{\varrho}$ and $\hat{\sigma}$. The constant of proportionality in Eq. (2.7) is conveniently chosen so that $\mathbb{P}$ is normalized to unity. The distance $D(\hat{\varrho},\hat{\sigma})$ must ensure that $\mathbb{P}(\hat{\varrho})$ satisfies some requirements motivated by both physical and mathematical concerns.

Roughly speaking, we can discern two different kinds of sets $\mathcal{S}$. The first one corresponds to unpolarized states, which operationally can be seen as the only ones that remain invariant under any polarization transformation [22]. The second possibility is to measure the distance between a given state and the set of all its SU(2) transformed, so this is a kind of sensitivity to polarization transformations and can be related to a generalized visibility [23].

We stress that these two sets represent quite distinct physical properties, so one could expect the corresponding optimal states to be very different. In the following we will treat these two instances separately.

III. DISTANCES TO UNPOLARIZED STATES

A. SU(2) polarization structure

We first observe that $[\hat{\mathbf{S}},\hat{S}_0]=0$, so each energy manifold can be treated separately. To bring out this point more clearly, it is advantageous to relabel the standard two-mode Fock basis in the form

$$|N,k\rangle = |k\rangle_H \otimes |N-k\rangle_V, \quad k=0,1,\ldots,N. \tag{3.1}$$

For each fixed total number of photons N, these states span an invariant subspace of dimension $N+1$ and the operators $\hat{\mathbf{S}}$ act therein according to

$$\hat{S}_+|N,k\rangle = 2\sqrt{(k+1)(N-k)}|N,k+1\rangle,$$

$$\hat{S}_-|N,k\rangle = 2\sqrt{k(N-k+1)}|N,k-1\rangle,$$

$$\hat{S}_3|N,k\rangle = 2(k-N/2)|N,k\rangle, \tag{3.2}$$

where $\hat{S}_\pm=\hat{S}_1\pm i\hat{S}_2$. These invariant subspaces will play a key role in the following.

Linear polarization transformations are generated by the Stokes operators (2.2). However, $\hat{S}_0$ induces only a common phase shift to all the states in any given subspace, which does not change the polarization and can thus be omitted. Therefore, we restrict ourselves to the SU(2) transformations, generated by $\hat{\mathbf{S}}$. Since $\hat{S}_1$ is related to $\hat{S}_2$ and $\hat{S}_3$ by the commutation relations, only the latter generators suffice. It is well known that $\hat{S}_2$ generates rotations around the direction of propagation, whereas $\hat{S}_3$ represents differential phase shifts between the modes. It follows then that any polarization transformation can be realized with linear optics (phase plates and rotators) and can be expressed as

$$\hat{U}_g(\phi,\theta,\psi) = e^{i(\phi/2)\hat{S}_3}e^{i(\theta/2)\hat{S}_2}e^{i(\psi/2)\hat{S}_3}. \tag{3.3}$$

As we have noticed before, unpolarized states are the only ones that remain invariant under any polarization transformation [22]. It turns out that this requirement imposes the density operator of these states to be of the form

$$\hat{\sigma} = \bigoplus_{N=0}^{\infty} r_N \,\mathbb{1}_N, \tag{3.4}$$

where $\mathbb{1}_N$ denotes the unity operator in the excitation manifold with N photons and the coefficients r_N are real and nonnegative and to meet the unit-trace condition of the density operator they must satisfy

$$\sum_{N=0}^{\infty} (N+1)r_N = 1. \tag{3.5}$$

Now we can reinterpret the general definition (2.7) in terms of the distance between $\hat{\varrho}$ and the set $\mathcal{U}$ of unpolarized states of the form (3.4). For a concrete analysis, we will consider first the Hilbert-Schmidt metric

$$D_{\mathrm{HS}}(\hat{\varrho},\hat{\sigma}) = \mathrm{Tr}[(\hat{\varrho}-\hat{\sigma})^2], \tag{3.6}$$

which has been previously studied in the context of entanglement [24].

According to the general strategy outlined in the definition (2.7), for a given state $\hat{\varrho}$ we should find the unpolarized state $\hat{\sigma}$ that minimizes the distance. After some elaboration we get

$$\mathbb{P}_{\mathrm{HS}}(\hat{\varrho}) = \mathrm{Tr}(\hat{\varrho}^2) - \sum_{N=0}^{\infty} \frac{p_N^2}{N+1}, \tag{3.7}$$

which depends on both the purity $0<\mathrm{Tr}(\hat{\varrho}^2)\leqslant 1$ and the photon-number distribution p_N. For Gaussian states, photon-counting experiments are sufficient to determine unambiguously these two quantities [25,26]. For a general mixed state, the purity can be obtained from an independent joint measurement on two copies of the state [27].

B. Scaling laws for the degree of polarization

It is obvious from Eq. (3.7) that for the states living in the manifold with exactly N photons, the optimum is reached for pure states [for which $\mathrm{Tr}(\hat{\varrho}^2)=1$]. These pure states can be written as

$$|\Psi_N\rangle = \sum_{k=0}^{N} c_{Nk}|N,k\rangle, \tag{3.8}$$

and all of them have the same degree of polarization,

$$\mathbb{P}_{\mathrm{HS}}(|\Psi_N\rangle) = \frac{N}{N+1} \simeq 1 - \frac{1}{N}, \tag{3.9}$$

where the last expression, showing the typical asymptotic scaling N^{-1}, holds when $N \gg 1$. The important SU(2) coherent states [28]

$$|N,\theta,\phi\rangle = \hat{R}(\theta,\phi)|N,0\rangle, \tag{3.10}$$

where

$$\hat{R}(\theta,\phi) = \exp[\theta(e^{-i\phi}\hat{S}_+ - e^{i\phi}\hat{S}_-)] \tag{3.11}$$

is the displacement operator on the Poincaré sphere, are a particular case of Eq. (3.8) with coefficients

$$c_{Nk}(\theta,\phi) = \binom{N}{k}^{1/2} \left(\sin\frac{\theta}{2}\right)^{N-k} \left(\cos\frac{\theta}{2}\right)^{k} e^{-ik\phi}, \tag{3.12}$$

as one can check using the disentangling theorem on Eq. (3.11). Here, θ and ϕ are the polar and azimuthal angles on the sphere, respectively.

However, this N^{-1} law can be surpassed. Perhaps the simplest example is when both modes are in (quadrature) coherent states. We denote this by $|\alpha_H,\alpha_V\rangle$. By reparametrizing the amplitudes as

$$\alpha_H = e^{-i\phi/2}\sqrt{\bar{N}}\sin(\theta/2), \quad \alpha_V = e^{i\phi/2}\sqrt{\bar{N}}\cos(\theta/2), \tag{3.13}$$

we can write them in the form

$$|\alpha_H,\alpha_V\rangle = \sum_{N=0}^{\infty} e^{-\bar{N}/2}\frac{\bar{N}^{N/2}}{\sqrt{N!}}|N,\theta,\phi\rangle, \tag{3.14}$$

where $\bar{N}=|\alpha_H|^2+|\alpha_V|^2$ is the average number of photons. Since for these states p_N is a Poissonian of mean $\bar{N}$, one can perform the sum in Eq. (3.7), with the result

$$\mathbb{P}_{\mathrm{HS}}(|\alpha_H,\alpha_V\rangle) = 1 - \frac{I_1(2\bar{N})}{\bar{N}}e^{-2\bar{N}} \simeq 1 - \frac{1}{2\sqrt{\pi}\bar{N}^{3/2}}, \tag{3.15}$$

where $I_1(z)$ is the modified Bessel function and the rightmost proportionality is valid for $\bar{N}\gg 1$. The natural question now is whether or not we can go beyond this bound.

C. Optimal states

The previous discussion suggests one find optimal states for a fixed average number of photons $\bar{N}$. Obtaining the whole optimal distribution p_N in Eq. (3.7) is exceedingly difficult, since it involves optimizing over an infinite number of variables. Our strategy to attack this problem is to truncate the Hilbert space and consider only photon numbers up to some value D, where we take the limit $D\to\infty$ at the end. In this truncated space, we need to find states that maximize Eq. (3.7) with the constraints

$$p_N \geqslant 0, \quad \sum_{N=0}^{D} p_N = 1, \quad \sum_{N=0}^{D} N p_N = \bar{N}. \tag{3.16}$$

It is clear that the optimal must be again a pure state. If we introduce the notations $p^T=(p_0,p_1,\ldots,p_D)$ and $H=2\,\mathrm{diag}[1,1/2,\ldots,1/(D+1)]$, the task can be recast as

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}p^T H p, \\ &\text{subject to} \quad Ap=b, \\ &\qquad\qquad\quad p \geqslant 0, \end{aligned} \tag{3.17}$$

where

$$b = \begin{pmatrix} 1 \\ \bar{N} \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & D \end{pmatrix}. \tag{3.18}$$

We deal then with a convex quadratic program, because H is positive definite [29]. The optimal point exists and it is unique: in fact, there are numerous algorithms that compute this optimum in a quite efficient manner. Alternatively, we may try to determine it analytically by incorporating the constraints by the method of Lagrange multipliers. The functional to be minimized is

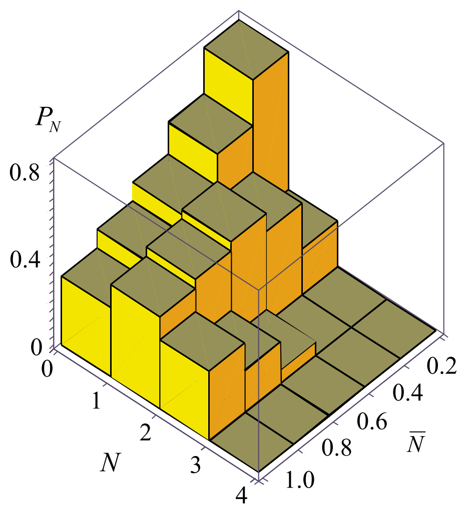

FIG. 1. (Color online) Optimal distribution p_N, obtained by solving numerically the quadratic program (3.17), plotted as a function of the average number of photons $\bar{N}$ and N. We have taken $\bar{N}$ running from 0.2 to 1 and the dimension of the space $D=4$.

$$\mathcal{L}(p,\lambda) = \frac{1}{2}p^T H p - \lambda^T (Ap - b). \tag{3.19}$$

The first-order optimality conditions $\nabla \mathcal{L}(p,\lambda)=0$ together with the initial equality constraint, give the system of linear equations

$$\begin{pmatrix} H & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} p \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \tag{3.20}$$

whose formal solution is

$$\lambda = (AH^{-1}A^T)^{-1} b, \quad p = H^{-1}A^T \lambda. \tag{3.21}$$

Before working out the analytical form of Eq. (3.21), in Fig. 1 we have plotted the numerical solution of the quadratic program (3.17) for some values in $0 \leqslant \bar{N} \leqslant 1$, obtained with the MINQ code implemented in MATLAB. The number of nonzero components of p_N is $[2\bar{N}+1]$, where the brackets denote the integer part. The distribution presents a clear skewness and one can check that it can be well fitted to a Poisson distribution, which in physical terms means that, in this range, a quadrature coherent state $|\alpha_H, \alpha_V\rangle$ can be considered as optimal. To better assess this behavior, we have calculated the Mandel $\mathcal{Q}$ parameter [30]

$$\mathcal{Q} = \frac{\langle (\Delta \hat{N})^2 \rangle}{\langle \hat{N} \rangle} - 1, \tag{3.22}$$

where the variance $\langle (\Delta \hat{N})^2 \rangle$ is a standard measure of the deviation from the Poisson statistics. In Fig. 2 we have represented $\mathcal{Q}$ in terms of $\bar{N}$. As we can see, $\mathcal{Q}$ increases linearly with $\bar{N}$ and is zero only for $\bar{N} \simeq 3$.

In Fig. 3 we have plotted the optimal distribution p_N for integer values of $\bar{N}$ running from 1 to 9. The truncation dimension D has been chosen to be 25 in all the cases, although it is sufficient to ensure, for each value of $\bar{N}$, that $p_N \simeq 0$ for $N > D$. Three distinctive features can be immediately discerned: for integer $\bar{N}$, the solutions are symmetric around $\bar{N}$, they are parabolic, and extend in a range from 0 to $2\bar{N}$. The two first facts are in full agreement with the symmetry properties of the original problem (3.17). The third one implies a variance that scales as $\bar{N}^2$, at difference of what happens for standard coherent optical processes presenting a variance linear with $\bar{N}$ (as for, e.g., in Poissonian or Gaussian statistics). In other words, the optimal-state photon-number distribution is extremely noisy and fluctuating. When $\bar{N}$ is not integer (or semi-integer), one can appreciate a small asymmetry that is less and less noticeable as $\bar{N}$ increases.

We thus conclude that the dimension D can be taken to be $2\bar{N}$. Given the very simple form of H and A, we can express the final solution (3.21) in a closed analytic form,

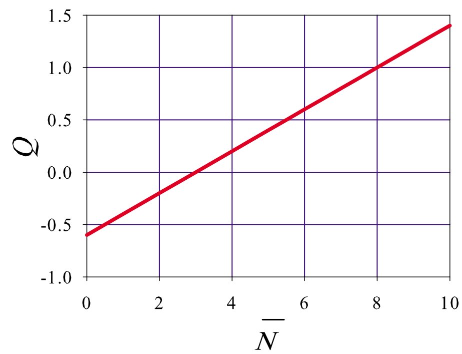

FIG. 2. (Color online) Plot of the Mandel $\mathcal{Q}$ parameter for the optimal distribution p_N obtained numerically from Eq. (3.17) in terms of the average number of photons $\bar{N}$ of the state.

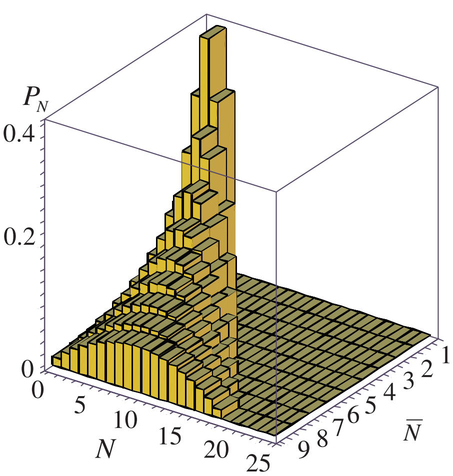

FIG. 3. (Color online) Optimal distribution p_N plotted as a function of the average number of photons $\bar{N}$ and N. We have taken $\bar{N}$ to be an integer running from 1 to 9 and the dimension of the space $D=25$.

$$p_N = 3\frac{(\bar{N}+1)^2 - (N-\bar{N})^2}{(2\bar{N}+1)(\bar{N}+1)(2\bar{N}+3)} \simeq \frac{3}{2\bar{N}}\left(\frac{N}{\bar{N}} - \frac{N^2}{2\bar{N}^2}\right), \tag{3.23}$$

which is properly normalized and shows all the aforementioned characteristics, with a maximum value of $p_{\bar{N}} \simeq 3/(4\bar{N})$ for $N=\bar{N}$. If we use the variable $x=N/(2\bar{N})$, which fulfills $0 \leqslant x \leqslant 1$ and can be considered as quasicontinuous when $\bar{N} \gg 1$, we can convert Eq. (3.23) into

$$p(x) = \frac{3}{\bar{N}}x(1-x), \tag{3.24}$$

and this is precisely the Beta distribution [31] of parameters (2,2).

For the solution (3.23), the corresponding degree of polarization is

$$\mathbb{P}_{\mathrm{HS}}^{\mathrm{opt}} = 1 - \frac{3}{(2\bar{N}+1)(2\bar{N}+3)} \simeq 1 - \frac{3}{4\bar{N}^2}. \tag{3.25}$$

This provides a full characterization of the optimal states we were looking for. However, their physical implementation stands as a serious problem. The crucial issue for the scaling in Eq. (3.25) is the fact that the distribution variance is proportional to $\bar{N}^2$. It turns out that, for the discrete (uniparametric) distributions usually encountered in physics, this is distinctive of the thermal (or geometric) distribution

$$p_N = \frac{1}{\bar{N}+1}\left(\frac{\bar{N}}{\bar{N}+1}\right)^N. \tag{3.26}$$

But this is the photon statistics associated with the states

$$|\xi\rangle = \sqrt{1-q^2}\sum_{n=0}^{\infty} q^n |n\rangle_H \otimes |n\rangle_V = \sqrt{1-q^2}\sum_{N=0,2,4,\ldots}^{\infty} q^{N/2}|N,N/2\rangle, \tag{3.27}$$

which are the twin modes generated in an optical parametric amplifier with a vacuum-state input. Here

$$q = \tanh\xi, \quad \bar{N} = 2\sinh^2\xi, \tag{3.28}$$

ξ being the squeezing parameter (which, for simplicity, has been assumed to be real) and $\bar{N}$ the average number of photons. The twin modes (3.27) are sometimes referred to as regularized EPR states, since they are maximally entangled and when $\xi \to \infty$ approach the idealized stated used in the original EPR proposal [32]. Notice that, although the photons in Eq. (3.27) are perfectly correlated, their statistics in each polarization mode alone is thermal [33].

The distribution (3.26) presents a skewness absent in the exact solution (3.23), but a calculation of the associated degree of polarization gives

$$\mathbb{P}_{\mathrm{HS}}(|\xi\rangle) \simeq 1 - \frac{\ln(\bar{N}/2)}{\bar{N}^2}. \tag{3.29}$$

Apparently, this is different from Eq. (3.25), but as soon as $\bar{N} \gg 1$ they both approach unity in essentially the same way, which means that the (maximally entangled) squeezed vacuum (3.27) is very close to an optimal polarization state when $\bar{N} \gg 1$.

To gain further insight into these states, phase-space quasidistributions constitute a very appropriate tool. The simplest one from a computational viewpoint is the SU(2) Q function, which is defined as [28]

$$Q(\theta,\phi) = \sum_{N=0}^{\infty} \frac{N+1}{4\pi}\langle N,\theta,\phi|\hat{\varrho}|N,\theta,\phi\rangle, \tag{3.30}$$

where $|N,\theta,\phi\rangle$ have been introduced in Eq. (3.10). This Q function is always non-negative and can be considered as a true probability distribution obtained by projection on the SU(2) coherent states, which can be regarded as the states with simultaneous minimum polarization fluctuations. Note also that the matrix elements of $\hat{\varrho}$ connecting different invariant subspaces do not contribute to Q.

For the states (3.27), one direct calculation shows that

$$Q(\theta,\phi) = \sum_{N=0,2,4,\ldots}^{\infty} p_N\, Q_N(\theta,\phi), \tag{3.31}$$

where p_N is the thermal distribution (3.26) and $Q_N(\theta,\phi)$ is the Q function for the state $|n\rangle_H \otimes |n\rangle_V = |N,N/2\rangle$, and is given by

$$Q_N(\theta,\phi) = \frac{N+1}{4\pi}\frac{1}{2^N}\binom{N}{N/2}\sin^N\theta. \tag{3.32}$$

Inserting this analytical form in Eq. (3.31) and performing the summation we get, after some manipulations,

$$Q(\theta,\phi) = \frac{1}{2\pi(\bar{N}+2)}\left(1 - \frac{\bar{N}}{\bar{N}+2}\sin^2\theta\right)^{3/2}. \tag{3.33}$$

This function is plotted in Fig. 4 for $\bar{N}=4$. It appears as a symmetric belt on the equator of the Poincaré sphere (the thinnest possible, in fact), showing the independence of the azimuthal angle ϕ. This shape indicates that the twin-photon beams have a random relative phase (even if each mode can be considered as a phase-coherent state [34]). On the contrary, one can check that the quadrature coherent states (3.14) have a sharp relative phase. This calls in question the classical notion that fully polarized fields have a perfectly defined relative phase between H- and V-polarized modes [18]. We note in passing that the states (3.27) have a highly nonclassical polarization behavior, even in the limit $\bar{N} \gg 1$.

D. Bures degree of polarization

One can think that the optimal states obtained in the previous section depend on the distance chosen. Consequently,

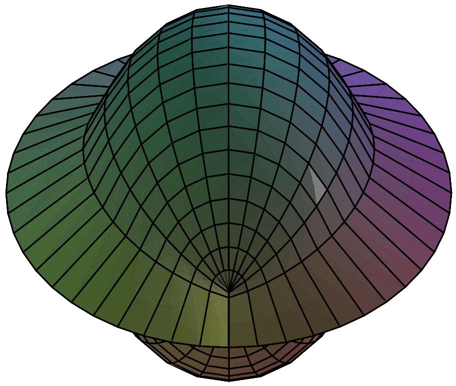

FIG. 4. (Color online) Plot of the Q function over the unit sphere for a highly squeezed vacuum state (3.27) with $\bar{N}=4$.

we next investigate what happens if the Bures metric is employed instead of the Hilbert-Schmidt; that is, now we take

$$D_{\mathrm{B}}(\hat{\varrho},\hat{\sigma}) = 2[1-\sqrt{F(\hat{\varrho},\hat{\sigma})}], \tag{3.34}$$

with

$$F(\hat{\varrho},\hat{\sigma}) = [\mathrm{Tr}(\hat{\sigma}^{1/2}\hat{\varrho}\hat{\sigma}^{1/2})^{1/2}]^2 \tag{3.35}$$

being the fidelity [35]. The origin of this distance can be seen intuitively by considering the case when $\hat{\varrho}$ and $\hat{\sigma}$ are both pure states: the Bures metric is just the Euclidean distance between the two pure states, with respect to the usual norm on the state space.

Since the larger the fidelity $F(\hat{\varrho},\hat{\sigma})$, the smaller the Bures distance $D_{\mathrm{B}}(\hat{\varrho},\hat{\sigma})$, we can define the Bures degree of polarization as

$$\mathbb{P}_{\mathrm{B}}(\hat{\varrho}) = 1 - \sup_{\hat{\sigma}\in\mathcal{U}} \sqrt{F(\hat{\varrho},\hat{\sigma})}. \tag{3.36}$$

Consider now the optimal states (3.27). For any generic $\hat{\sigma}$ as in Eq. (3.4) we can easily calculate $\hat{\sigma}^{1/2}\hat{\varrho}\hat{\sigma}^{1/2}$, obtaining

$$\sqrt{F(\hat{\varrho},\hat{\sigma})} = (1-q^2)\sum_{N=0}^{\infty} q^N p_N. \tag{3.37}$$

The minimum of this fidelity is clearly attained when $p_N=\delta_{N,0}$, and therefore

$$\mathbb{P}_{\mathrm{B}}(|\xi\rangle) \simeq 1 - \frac{1}{2\bar{N}^2}, \tag{3.38}$$

which shows again essentially the same scaling as before when $\bar{N}\gg 1$.

IV. OPTIMAL SU(2) DISTINGUISHABLE STATES

As we have discussed in Sec. III, the other sensible approach to the problem at hand is to interpret Eq. (2.7) as

$$\mathbb{P}_{\mathrm{dis}}(\hat{\varrho}) = \frac{1}{2}\inf_{\hat{U}_g} D_{\mathrm{HS}}(\hat{\varrho},\hat{U}_g\hat{\varrho}\hat{U}_g^{\dagger}), \tag{4.1}$$

where $\inf_{\hat{U}_g}$ denotes the infimum under any SU(2) transformation as expressed in Eq. (3.3). For definiteness, we have chosen the Hilbert-Schmidt metric and we have inserted a factor 1/2 to guarantee the proper normalization. Any state that is not invariant under all possible linear polarization transformations has then some degree of quantum polarization.

For a pure state $|\Psi\rangle$, Eq. (4.1) reduces to

$$\mathbb{P}_{\mathrm{dis}}(|\Psi\rangle) = 1 - \inf_{\hat{U}_g} |\langle\Psi|\hat{U}_g|\Psi\rangle|^2. \tag{4.2}$$

In this way, this degree appears as a measure of the minimum overlap between the state and the set of its rotated counterparts. As discussed in Ref. [23], this magnitude can be directly determined as the visibility of an interferential experiment. Moreover, it has been shown [10] that for all N-photon pure states $|\Psi_N\rangle$, one can transform them into an orthogonal state, which yields

$$\mathbb{P}_{\mathrm{dis}}(|\Psi_N\rangle) = 1. \tag{4.3}$$

That is, any N-photon pure state is fully polarized.

By expanding the pure state $|\Psi\rangle$ in terms of its photon-number components

$$|\Psi\rangle = \sum_{N=0}^{\infty} a_N |\Psi_N\rangle, \tag{4.4}$$

and denoting $p_N=|a_N|^2$, we immediately can reformulate the definition (4.2) in the form

$$\mathbb{P}_{\mathrm{dis}}(|\Psi\rangle) = 1 - \inf_{\hat{U}_g} \sum_{N=0}^{\infty} p_N^2 \, |\langle\Psi_N|\hat{U}_g|\Psi_N\rangle|^2. \tag{4.5}$$

Let us now be more specific and take SU(2) coherent states as a particular case of N-photon states in Eq. (4.4). If we observe that $\hat{R}^{\dagger}(\theta,\phi)\hat{U}(\theta_0,\phi_0,\psi_0)\hat{R}(\theta,\phi)=\hat{U}(\theta',\phi',\psi')$, where θ' and ϕ' have an involved expression of no interest for our purposes here, and recall Eq. (3.11), we get

$$\mathbb{P}_{\mathrm{dis}} = 1 - \inf \sum_{N=0}^{\infty} p_N^2 \, |d^N_{N/2N/2}(\theta')|^2, \tag{4.6}$$

where $d^N_{k'k}(\theta)$ is the Wigner d function [36]

$$d^N_{k'k}(\theta) = \langle N,k'|\exp(i\theta\hat{S}_2)|N,k\rangle. \tag{4.7}$$

Taking into account that $d^N_{N/2N/2}(\theta)=\cos^N(\theta/2)$, we immediately get that $\mathbb{P}^{\mathrm{opt}}_{\mathrm{dis}}=1$ whenever $\theta'=\pi$, which means that the state is maximally polarized if it is generated from the lower (or higher) state in each invariant subspace by applying a polarization transformation. Note that the two-mode quadrature coherent states (3.14) constitute an example of this kind of maximally polarized states with p_N Poissonian.

Next, we reconsider the highly squeezed vacuum (3.27) from the perspective of this distance. Instead of a brute-force attack, we observe that, according to our previous discussion, each state $|N,N/2\rangle$ is represented as a belt on the equator of the Poincaré sphere. It is then clear that the operator $\hat{U}_g$ that minimizes the overlap $|\langle\Psi_N|\hat{U}_g|\Psi_N\rangle|^2$ is a rotation of angle $\pi/2$ around the axis Y (or X). In consequence,

$$\mathbb{P}_{\rm dis}(|\xi\rangle) = 1 - \sum_{N=0,2,4,\ldots}^{\infty} p_N^2 \, |d^N_{N/2 N/2}(\pi/2)|^2, \tag{4.8}$$

p_N being the thermal distribution (3.26). Since $d^N_{N/2N/2}(\theta)$ reduces to the Legendre polynomial $P_N(\theta)$, we have, after some calculations,

$$\mathbb{P}_{\rm dis}(|\xi\rangle) = 1 - \frac{2}{\pi}\left(\frac{1}{\bar{N}+1}\right)^2 K([\bar{N}/(\bar{N}+1)]^4), \tag{4.9}$$

where $K(x)$ is the elliptic integral [37]. Considering the asymptotic limit of this function we get

$$\mathbb{P}_{\rm dis}(|\xi\rangle) \simeq 1 - \frac{1}{\pi}\frac{\ln \bar{N}}{\bar{N}^2}, \tag{4.10}$$

which is the same scaling as in the degrees of polarization defined in the previous section.

In fact, one could argue that the state with eigenvalue zero in all even subspaces (i.e., $\hat{S}_3|\zeta\rangle = 0$) is an eigenstate of any linear combination of the Stokes operators. This state can be expressed as

$$|\zeta\rangle = \sum_{N=0,2,4,\ldots}^{\infty} b_N |N, N/2\rangle, \tag{4.11}$$

and can be thought classically (and the whole family of states $\hat{U}_g|\zeta\rangle$) as a fully polarized state. Obviously, in the quantum domain the fluctuations prevent such a state to be fully polarized. We stress that Eq. (4.11) is analogous to Eq. (3.27), but with arbitrary coefficients b_N. The analysis performed before can be easily extended to this case and we obtain that asymptotically the degree of polarization (4.5) gives the same results as the Hilbert-Schmidt measure studied previously.

V. CONCLUSIONS

We have explored the use of a degree of polarization based on the distance to an appropriately chosen set of states. When the unpolarized states are used as the reference set, the twin beams generated in an optical amplifier can be taken as maximally polarized to a good approximation. We have also considered the distance of a state to its SU(2) transformed, finding that now a linear superposition of SU(2) coherent states makes this degree to be unity.

ACKNOWLEDGMENTS

We thank J. Söderholm, A. Felipe, and M. Curty for helpful discussions. This work was supported by the Swedish Foundation for International Cooperation in Research and Higher Education (STINT), the Swedish Foundation for Strategic Research (SSF), the Swedish Research Council (VR), the CONACyT Grant No. 45704, and the Spanish Research Project No. FIS2005-0671.

[1] G. G. Stokes, Trans. Cambridge Philos. Soc. **9**, 399 (1852).
[2] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University Press, Cambridge, England, 1999).
[3] A. Picozzi, Opt. Lett. **29**, 1653 (2004).
[4] P. Réfrégier and F. Goudail, J. Opt. Soc. Am. A **19**, 1223 (2002); P. Réfrégier, F. Goudail, P. Chavel, and A. T. Friberg, *ibid.* **21**, 2124 (2004); P. Réfrégier, M. Roche, and F. Goudail, *ibid.* **23**, 124 (2006); P. Réfrégier and F. Goudail, *ibid.* **23**, 671 (2006).
[5] T. Setälä, A. Shevchenko, M. Kaivola, and A. T. Friberg, Phys. Rev. E **66**, 016615 (2002); T. Setälä, M. Kaivola, and A. T. Friberg, Phys. Rev. Lett. **88**, 123902 (2002); T. Saastamoinen and J. Tervo, J. Mod. Opt. **51**, 2039 (2004).
[6] J. Ellis, A. Dogariu, S. Ponomarenko, and E. Wolf, Opt. Commun. **248**, 333 (2005).
[7] A. S. Chirkin, A. A. Orlov, and D. Yu. Paraschuk, Kvantovaya Elektron. (Moscow) **20**, 999 (1993) [Quantum Electron. **23**, 870 (1993)].
[8] A. P. Alodjants and S. M. Arakelian, J. Mod. Opt. **46**, 475 (1999).
[9] A. Luis, Phys. Rev. A **66**, 013806 (2002); **71**, 063815 (2005); Opt. Commun. **253**, 10 (2005).
[10] A. Sehat, J. Söderholm, G. Björk, P. Espinoza, A. B. Klimov, and L. L. Sánchez-Soto, Phys. Rev. A **71**, 033818 (2005).
[11] A. B. Klimov, L. L. Sánchez-Soto, E. C. Yustas, J. Söderholm, and G. Björk, Phys. Rev. A **72**, 033813 (2005); L. L. Sánchez-Soto, J. Söderholm, E. C. Yustas, A. B. Klimov, and G. Björk, J. Phys.: Conf. Ser. **36**, 177 (2006).
[12] M. Hillery, Phys. Rev. A **35**, 725 (1987); V. V. Dodonov, O. V. Man'ko, V. I. Man'ko, and A. Wünsche, J. Mod. Opt. **47**, 633 (2000); P. Marian, T. A. Marian, and H. Scutaru, Phys. Rev. A **69**, 022104 (2004).
[13] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997); V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[14] C. A. Fuchs and C. M. Caves, Phys. Rev. Lett. **73**, 3047 (1994); B. Schumacher, Phys. Rev. A **51**, 2738 (1995); Č. Brukner and A. Zeilinger, Phys. Rev. Lett. **83**, 3354 (1999); T. Rudolph and B. C. Sanders, *ibid.* **87**, 077903 (2001); A. Gilchrist, N. K. Langford, and M. A. Nielsen, Phys. Rev. A **71**, 062310 (2005).
[15] H. Maassen and J. B. M. Uffink, Phys. Rev. Lett. **60**, 1103 (1988); A. Anderson and J. J. Halliwell, Phys. Rev. D **48**, 2753 (1993); S. Gnutzmann and K. Zyczkowski, J. Phys. A **34**, 10123 (2001).
[16] W. K. Wootters, Phys. Rev. D **23**, 357 (1981).
[17] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, MA, 1959); E. Collett, Am. J. Phys. **38**, 563 (1970); A. Luis and L. L. Sánchez-Soto, Prog. Opt. **41**, 421 (2000).
[18] C. Brosseau, *Fundamentals of Polarized Light: A Statistical Optics Approach* (Wiley, New York, 1998).
[19] P. Grangier, R. E. Slusher, B. Yurke, and A. LaPorta, Phys. Rev. Lett. **59**, 2153 (1987); M. Kitagawa and M. Ueda, Phys.

Rev. A **47**, 5138 (1993); G. S. Agarwal and R. R. Puri, *ibid.* **49**, 4968 (1994); D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, *ibid.* **50**, 67 (1994).

[20] V. P. Karassiov, J. Phys. A **26**, 4345 (1993).

[21] D. N. Klyshko, Usp. Fiz. Nauk **166**, 613 (1996).

[22] H. Prakash and N. Chandra, Phys. Rev. A **4**, 796 (1971); G. S. Agarwal, Lett. Nuovo Cimento Soc. Ital. Fis. **1**, 53 (1971); J. Lehner, U. Leonhardt, and H. Paul, Phys. Rev. A **53**, 2727 (1996); J. Söderholm, G. Björk, and A. Trifonov, Opt. Spectrosc. **91**, 540 (2001).

[23] G. Björk, S. Inoue, and J. Söderholm, Phys. Rev. A **62**, 023817 (2000).

[24] C. Witte and M. Trucks, Phys. Lett. A **257**, 14 (1999); M. Ozawa, *ibid.* **268**, 158 (2000); R. A. Bertlmann, H. Narnhofer, and W. Thirring, Phys. Rev. A **66**, 032319 (2002).

[25] M. G. A. Paris, F. Illuminati, A. Serafini, and S. De Siena, Phys. Rev. A **68**, 012314 (2003).

[26] J. Fiurášek and N. J. Cerf, Phys. Rev. Lett. **93**, 063601 (2004); J. Wenger, J. Fiurášek, R. Tualle-Brouri, N. J. Cerf, and P. Grangier, Phys. Rev. A **70**, 053812 (2004).

[27] R. B. A. Adamson, L. K. Shalm, and A. M. Steinberg, Phys. Rev. A **75**, 012104 (2007).

[28] A. M. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986).

[29] S. Boyd and L. Vandenberghe, *Convex Optimization* (Cambridge University Press, Cambridge, England, 2004).

[30] L. Mandel and E. Wolf, *Coherence and Quantum Optics* (Cambridge University Press, Cambridge, England, 1995).

[31] M. Evans, N. Hastings, and B. Peacock, *Statistical Distributions*, 3rd ed. (Wiley, New York, 2000).

[32] J. A. Larsson, Phys. Rev. A **67**, 022108 (2003).

[33] C. A. Holmes, G. J. Milburn, and D. F. Walls, Phys. Rev. A **39**, 2493 (1989).

[34] J. H. Shapiro, S. R. Shepard, and N. C. Wong, Phys. Rev. Lett. **62**, 2377 (1989); G. M. D'Ariano, M. G. A. Paris, and M. F. Sacchi, Phys. Rev. A **57**, 4894 (1998).

[35] A. Uhlmann, Rep. Math. Phys. **9**, 273 (1976); R. Jozsa, J. Mod. Opt. **41**, 2315 (1994); M. Raginsky, Phys. Lett. A **290**, 11 (2001); P. M. Alberti, Rep. Math. Phys. **51**, 87 (2003).

[36] D. Varshalovich, A. Moskalev, and V. Khersonksii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988). This reference employs the standard angular momentum notation, instead of our basis $|N,k\rangle$. Changing from one to the other basis can be easily accomplished.

[37] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1984).