PHYSICAL REVIEW A **76**, 043810 (2007)

Effects of the vacuum state on the statistics of nonclassical states

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(Received 18 April 2007; published 10 October 2007)

Based on the calculation of the Wigner function, some statistical properties of the superposition of two coherent states with a vacuum state are demonstrated. The distance variation difference function is calculated for these states. Application of homodyne statistics shows that the addition (subtraction) of the vacuum state can improve the classical channel capacity of a noiseless binary symmetric system.

DOI: 10.1103/PhysRevA.76.043810 PACS number(s): 42.50.Ar

I. INTRODUCTION

In a previous paper [1], we analyzed the information and detection performance of systems of optical communications in which entangled states are received by various detectors. These states, that are a superposition of two coherent states [2], have been taken to be pure states and are modulated in phase. We showed that, for such ideal classical systems, the performance is limited only by the type of detector. A photon counting system, for example, is closest to the optimal.

In order to describe more realistic channels, we model a noisy channel as a coupler which adds (or subtracts) the vacuum state to the superposition of the two coherent states. This paper uses the Wigner function to calculate the statistical properties of such states. The probability distribution functions (PDFs) of the field and of its intensity are shown to be complementary. We then evaluate the effects of the vacuum state on the transmission of the information using a typical homodyne detector. The paper is organized as follows.

Section II is devoted to the fundamental equations. First we explain how to use the PDF of the photon number to choose the coefficient of the vacuum state. Because we are mainly interested in the nonclassical properties of the state, we use the Wigner function which is a quasiprobability distribution that can take negative values. In order to complete the characterization of the states, we establish some properties of the variances of the field and its intensity. We show that, depending on the vacuum state coefficient, a variety of nonclassical states can be generated.

The distance variation difference (DVD) that measures the deviation of any state from its corresponding coherent state is calculated for the superposition of the coherent states with the vacuum state in Sec. III. An approximation of the DVD is calculated for low power input radiation.

Section IV presents an application of the results of the preceding sections. It shows that the superposition of coherent states with the vacuum state can be used to improve the transmission of information in a noiseless binary symmetric channel. The variations of the classical channel capacity with respect to the input power is explained using the DVD function. Finally, Sec. V is a discussion of the results.

II. BASIC RELATIONS

We consider a pure state of light and its density operator expressed as

$$|\Psi\rangle = d(|\alpha\rangle + |-\alpha\rangle + \gamma|0\rangle), \tag{2.1a}$$

$$\hat{\rho}_\psi = |\Psi\rangle\langle\Psi| = |d|^2(\hat{\rho}_+ + \hat{\rho}_- + |\gamma|^2\hat{\rho}_0 + \hat{A} + \hat{A}^\dagger + \hat{\Gamma} + \hat{\Gamma}^\dagger), \tag{2.1b}$$

where $\alpha=\alpha_0 e^{i\theta}$ and $|\pm\alpha\rangle$ are the coherent states obeying $\langle\alpha_1|\alpha_2\rangle = e^{-1/2|\alpha_1-\alpha_2|^2}$, $\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|d^2\alpha=\hat{1}$, and $d^2\alpha \equiv d(\mathrm{Re}\,\alpha)d(\mathrm{Im}\,\alpha)$, where $\hat{1}$ is the identity operator. In Eq. (2.1a), $|0\rangle$ is the vacuum state. We also set $\hat{\rho}_+=|\alpha\rangle\langle\alpha|$, $\hat{\rho}_-=|-\alpha\rangle\langle-\alpha|$, $\hat{\rho}_0=|0\rangle\langle 0|$, $\hat{A}=|\alpha\rangle\langle-\alpha|$, and $\hat{\Gamma}=\gamma(|0\rangle\langle\alpha|+|0\rangle\langle-\alpha|)$. For an operator $\hat{O}$, $\hat{O}^\dagger$ denotes its conjugate. The last four terms of Eq. (2.1b) show the interference between the coherent states and between the coherent states and the vacuum state. The terms corresponding to Γ are a little more complicated than those due to A.

We assume at first that γ is an arbitrary real constant and limit ourselves to the specific values $\theta=\frac{\pi}{2}$, and $\alpha=i\alpha_0$. The coefficient d in Eq. (2.1a) is then determined by normalizing the state $|\Psi\rangle$ such that

$$|d|^2 = \frac{1}{2\left(1+e^{-\beta^2}+2\gamma e^{-\beta^2/4}+\frac{\gamma^2}{2}\right)}, \tag{2.2}$$

where $\beta=\alpha_0\sqrt{2}$. Before calculating the probability and the quasiprobability functions needed here, we must specify the coefficient γ which is a parameter of the model.

A. Coefficient of the vacuum state

Let us first consider the case where γ is an arbitrary constant. Hereafter, we set $\hbar\omega=1$. The conventional creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, respectively, obey the commutator $[\hat{a},\hat{a}^\dagger]=\hat{1}$, and the equation $\hat{a}|\alpha\rangle=\alpha|\alpha\rangle$. These

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operators can be used to describe the momentum operator $\hat{P}=i\frac{\hat{a}^\dagger-\hat{a}}{\sqrt{2}}$ and the coordinate operator $\hat{X}=\frac{\hat{a}^\dagger+\hat{a}}{\sqrt{2}}$, which is sometimes called the field operator [3].

We need the following definitions, notations, and relations $\hat{X}|x\rangle=x|x\rangle, \int_{-\infty}^{\infty}|x\rangle\langle x|dx=\hat{1}$, and $\hat{P}|p\rangle=p|p\rangle, \int_{-\infty}^{\infty}|p\rangle\langle p|dp=\hat{1}$. The commutator is given by $[\hat{X},\hat{P}]=i$. We also repeatedly use the scalar products $\langle n|\alpha\rangle=\frac{\alpha^n}{\sqrt{n!}}e^{-|\alpha|^2/2}$, $\langle x|\alpha\rangle=(\frac{1}{\pi})^{1/4}e^{-1/2x^2+\alpha\sqrt{2}x-\alpha\,\mathrm{Re}(\alpha)}$, $\langle p|\alpha\rangle=(\frac{1}{\pi})^{1/4}e^{-1/2p^2-i\alpha\sqrt{2}p+i\alpha\,\mathrm{Im}(\alpha)}$, and $\langle x|p\rangle=\frac{1}{\sqrt{2\pi}}e^{ixp}$ [3–5].

Suppose we are measuring the Hermitian number operator $\hat{N}=\hat{a}^\dagger\hat{a}=\hat{N}^\dagger$ in the state $|\Psi\rangle$. From the above formulas, we easily demonstrate that the PDF for the discrete outcomes n is given by

$$\begin{aligned} p_\psi(n;\beta,\gamma) &= \mathrm{Tr}\,\hat{\rho}_\psi\hat{N} \\ &= |\langle n|\psi\rangle|^2 \\ &= \frac{\frac{\left(\frac{\beta^2}{2}\right)^n(1+(-1)^n)}{n!}e^{-\beta^2/2}+\left(2\gamma e^{-\beta^2/4}+\frac{\gamma^2}{2}\right)\delta(n)}{1+e^{-\beta^2}+2\gamma e^{-\beta^2/4}+\frac{\gamma^2}{2}}, \end{aligned} \tag{2.3}$$

where $\delta(n)$ is such that $\delta(0)=1$ and $\delta(n>0)=0$. This PDF has the main property that $p_\psi(n_c)=0$, $n_c=2\ell+1$, and $\ell=0,1,\ldots$. This is sometimes seen as indicating the nonclassicality [6]. The first two moments of the photon number can be calculated taking into account that the second term of the numerator of Eq. (2.3) has no contribution to the moments of n. We have

$$\langle n\rangle = \frac{\frac{\beta^2}{2}(1-e^{-\beta^2})}{1+e^{-\beta^2}+2\gamma e^{-\beta^2/4}+\frac{\gamma^2}{2}}, \tag{2.4a}$$

$$\langle n^2\rangle = \frac{\frac{\beta^2}{4}[\beta^2+2+(\beta^2-2)e^{-\beta^2}]}{1+e^{-\beta^2}+2\gamma e^{-\beta^2/4}+\frac{\gamma^2}{2}}. \tag{2.4b}$$

We now consider the case where γ depends on β. In fact, because the first two moments of the photon number are $\frac{\beta^2}{2}$ and $\frac{\beta^2}{4}(\beta^2+2)$ for the coherent state, we derive the values of $\gamma(\beta)$ from the solutions of the equations (i): $\langle n\rangle-\frac{\beta^2}{2}=0$ and (ii): $\langle n^2\rangle-\frac{\beta^2}{4}(\beta^2+2)=0$, $\langle n\rangle$ and $\langle n^2\rangle$ being given by Eqs. (2.4a) and (2.4b). We get from (i)

$$\gamma_\pm^* = -2e^{-\beta^2/4}(1\mp e^{-\beta^2/4}\sqrt{e^{\beta^2/2}-1}). \tag{2.5}$$

The condition (ii) leads to

$$\gamma_\pm = -2e^{-\beta^2/4}\left(1\mp\sqrt{1-\frac{2e^{-\beta^2/2}}{2+\beta^2}}\right) \simeq -2\left(1\mp\beta-\frac{1}{4}\beta^2\right), \tag{2.6}$$

where $\gamma_\pm(0)=\gamma_\pm^*(0)=-2$.

The second order moment being more significant physically, we choose to select Eq. (2.6) to define the dependence of γ on β. We then have $\forall\beta, \gamma_-\le\gamma_+\le 0, (|\gamma_-|\ge|\gamma_+|)$. We can also show that $\gamma_\pm(\beta\to\infty)\to 0$. Notice finally that γ_+ is a monotonic function of β while γ_- has a minimum for $\beta^2\simeq\frac{1}{2}$.

The calculation of the variance of the photon number at moderate power is easily deduced from Eqs. (2.4a) and (2.4b),

$$\sigma_N^2(\beta,\gamma) \simeq \frac{\beta^4}{(2+\gamma)^2}\left(2-\frac{\gamma}{2+\gamma}\beta^2\right) \tag{2.7}$$

yielding

$$\sigma_N^2(\beta,\gamma) \simeq \begin{cases} \frac{1}{2}\beta^4, & \gamma=0, \\ \frac{1}{2}\beta^2\left(1\pm\frac{1}{2}\beta\right), & \gamma=\gamma_\pm, \end{cases} \tag{2.8}$$

proving that $\sigma_N^2(\beta,\gamma_-)$ and $\sigma_N^2(\beta,0)$ are less than the variance of a coherent state.

Having determined the values of γ that are of interest here from the photon counting distribution, we now want to study the statistical properties of the field and its intensity, using their PDFs. The appropriate method to calculate these PDFs is to derive them from the Wigner distribution.

B. Wigner distribution

The Wigner distribution in the phase space of the density operator (2.1b) is given by

$$W(x,p) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\langle x+\xi|\hat{\rho}_\psi|x-\xi\rangle e^{-2ip\xi}d\xi, \tag{2.9}$$

where $|x\pm\xi\rangle$ are the eigenvectors of the operator $\hat{X}$. Notice that $W(x,p)$ is a real function normalized to unity. We denote

$$W(x,p) = \frac{2|d|^2}{\pi}(w_+ + w_- + \gamma^2 w_0 + \gamma w_v + w_e), \tag{2.10}$$

where the w_+, w_-, and w_0 are the functions associated with the density operators $\hat{\rho}_+$, $\hat{\rho}_-$, and $\hat{\rho}_0$, respectively [see Eq. (2.1b)]. The other terms of Eq. (2.10) are

$$w_v = \int_{-\infty}^{\infty}\langle x+\xi|(|\alpha\rangle\langle-\alpha|+|-\alpha\rangle\langle\alpha|)|x-\xi\rangle\exp(-2ip\xi)d\xi \tag{2.11}$$

and

$$\begin{aligned} w_e = \int_{-\infty}^{\infty}&\langle x+\xi|(|-\alpha\rangle\langle 0|+|-\alpha\rangle\langle 0|+|0\rangle\langle\alpha|+|0\rangle\langle-\alpha|)|x-\xi\rangle \\ &\times\exp(-2ip\xi)d\xi, \end{aligned} \tag{2.12}$$

and correspond to the interference between the coherent and vacuum states, respectively. We obtain

$$\begin{aligned} W(x,p;\beta,\gamma) = \frac{2|d|^2}{\pi}e^{-(x^2+p^2)}\Big[&e^{-\beta^2}\cosh(2p\beta)+\cos(2\beta x) \\ &+2\gamma e^{-\beta^2/4}\cosh(p\beta)\cos(\beta x)+\frac{\gamma^2}{2}\Big]. \end{aligned} \tag{2.13}$$

Typical plots of these functions versus x for various γ with

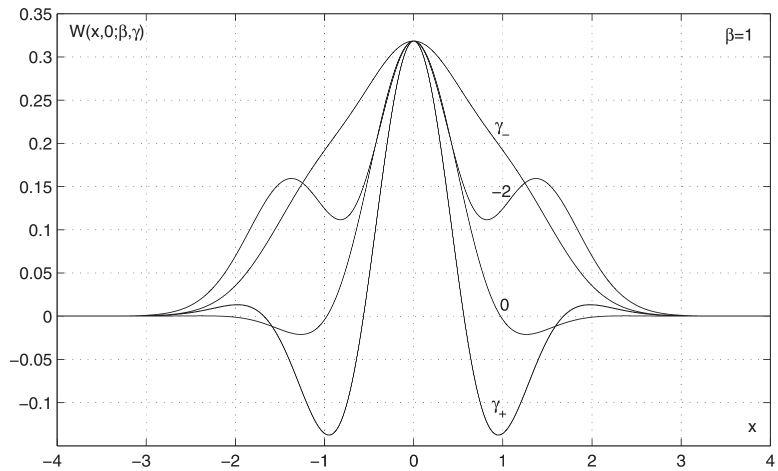

FIG. 1. The Wigner functions $W(x,0;\beta,\gamma)$ are plotted versus the coordinate x (field amplitude) for $\beta=1$. The curves are indexed with the value of $\gamma=\gamma_+,\ 0,\ -2,\gamma_-$.

$\beta=1$ are given in Fig. 1. Other curves for higher values of β are displayed in Fig. 2.

In order to explain the behavior of $W(x,p;\beta,\gamma)$, some useful closed expressions can be obtained when the light intensity is low ($\beta^2 \lesssim 1$). Thus

$$W(x,0;\beta,\gamma) \simeq \frac{e^{-x^2}}{\pi}\left(1-\frac{x^2}{1+\frac{\gamma}{2}}\beta^2\right) \tag{2.14}$$

for arbitrary constant $\gamma > -2$. It clearly shows a negative part within the domains

$$-\infty \le x \le -\frac{1}{\beta}\sqrt{1+\frac{\gamma}{2}}, \quad \frac{1}{\beta}\sqrt{1+\frac{\gamma}{2}} \le x \le \infty, \tag{2.15}$$

illustrating a nonclassical behavior even for $\gamma \neq 0$. However, for the special values of $\gamma_\pm$, we have

$$W(x,0;\beta,\gamma_\pm) \simeq \frac{e^{-x^2}}{\pi}[1 \mp 2x^2\beta + (x^4-2x^2)\beta^2], \tag{2.16}$$

which clearly differs from Eq. (2.14). In particular, $W(x,0;\beta,\gamma_-)$ is ≥ 0, as opposed to $W(x,0;\beta,\gamma_+)$ which is ≤ 0, within the domains

$$-\frac{1}{\sqrt{\beta}}-\frac{\sqrt{2}}{2} \le x \le -\frac{1}{\sqrt{\beta}}+\frac{\sqrt{2}}{2}, \tag{2.17a}$$

$$\frac{1}{\sqrt{\beta}}-\frac{\sqrt{2}}{2} \le x \le \frac{1}{\sqrt{\beta}}+\frac{\sqrt{2}}{2}. \tag{2.17b}$$

After having examined the conditions for nonclassicality directly from the quasiprobability distribution in phase space, we wish to show that more can be derived from the variance of the measured coordinate operator. This is deduced from the photon field distribution.

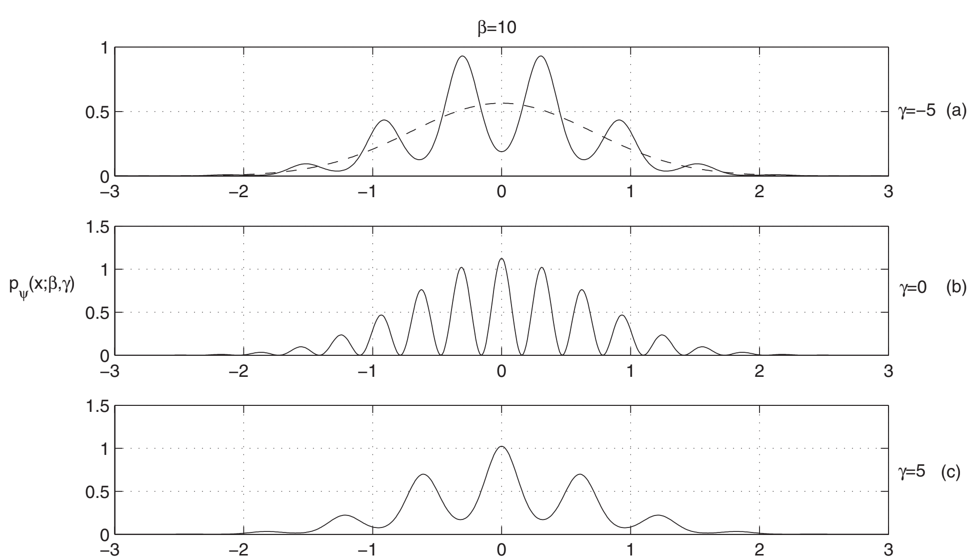

FIG. 2. The PDFs $p_\psi(x;\beta,\gamma)$ are plotted versus the coordinate x (field amplitude) for $\beta=10$ and $\gamma=-5,\ 0,\ 5$. The Gaussian PDF $p_\psi(x;0,0)$ is plotted as a dashed line in the upper curve.

C. Photon field distribution

Basically, homodyne detection is the measurement of the operator $\hat{X}$ whose continuous outcome x can be seen as the random variable (RV) of the light field [3]. The statistical properties of $\hat{X}$ can be summarized with the study of $p_\psi(x;\beta,\gamma)$, the PDF of the field, which is necessarily positive. In order to find the PDF of the field, we must integrate the Wigner functions with respect to the momentum p. In fact,

$$\begin{aligned}\bar{W}(x;\beta,\gamma) &= \int_{-\infty}^{\infty} W(x,p;\beta,\gamma)dp \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-2ip\xi}dp\right)\langle x+\xi|\hat{\rho}_\psi|x-\xi\rangle d\xi \\ &= \int_{-\infty}^{\infty}\delta(\xi)\langle x+\xi|\hat{\rho}_\psi|x-\xi\rangle d\xi \\ &= \langle x|\hat{\rho}_\psi|x\rangle = |\langle x|\psi\rangle|^2 = p_\psi(x;\beta,\gamma), \qquad (2.18)\end{aligned}$$

where $\delta(\xi)$ is the Dirac function. From Eq. (2.18), we have

$$p_\psi(x;\beta,\gamma) = p_\alpha(x)\frac{1+\cos(2\beta x)+2\gamma\cos(\beta x)+\frac{\gamma^2}{2}}{1+e^{-\beta^2}+2\gamma e^{-\beta^2/4}+\frac{\gamma^2}{2}}, \qquad (2.19a)$$

$$p_\alpha(x) = \frac{e^{-x^2}}{\sqrt{\pi}}. \qquad (2.19b)$$

A more general expression is given in the Appendix.

We obtain from Eq. (2.19a), $p_\psi(x;\beta,0)=0$ for $x_\ell=\pm\frac{(\ell+1)\pi}{2\beta},\beta\neq 0$. It shows a nonclassicality within the domains that include those that have been previously determined [Eqs. (2.15)–(2.17)].

The moments of x can then easily be derived. Notice, first, that due to the parity of Eq. (2.19a), we have for $\ell=0,1,\ldots$

$$\langle x^{2\ell+1}\rangle = 0. \qquad (2.20)$$

The first two moments that are $\neq 0$ are then easily obtained,

$$\langle x^2\rangle = \frac{1}{2}\frac{1+\frac{\gamma^2}{2}+(1-2\beta^2)e^{-\beta^2}+\gamma(2-\beta^2)e^{-\beta^2/4}}{1+e^{-\beta^2}+2\gamma e^{-\beta^2/4}+\frac{\gamma^2}{2}}, \qquad (2.21a)$$

$$\langle x^4\rangle = \frac{3}{4}\frac{1+\frac{\gamma^2}{2}+\left(1-4\beta^2+\frac{4}{3}\beta^4\right)e^{-\beta^2}+2\gamma\left(1-\beta^2+\frac{1}{12}\beta^4\right)e^{-\beta^2/4}}{1+e^{-\beta^2}+2\gamma e^{-\beta^2/4}+\frac{\gamma^2}{2}}, \qquad (2.21b)$$

yielding the variance $\sigma_X^2(\beta,\gamma)=\langle x^2\rangle$. Note that $\sigma_X^2(0,0)=\frac{1}{2}$ corresponding to the fluctuations of the vacuum state characterizes a lower limit of the classical field variance. Note also that $\sigma_X^2(\beta,\gamma_c)=\frac{1}{2}$ for

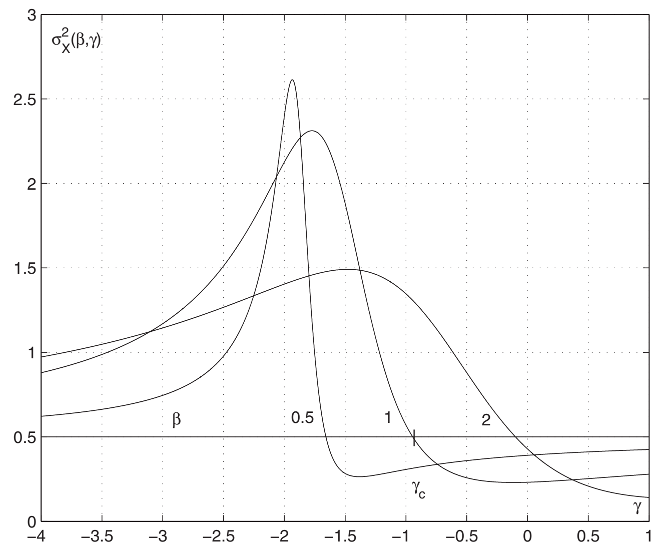

FIG. 3. Plots of $\sigma_X^2(\beta,\gamma)$, the variance of the coordinate X (field) versus γ the coefficient of the vacuum state added to the superposition of two coherent states. The curves are indexed with the value of β: 0.5, 1, 2. The parameter β is such that $\beta^2=2\alpha^2$, where α^2 is the power of the coherent state. The abscissa γ_c is given by $\gamma_c=-2e^{-3\beta^2/4}$.

$$\gamma_c = -2e^{-3/4\beta^2}. \qquad (2.22)$$

Because of the term in e^{-x^2}, the domain of interest of Eq. (2.19a) can be restricted to small values of x, say $x\in[-4,4], \forall\beta,\gamma$. Up to β^4 and $\gamma\neq -2$, a useful expression for the variance can be obtained,

$$\sigma_X^2(\beta,\gamma) \simeq \frac{1}{2}[a_0+a_2(\gamma)\beta^2+a_4(\gamma)\beta^4], \qquad (2.23)$$

where $a_0=1$, $a_2(\gamma)=-\frac{1}{1+\frac{\gamma}{2}}$, and $a_4(\gamma)=\frac{1+\frac{\gamma}{4}}{2(1+\frac{\gamma}{2})^2}$. Furthermore, for the special values of γ, it can be shown

$$\sigma_X^2(\beta,\gamma) \simeq \begin{cases}\frac{1}{2}\left(1-\beta^2+\frac{1}{2}\beta^4\right), & \gamma=0,\\ \frac{1}{2}\left(1-\beta+\frac{1}{2}\beta^2-\frac{1}{16}\beta^3\right), & \gamma=\gamma_+,\\ \frac{1}{2}\left(1+\beta+\frac{1}{8}\beta^3\right), & \gamma=\gamma_-.\end{cases} \qquad (2.24)$$

When $\gamma=\gamma_+$, the field variance starts by decreasing as β increases. When $\gamma=\gamma_-$, the field variance starts increasing with β as shown in Figs. 3 and 4.

Although the above expressions are valid only for low values of β, they appear good enough to approximate β^* the positive abscissa of the extrema of σ_X^2. We easily obtain $\beta^*=0$ and 1 for $\gamma=0$. For $\gamma=\gamma_+$, we get $\beta^*=\frac{4\sqrt{7}-8}{3}\simeq 0.86$ which is reasonably close to the numerical value $\frac{3}{4}$. The variance is continuously increasing in the domain of approximation for $\gamma=\gamma_-$. More comments are given in Sec. V. We now want to

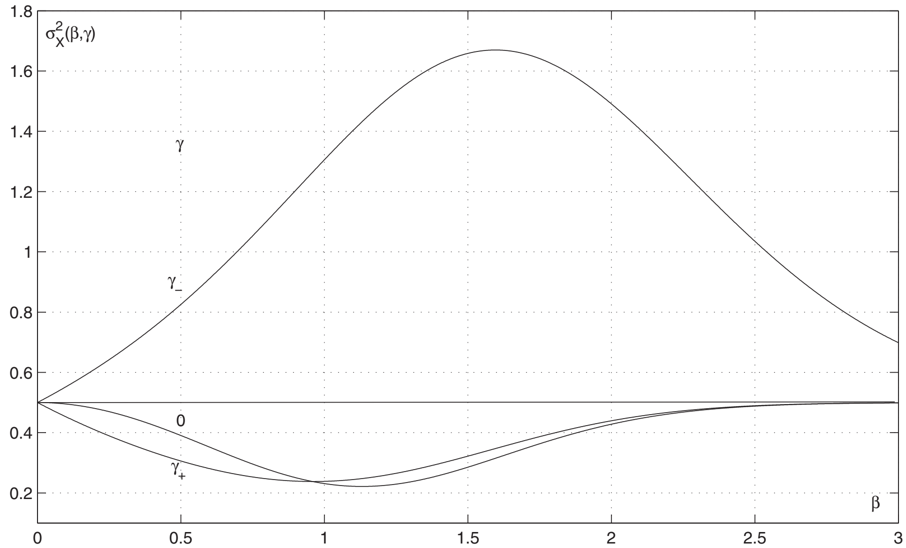

FIG. 4. Plots of $\sigma_X^2(\beta,\gamma)$, the variance of the coordinate X (field) versus $\beta^2=2\alpha^2$, α^2 being the power of the coherent state. The curves are indexed with the value of γ: γ_-, 0, and γ_+.

complete these results about the characterization of the nonclassicality by studying the properties of the light intensity.

D. Photon intensity distribution

As in the treatment of the classical optical coherence of electromagnetic fields, it is convenient to introduce the concept of intensity. This is defined, at position $\vec{r}$ and time t, as

$$J(\vec{r},t) = V^*(\vec{r},t)V(\vec{r},t), \tag{2.25}$$

where $V(\vec{r},t)$ is the analytic signal associated with the instantaneous real field $V^{(r)}(\vec{r},t)$ via the relation $V^{(r)}(\vec{r},t)=2\,\mathrm{Re}[V(\vec{r},t)]$ [7]. The quantum-mechanical construction in terms of operators closely follows this description.

We consider a field component of the radiation of a given polarization, with wavelength λ, confined in a volume Ω such that $\Omega \gg \lambda^3$. The intensity operator, up to a scalar χ, is given by

$$\hat{J}(\vec{r},t) = \chi \hat{a}^\dagger \hat{a}. \tag{2.26}$$

This operator, however, cannot be seen as an operator that localizes photons precisely. It is associated with the photon number operator $\hat{N}$ such that

$$\int_\Omega \hat{J}(\vec{r},t) d^3\vec{r} = \hat{N}. \tag{2.27}$$

Here, we establish some results concerning the properties of the intensity $J=|E|^2$.

For a quasimonochromatic field $\vec{E}$, the components which can be measured are the RVs $E_x=|E|\cos\vartheta$ and $E_y=|E|\sin\vartheta$. They are assumed to be independent and to have the same distribution (2.19a). The distribution of the random phase ϑ is supposed uniform over $\vartheta\in[0,2\pi]$, $p(\vartheta)d\vartheta=\frac{d\vartheta}{2\pi}$, so that the PDF of J is given by

$$\begin{aligned} P_\psi(J;\beta,\gamma) = K\exp(-J)\int_0^{2\pi} & \Bigg[1+\cos(2\beta\sqrt{J}\cos\vartheta) \\ & + 2\gamma\cos(\beta\sqrt{J}\cos\vartheta) + \frac{\gamma^2}{2}\Bigg] \\ & \times\Bigg[1+\cos(2\beta\sqrt{J}\sin\vartheta) \\ & + 2\gamma\cos(\beta\sqrt{J}\sin\vartheta) + \frac{\gamma^2}{2}\Bigg]d\vartheta, \end{aligned} \tag{2.28}$$

where K is a constant of normalization such that $\int_0^\infty P_\psi(J)dJ=1$.

In the general case, the PDF of the intensity can be calculated only numerically. However, we can prove the following expression for terms up to β^4:

$$P_\psi(J;\beta,\gamma) \simeq [a_0(\beta,\gamma)+a_1(\beta,\gamma)J+a_2(\beta,\gamma)J^2]\exp(-J), \tag{2.29a}$$

$$a_0(\beta,\gamma) = 1+\frac{2}{\gamma+2}\beta^2+\frac{4-\gamma}{4(\gamma+2)^2}\beta^4,$$

$$a_1(\beta,\gamma) = -\frac{2}{\gamma+2}\beta^2-\frac{4}{(\gamma+2)^2}\beta^4,$$

$$a_2(\beta,\gamma) = \frac{\gamma+12}{(\gamma+2)^2}\beta^4. \tag{2.29b}$$

From this approximation, it can be seen that there must exist several values of intensity $J^*\neq 0$ leading to $P_\psi(J^*;\beta,\gamma)\to 0$. This is, as has been mentioned above, a necessary condition for nonclassicality. An approximation of J^* is easily evaluated from Eqs. (2.29a) and (2.29b) to be

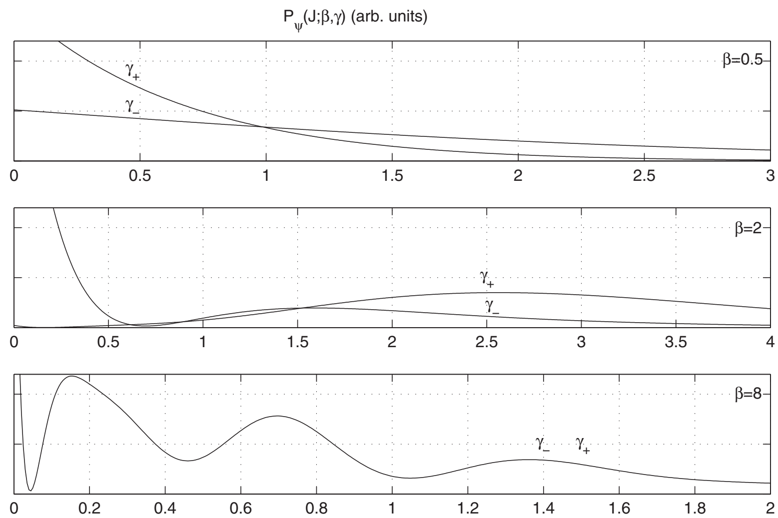

FIG. 5. Plots of the PDF $P_\psi(J;\beta,\gamma)$ versus J the intensity, for various values of β and for the two special values of γ: γ_+ and γ_-. Remark that for high values of β, $P_\psi(J;\beta,\gamma)=P_\psi(J;\beta,0)$.

$$J^*(\gamma) \simeq 1 + \frac{2+\gamma}{2\beta^2}. \tag{2.30}$$

The agreement of the estimated values with the computed values is rather poor as can be seen in Fig. 5. This approximation is, however, useful since it shows that $J^*(\gamma_+) > J^*(\gamma_-)$ for a given $\beta \neq 0$.

We can now derive some approximate results which are in good agreement with the exact results. Thus we have shown

$$\sigma_J^2(\beta,\gamma) \simeq \begin{cases} 1 - 2\beta^2 + \dfrac{3}{2}\beta^4, & \gamma = 0, \\ 1 - 2\beta + \dfrac{5}{4}\beta^2 + \dfrac{1}{4}\beta^3, & \gamma = \gamma_+, \\ 1 + 2\beta + \dfrac{5}{4}\beta^2 + \dfrac{9}{4}\beta^3, & \gamma = \gamma_-. \end{cases} \tag{2.31}$$

As for σ_X^2, the variance of the intensity starts decreasing with respect to β for $\gamma=0$ and γ_+, whereas it starts increasing for γ_-.

Here again, although the previous expressions are valid only for low values of β, they appear good enough to approximate β^*, the positive abscissa of the extrema of σ_J^2. We easily obtain $\beta^*=0$ and $\frac{\sqrt{6}}{3}$ for $\gamma=0$. For $\gamma=\gamma_+$ the minimum occurs at $\beta^*=\frac{2}{3}$. For $\gamma=\gamma_-$ the variance continuously increases in the domain of approximation.

Because of the relation between the operators $\hat{J}$ and $\hat{N}$, it is tempting to make a comparison of the corresponding variances (2.8) and (2.31). This is briefly discussed in Sec. V.

As seen, several functions can be defined to measure the difference between classical and nonclassical behaviors. We want to introduce the distance variation difference, a criterion that was recently shown to be efficient.

III. DISTANCE VARIATION DIFFERENCE

The distance variation difference (DVD), D, was introduced for other purposes and for a discrete variable in [8]. The DVD is a measure of the difference between classical and quantum states. We recently applied it to the measurement of the photon number operator $\hat{N}$ and concluded that $D<0$ is the condition for nonclassicality [9].

Because the DVD is bounded by the Fisher entropy, it is applicable for a continuous variable such as the photon field x, as well. Therefore we will set

$$D_\psi(\beta,\gamma) = \int_{-\infty}^{\infty} \sqrt{p_\psi(x;\beta,\gamma)}dx - \int_{-\infty}^{\infty} \sqrt{p_\alpha(x;\beta)}dx, \tag{3.1}$$

where $p_\psi(x;\beta,\gamma)$ and $p_\alpha(x;\beta)$ are the PDFs of the field X measured in the state $|\psi\rangle$ and the coherent state $|\alpha\rangle$, respectively. We have $p_\alpha(x;\beta)=\frac{1}{\sqrt{\pi}}e^{-x^2}$.

Although Eq. (3.1) obeys two properties of a distance function, it is not a distance because it can be negative. Only a few exact results of Eq. (3.1) seem simple to establish. For instance, we can show $\forall\gamma$,

$$D_\psi(0,\gamma) = 0, \tag{3.2a}$$

$$D_\psi(\beta\to\infty,\gamma) \to 0. \tag{3.2b}$$

However, approximate expressions for Eq. (3.1) when $\beta^2 \lesssim \frac{1}{2}$ are of interest. Thus $p_\psi(x;\beta,\gamma)$ can be approximated by Eq. (2.19a) which is then inserted into Eq. (3.1). So, for the special values of γ, we obtain

$$D_\psi(\beta,0) \simeq -\frac{\sqrt{\pi}}{2}\beta^2, \tag{3.3a}$$

$$D_\psi(\beta,\gamma_\pm) \simeq \mp(2\sqrt{\pi})^{1/2}\frac{\beta}{\beta+4}, \tag{3.3b}$$

showing a nonclassicality ($D<0$) only for $\gamma=0$ and for γ_+ in accordance with the previous results.

These approximate expressions (3.3a) and (3.3b) of the DVD functions are good enough when compared with the exact ones which are displayed in Fig. 6. Recall that we are considering only the pure imaginary case $\alpha=i\alpha_0$.

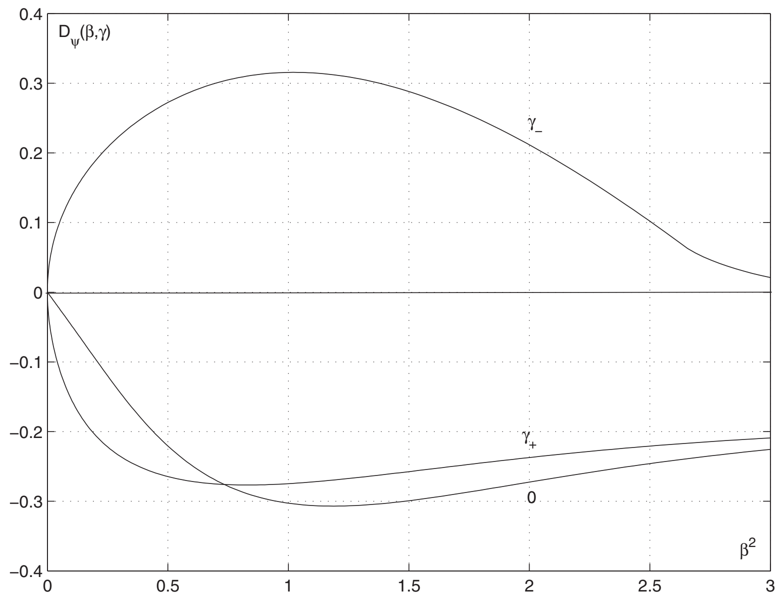

FIG. 6. Plots of the DVD functions of the states analyzed in this paper versus $\beta^2 = 2\alpha^2$, α^2 being the power of the coherent state, for various values of $\gamma=0$, γ_+, and γ_-.

So far, such a source of superposed coherent states has been analyzed from the nonclassical statistics point of view. We now want to see how it can be used to transmit information and see how the D function is involved.

IV. HOMODYNE PROCESSING

As an application of the previous results, let us consider a binary system of communication where the source delivers the light state (2.1a). Here the channel is noiseless and the receiver is a homodyne one. The channel is symmetric and the transition probabilities $P(\text{output}|\text{input})$ are such that [1]

$$P(0|0) = p_\alpha(x), \quad P(1|1) = p_\psi(x;\beta,\gamma),$$

$$P(0|1) = P(1|0) = P_{er}, \tag{4.1}$$

where $p_\psi(x;\beta,\gamma), p_\alpha(x)$ are given by Eqs. (2.19a) and (2.19b), respectively, and P_{er} is the probability of error. It is given by

$$P_{er} = \tfrac{1}{2}(1 + P_f - P_d), \tag{4.2}$$

where the P_f and P_d are the false-alarm and detection probabilities that result from a decision threshold μ [3]. This threshold μ, calculated such that $p_\psi(x;\beta,\gamma) > p_\psi(x;0)$, determines, in general, two domains of decision $x \gtrless \mu$, such that

$$P_f = \int_\mu^\infty p_\psi(x;0)dx, \quad P_d = \int_\mu^\infty p_\psi(x;\beta,\gamma)dx. \tag{4.3}$$

The classical channel capacity is

$$C_X(\beta,\gamma) = \log 2 + P_{er}\log P_{er} + (1-P_{er})\log(1-P_{er}). \tag{4.4}$$

Here, the domain of the decision is the union of several disjoint intervals determined by the intersection of $p_\psi(x;\beta,\gamma)$ and $p_\psi(x;0)$ as can be seen in Fig. 2 [curves (a)]. The results of the exact calculation of Eq. (4.4) are plotted in Fig. 7.

Following our method [1], some approximations valid for $\beta \lesssim 1$ will be useful to explain the numerical results. These approximations are based on setting the threshold at

$$\mu_\gamma^\pm \simeq \pm\frac{\sqrt{2}}{2}[1 - \epsilon(\gamma)\beta^2] + o(\beta^3), \tag{4.5a}$$

$$\epsilon(\gamma) = \frac{1}{24}\frac{\gamma+8}{\gamma+2}, \quad \gamma \neq -2, \tag{4.5b}$$

yielding to calculate the probability of error P_{er} (4.2) and then inserting the probability of error into Eq. (4.4) to bound the capacity $C_X(\beta,\gamma)$. After some calculation done for $\beta^2 \lesssim \frac{1}{2}$, we get

$$P_{er}(\beta,\gamma) \lesssim \begin{cases} \dfrac{1}{2} - \dfrac{E_0}{8}\beta^2, & \gamma = 0, \\ \dfrac{1}{2} - \dfrac{E_0}{8}\beta, & \gamma = \gamma_+, \\ \dfrac{1}{2} + \dfrac{E_0}{2}\dfrac{\beta}{\beta-4}, & \gamma = \gamma_-. \end{cases} \tag{4.6}$$

Here we set $E_0 = \text{erf}(\frac{\sqrt{2}}{2}) \approx \frac{\sqrt{2}}{2}$, where $\text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$. Denoting $\Delta_0 = \frac{E_0^2}{32}$, we obtain with the help of $P_{er}(\beta,\gamma)$, a lower bound to $C_X(\beta,\gamma)$,

$$C_X(\beta,\gamma) \gtrsim \begin{cases} \Delta_0\beta^4 + o(\beta^6), & \gamma = 0, \\ \Delta_0\beta^2 + o(\beta^4), & \gamma = \gamma_+, \\ \Delta_0\beta^2 + \dfrac{\Delta_0}{2}\beta^3 + o(\beta^4), & \gamma = \gamma_-. \end{cases} \tag{4.7}$$

An upper bound to $C_X(\beta,\gamma)$ based on the maximum of the P_d is attained for $\gamma \simeq -2 + \beta^2$, yielding $P_d \simeq \frac{e_0 + f_0\beta^2}{4-\beta^2}$ where $e_0 = \text{erf}(\sqrt{2}) \approx 1$ and $f_0 = \frac{1}{24}\sqrt{\frac{2}{\pi}}$. Therefore

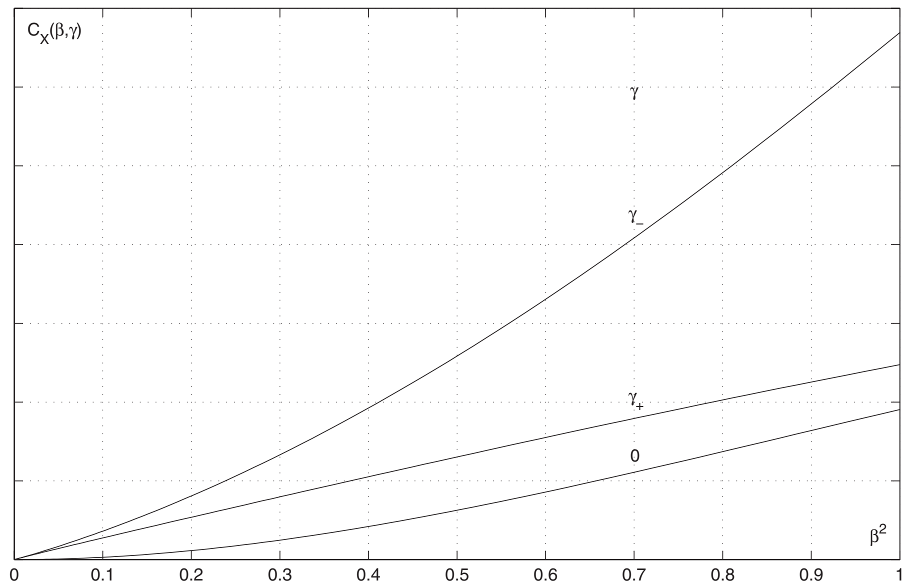

FIG. 7. Plots of channel capacity for a binary system of communication with a homodyne receiver and a source of nonclassical states versus $\beta^2=2\alpha^2$, α^2 being the power of the coherent state. The curves are indexed with the value of γ: 0, γ_+, and γ_-.

$$C_X(\beta,\gamma) \lesssim \frac{e_0}{16}\beta^2. \tag{4.8}$$

Combining Eqs. (4.7) and (4.8), we can prove that

$$C_X(\beta,0) \leq C_X(\beta,\gamma_+) \leq C_X(\beta,\gamma_-), \tag{4.9}$$

as can be seen in the results displayed in Fig. 7.

V. RESULTS AND DISCUSSION

The main results are summarized in Figs. 1–7. Some examples of the functions $W(x,0;\beta,\gamma)$ given by Eq. (2.13) for moderate power ($\beta=1$) and for different values of γ are displayed in Fig. 1. The main property of the nonclassicality, i.e., the negativity of these functions, is clearly shown for the values of γ_+ and $\gamma=0$.

In Fig. 2, the PDF $p_\psi(x;\beta,\gamma)$ of the light field is plotted for the high value of $\beta=10$ and three values of $\gamma=-5,\ 0,\ 5$. There are several values of $x=x_c=(2\ell+1)\frac{\pi}{2\beta},\ \ \ell=0,1,\ldots$ yielding $p_\psi(x;\beta,\gamma)=0$.

In Fig. 3, the variance of the light field $\sigma_X^2(\beta,\gamma)$ is plotted versus γ for various values of β^2. It is clearly seen that $\sigma_X^2 \geq \frac{1}{2},\ \ \forall \beta \neq 0$, provided that $\gamma \leq \gamma_c$ where γ_c is given by Eq. (2.22). Moreover and as expected, $\sigma_X^2 \to \frac{1}{2}$ when $\gamma \to \pm\infty$. Notice that, depending on γ, the field exhibits both classical and nonclassical properties. This is not the case when we plot σ_X^2 versus β considering γ as an independent parameter. In fact, in Fig. 4, we plot the same variance σ_X^2 versus β for the three specific values of γ. It is first seen that $\forall \beta,\ \sigma_X^2(\beta,\gamma_+) \leq \frac{1}{2}$, and $\sigma_X^2(\beta,\gamma_-) \geq \frac{1}{2}$. Furthermore, the approximation (2.24) leads to $\beta_m \simeq 1$ as the abscissa of the minimum which is close the numerical value.

In Fig. 5, the PDF of the intensity, given by Eq. (2.28), is plotted versus the intensity J for the two special values of γ and for various values of β. Notice that for $\beta \gg 1$, $P_\psi(J;\beta,\gamma)$ behaves like $[1+\cos(2\beta\sqrt{J})]e^{-J}$. The extrema can then be easily evaluated, $J_m=(\frac{\pi}{2\beta})^2$ and $J_M=(\frac{\pi}{\beta})^2$. We notice that $P_\psi(J;\beta,\gamma)$ vanishes for a few values of the intensity $J \neq 0$, whereas for its discrete counterpart $p_\psi(n;\beta,\gamma)$, there are several n_c yielding $p_\psi(n_c;\beta,\gamma)=0,\ \forall \beta$.

Furthermore, from Eqs. (2.8) and (2.31), we observe that $\sigma_N^2(\beta,\gamma_\pm)-\sigma_N^2(\beta,0)=\frac{1}{2}\beta^2+o(\beta^3)$ and $\sigma_J^2(\beta,\gamma_+)-\sigma_J^2(\beta,0)=-2\beta+o(\beta^2)$. It is interesting to compare these expressions with those corresponding to classical and nonclassical states.

For the first type, e.g., thermal light, we have $\sigma_N^2(\beta)=\frac{1}{2}\beta^2+\frac{1}{4}\beta^4$ and $\sigma_J^2(\beta)=\frac{1}{4}\beta^4$. For the second one, e.g., squeezed light having $\varrho \geq 0$ as a squeezing parameter, we have $\sigma_N^2(\beta,\varrho)=\frac{1}{2}\beta^2 e^{-2\varrho}+\frac{1}{2}\sinh^2 2\varrho$ and $\sigma_J^2(\beta,\varrho)=-\beta^2 e^{-\varrho}\sinh \varrho+\sinh^4 \varrho+\frac{1}{4}\sinh^2 2\varrho$ [10].

At low power, for all states, the photon counting fluctuations increase with the power. On the contrary, the nonclassical states intensity fluctuations decrease with the power. This fact that can make questionable the definition of the intensity is, however, corroborated by the variance of the field for which we get, using Eq. (2.24), $\sigma_X^2(\beta,\gamma_+)-\sigma_X^2(\beta,0)=-\frac{1}{2}\beta+o(\beta^2)$.

Figure 6 plots the DVD functions of the state $|\psi\rangle$ versus the power of the coherent state $|\alpha\rangle$, for various values of γ. The approximations given by Eqs.(3.3a) and (3.3b) are in a good agreement with the numerical results. We also remark that the highest DVD of the states analyzed in this paper occurs when $\gamma=\gamma_-$.

In Fig. 7, it is clearly shown that the classical channel capacity is significantly improved when the vacuum state is added, with a coefficient that depends on the input power, to the superposed pure coherent states. The highest channel capacity is attained when the nonlinear parameter is such that $\gamma=\gamma_-$. The lowest channel capacity, when $\beta^2 \gtrsim \frac{1}{2}$ is attained for $\gamma=0$ which then corresponds to the pure Schrödinger state.

Finally, we conclude that adding the vacuum state to the superposition of two coherent states generates a variety of classical and nonclassical states that can be used to improve the transmission of information over a noiseless binary symmetric channel. The variation of the classical channel capacity with respect to the input power is more appropriately described by the D function than by the variance. Moreover, in view of Figs. 6 and 7, the behavior of $C_X(\beta,\gamma)$ versus $D_\psi(\beta,\gamma)$ is nonlinear as for $\beta^2 \lesssim \frac{1}{2}$, we have Eq. (4.9) for $D_\psi(\beta,\gamma_+) \leq D_\psi(\beta,0) \leq D_\psi(\beta,\gamma_-)$.

ACKNOWLEDGMENTS

The Laboratoire des Signaux et Systèmes is a joint laboratory of C.N.R.S. and École Supérieure d'Électricité and associated with the Université Paris–Orsay, France.

APPENDIX: PHOTON FIELD DISTRIBUTION

Instead of Eq. (2.1a), we may consider the state

$$|\Psi\rangle = d(|\alpha e^{i\theta}\rangle + e^{-i\varphi}|-\alpha e^{i\theta}\rangle + \gamma|0\rangle), \tag{A1}$$

where d is chosen so that $\langle\Psi|\Psi\rangle = 1$. Thus

$$|d|^2 = \frac{1}{2}\frac{1}{1+e^{-\beta^2}\cos\varphi + \gamma e^{-\beta^2/4}(1+\cos\varphi)+\frac{\gamma^2}{2}}, \tag{A2}$$

where we denote $\beta = |\alpha|\sqrt{2}$. The PDF of the field is obtained as usual,

$$\begin{aligned} p_\Psi(x) &= |\langle x|\Psi\rangle|^2 \\ &= 2|d|^2 \frac{e^{-x^2}}{\sqrt{\pi}} \Bigg\{ e^{-\beta^2\cos^2\theta}[\cosh(2\beta x\cos\theta) \\ &\quad + \cos(2\beta x \sin\theta + \varphi)] \\ &\quad + \gamma e^{-\beta^2/2\cos^2\theta}[e^{-\beta x\cos\theta}\cos(\beta x\sin\theta+\varphi) \\ &\quad + e^{\beta x\cos\theta}\cos(\beta x\sin\theta)] + \frac{\gamma^2}{2}\Bigg\}. \end{aligned} \tag{A3}$$

We recall that Eq. (2.18) can be deduced from Eq. (A3) for $\varphi=0, \theta=\frac{\pi}{2}$.

Approximated closed expressions of the first two moments can easily be obtained for $\beta \ll 1$. They are derived from the PDF of x in the state (A1),

$$p_\Psi(x) \simeq K_1 \frac{e^{-x^2}}{\Delta\sqrt{\pi}}(\Delta - \Gamma\beta + \Lambda\beta^2), \tag{A4}$$

where $K_1 = 1 + o(\beta^2)$ and

$$\Gamma = \left[\sin\theta\sin\varphi + \gamma\sin\left(\theta - \frac{\varphi}{2}\right)\sin\frac{\varphi}{2}\right]x \equiv \Gamma_0 x, \tag{A5a}$$

$$\Delta = 1 + \cos\varphi + \gamma(1+\cos\varphi) + \frac{\gamma^2}{2}, \tag{A5b}$$

$$\Lambda = c_0 + c_1 x^2, \tag{A5c}$$

$$c_0 = \cos\varphi - (1+\cos\varphi)\left(\cos^2\theta + \frac{\gamma}{4}\cos(2\theta)\right),$$

$$c_1 = 2(1+\cos\varphi)\cos^2\theta + \frac{\gamma}{2}[\cos(2\theta) + \sin\varphi\sin(2\theta)]$$

leading to

$$\langle x\rangle = -\frac{\Gamma_0}{\Delta}\beta, \tag{A6a}$$

$$\langle x^2\rangle = \frac{1}{2} + \frac{c_0 + \frac{3}{2}c_1}{2\Delta}\beta^2, \tag{A6b}$$

which show a nonclassical behavior for $\theta = \frac{\pi}{2}$ and

$$\gamma < \frac{2\cos\varphi - (1+\cos\varphi)\cos^2\theta}{3\sin\varphi\sin(2\theta) + \frac{11}{4}(1+\cos\varphi)\cos(2\theta)}. \tag{A7}$$

Note that in the limit of $\beta \ll 1$, we have

$$\gamma_0 = -2\frac{\sin\theta\cos\frac{\varphi}{2}}{\sin\left(\theta - \frac{\varphi}{2}\right)} \to \langle x\rangle = 0, \tag{A8a}$$

$$\gamma_1 = -\frac{8}{3}\frac{(1+\cos\varphi)\cos^2\theta - \cos\varphi}{(1+\cos\varphi)\cos(2\theta) + \sin\varphi\sin(2\theta)} \to \langle x^2\rangle = \frac{1}{2}, \tag{A8b}$$

provided that $\theta - \frac{\varphi}{2} \neq \ell\pi, \ell$ is an integer.

Similarly, for $\beta \gg 1$, we have for a particular value of $\theta = \frac{\pi}{2}$,

$$p_\Psi(x) \simeq K_2 \frac{e^{-x^2}}{\sqrt{\pi}}\left(1 + \frac{\cos(\beta x + \varphi) + 2\gamma\cos\left(\frac{\beta x}{2}\right)\cos\left(\frac{\beta x}{2} + \varphi\right)}{1 + \frac{\gamma^2}{2}}\right), \tag{A9}$$

where $K_2 = 1 + \frac{\gamma\cos\varphi}{1+\frac{\gamma^2}{2}}\left[1 + e^{-\beta^2/4}\left(1 + \frac{1}{\gamma}\right)\right]$ is such that Eq. (A9) is normalized with respect to $x \in]-\infty, \infty[$. One has

$$\langle x\rangle \simeq -\frac{1}{2}\frac{(1+\gamma)\sin\varphi}{1 + \gamma\cos\varphi + \frac{\gamma^2}{2}}\beta e^{-\beta^2/4}, \tag{A10a}$$

$$\langle x^2\rangle \simeq \frac{1}{2} - \frac{1}{2}\frac{(1+\gamma)\cos\varphi}{1+\gamma\cos\varphi + \frac{\gamma^2}{2}}\left(1 - \frac{\beta^2}{2}\right)e^{-\beta^2/4}, \tag{A10b}$$

which characterizes a nonclassical state as long as $(1+\gamma)\cos\varphi < 0$, i.e., $\gamma < -1$ for the case studied here.

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