Spin squeezing and maximal-squeezing time

Guang-Ri Jin* and Sang Wook Kim†

Department of Physics Education and Department of Physics, Pusan National University, Busan 609-735, Korea

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Spin squeezing of a nonlinear interaction model with Josephson-like coupling is studied to obtain the time scale of maximal squeezing. Based upon two exactly solvable cases for two and three particles, we find that the maximal-squeezing time depends on the level spacing between the ground state and its next-neighbor eigenstate.

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I. INTRODUCTION

The phenomenon of spin squeezing in collective spin systems has attracted much attention for decades not only because of fundamental physical interests [1–9], but also for its possible applications in atomic clocks for reducing quantum noise [2–5] and quantum information [10–14]. The occurrence of spin squeezing is due to quantum correlations among individual spins, which requires at least two spins and a nonlinear interaction between them. Kitagawa and Uea have studied the spin squeezing generated by the so-called one-axis twisting (OAT) model with Hamiltonian $\hat{H}_{\mathrm{OAT}}=2\kappa\hat{J}_z^2$ [1]. Possible realization of the OAT-type squeezing in a two-component Bose-Einstein condensate (TBEC) [10,15] and atomic ensemble system in a dispersive regime [16] has been investigated recently. Sørensen *et al.* also proposed that spin squeezing can be used as a measure of many-particle quantum entanglement [10].

So far, OAT-type spin squeezing was mainly studied within the Heisenberg picture. As a result, an explicit expression of the spin-squeezed state (SSS) is unknown. Moreover, the direction that spin squeezing is observed varies with time [11]. Jaksch *et al.* have shown that an OAT-type SSS can be stored for arbitrarily long time by removing the self-interaction [17]. However, it might not be easy to handle in experiment since precisely designed additional pulses are crucially required. In Refs. [18,19], the authors proposed the constant-coupling scheme by introducing additional Josephson-like coupling $\Omega\hat{J}_x$ to the OAT model. It was shown that the Josephson interaction results in an enhancement of spin squeezing compared with that of the OAT. Moreover, the strongest squeezing appears in the z direction [18], which, however, is valid only at the maximal-squeezing time (MST). Although some formulas of the MST for extremely small [1] or large coupling [18–22] are already known, it is challenging to determine the MST within an intermediate coupling $1<\Omega/\kappa\ll N$ (where N is the total particle number).

In this paper, we reconsider the constant-coupling scheme [18,19] with the purpose to determine the MST. We find all analytic solutions for two- and three-particle cases. Motivated by exactly solvable cases, we show that the MST depends on the level spacing between the ground state and its next-neighbor eigenstate. We explain it by investigating the spectral distribution of the spin state and find that only the two lowest available levels are predominantly occupied. Our paper is organized as follows. In Sec. II, we introduce theoretical model and derive some basic formulas. To proceed, in Sec. III, we give some analytic expressions for the cases of $N=2$ and $N=3$. In Sec. IV, we study spin squeezing for many-particle cases and present an exact-diagonalization method to obtain the MST. Moreover, we compare our result with its analytic solution. Finally, a summary of our paper is presented.

II. THEORETICAL MODEL

Formally, a two-level atom can be regarded as a fictitious spin-1/2 particle with spin operators $s_z^{(i)}=(|b\rangle_{ii}\langle b|-|a\rangle_{ii}\langle a|)/2$ and $s_+^{(i)}=(s_-^{(i)})^\dagger=|b\rangle_{ii}\langle a|$, where $|a\rangle_i$ and $|b\rangle_i$ are the internal states of the ith atom. We consider an ensemble of N atoms with its dynamics described by a collective spin operator: $\hat{J}=\Sigma_{i=1}^N s^{(i)}$. Spin squeezing is quantified by a parameter [1]:

$$\xi=\frac{\sqrt{2}(\Delta\hat{J}_{\mathbf{n}})_{\min}}{j^{1/2}}, \tag{1}$$

where $j=N/2$ and $(\Delta\hat{J}_{\mathbf{n}})_{\min}$ represents the smallest variance of a spin component $\hat{J}_{\mathbf{n}}=\hat{\mathbf{J}}\cdot\mathbf{n}$ normal to the mean spin $\langle\hat{J}\rangle$. For a coherent spin state (CSS), the variance $(\Delta\hat{J}_{\mathbf{n}})_{\min}=\sqrt{j/2}$ and $\xi=1$. In general, a spin state is called spin squeezed state if the variance of the spin component, $\hat{J}_{\mathbf{n}}$, is smaller than that of the CSS—i.e., $\xi<1$.

Follow Refs. [23–30], we consider a nonlinear spin system governed by

$$\hat{H}=\Omega\hat{J}_x+2\kappa\hat{J}_z^2, \tag{2}$$

which can be realized in the TBEC [31,32]. The first term is Josephson-like coupling induced by a microwave (radio frequency) field. The Rabi frequency Ω can be controlled by the strength of the external field. The second term is the self-interaction aroused from nonlinear collisions between atoms. An initial coherent spin state $|j,-j\rangle_x=e^{-i\pi J_y/2}|j,-j\rangle$ will be considered in this paper. Physically, the Dicke state $|j,-j\rangle$

*Present address: Department of Physics, School of Science, Beijing Jiaotong University, Beijing 100044, China.

†swkim0412@pusan.ac.kr

represents all atoms occupying in the internal ground state $|a\rangle$. By applying a short $\pi/2$ pulse to the Dicke state, one can obtain the CSS with each spin aligned along the negative x direction [10]. After that, one switches on the Josephson-like coupling Ω immediately; then, the dynamics of the spin system is governed by the Hamiltonian (2). Note that we will consider only the positive κ case. However, our results remain valid in the opposite case by using the initial maximum-weight state of $\hat{J}_x$—i.e., $|j,j\rangle_x$.

The state vector at any time t can be expanded in terms of eigenstates of $\hat{J}_z$: $|\psi(t)\rangle=\Sigma_m c_m(t)|j,m\rangle$, where $-j\le m\le j$. The probability amplitudes $c_m(t)$ can be solved by the time-dependent Schrödinger equation, obeying

$$i\dot{c}_m = E_m c_m + X_m c_{m-1} + X_{-m} c_{m+1}, \tag{3}$$

where $E_m=2\kappa m^2$ and $X_m=\frac{\Omega}{2}\sqrt{(j+m)(j-m+1)}$ with $X_{-j}=0$ and $X_{\pm m}=X_{\mp m+1}$. The probability amplitudes of the initial CSS,

$$c_m(0) = \frac{(-1)^{j+m}}{2^j}\sqrt{\frac{(2j)!}{(j-m)!\,(j+m)!}}, \tag{4}$$

satisfy $c_{-m}(0)=c_m(0)$ for even N and $c_{-m}(0)=-c_m(0)$ for odd N. Due to the symmetry properties of the elements $X_{\pm m}$ and the initial amplitudes $c_m(0)$, we obtain the simple expressions $c_{-m}(t)=\pm c_m(t)$, which in turn result in $\langle\hat{J}_y\rangle=\langle\hat{J}_z\rangle=0$ and $\langle\hat{J}_x\rangle\neq 0$; i.e., the mean spin $\langle\hat{J}\rangle$ is always along the x axis. The spin component normal to the mean spin is $\hat{J}_{\mathbf{n}}=\hat{J}_y\sin\theta+\hat{J}_z\cos\theta$, and its variance is $(\Delta\hat{J}_{\mathbf{n}})^2=\langle\hat{J}_{\mathbf{n}}^2\rangle-\langle\hat{J}_{\mathbf{n}}\rangle^2\equiv\frac{1}{2}C+\frac{\cos 2\theta}{2}A+\frac{\sin 2\theta}{2}B$, where $A=\langle\hat{J}_z^2-\hat{J}_y^2\rangle$, $B=\langle\hat{J}_z\hat{J}_y+\hat{J}_y\hat{J}_z\rangle$, and $C=\langle\hat{J}_z^2+\hat{J}_y^2\rangle$. By minimizing the variance $(\Delta\hat{J}_{\mathbf{n}})^2$ with respect to θ, we get the squeezing angle

$$\theta_{\min} = \frac{1}{2}\tan^{-1}(B/A) \tag{5}$$

and the smallest variance

$$(\Delta\hat{J}_{\mathbf{n}})^2_{\min} = \frac{1}{2}C - \frac{1}{2}\sqrt{A^2+B^2}, \tag{6}$$

from which one also obtains the squeezing parameter, Eq. (1). We consider spin squeezing in the intermediate-coupling regime—namely, $1<\Omega/\kappa\ll N$—where no analytic solutions are available for the nonlinear spin system [18,33]. However, we can exactly solve two- and three-particle cases. Some of the important physics can be extended to many-particle cases.

III. EXACT SOLVABLE CASES

In this section, we study spin squeezing based on two exactly solvable cases with $N=2$ and $N=3$. Though simple, it is of general interest to investigate the relationship between spin squeezing and quantum entanglement [10–13,34–37]. Such a relationship for two-particle (two-qubit) [3,4,35,36] and three-particle [5,37] cases has been studied recently. Here, we focus on the dynamical behavior of the spin system to show the conditions of the optimal squeezing and its time scale.

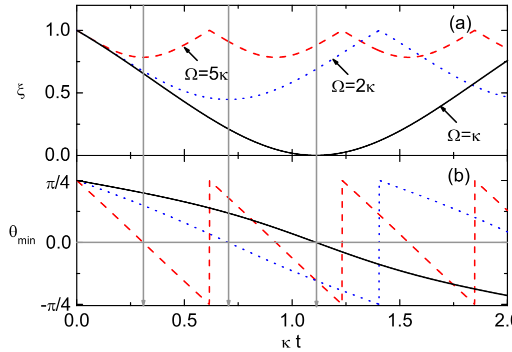

FIG. 1. (Color online) Time evolution of (a) the squeezing parameter and (b) the squeezing angle for $N=2$ and various Rabi frequencies: $\Omega=5\kappa$ (dashed red lines), $\Omega=2\kappa$ (dotted blue lines), and $\Omega=\kappa$ (solid black lines). The maximal-squeezing times t_0 for different Ω are indicated by the vertical lines.

A. Two-particle case

For the simplest case $N=2$ $(j=1)$, only three spin projections $(m=-1,0,+1)$ are involved. From Eq. (3), we obtain

$$i\begin{pmatrix}\dot{p}_0^{(+)}\\ \dot{p}_1^{(+)}\end{pmatrix} = \begin{pmatrix}E_0 & 2X_1\\ X_1 & E_1\end{pmatrix}\begin{pmatrix}p_0^{(+)}\\ p_0^{(+)}\end{pmatrix}, \tag{7}$$

where E_m and X_m are defined in Eq. (3) and we have introduced the linear combinations of the probability amplitudes $p_1^{(+)}(t)=c_1(t)+c_{-1}(t)$ and $p_0^{(+)}(t)=2c_0(t)$, with the initial conditions $p_1^{(+)}(0)=1$ and $p_0^{(+)}(0)=-\sqrt{2}$. Similarly, we also introduce $p_1^{(-)}(t)=c_1(t)-c_{-1}(t)$. However, its solution $p_1^{(-)}(t)=e^{-i2\kappa t}p_1^{(-)}(0)\equiv 0$ due to $p_1^{(-)}(0)=0$. Therefore, we obtain $c_1(t)=c_{-1}(t)\equiv p_1^{(+)}/2$, which gives $\langle\hat{J}_z\rangle=|c_{+1}|^2-|c_{-1}|^2\equiv 0$ and $\langle\hat{J}_+\rangle=\sqrt{2}(c_{-1}c_0^*+c_0c_1^*)=2\sqrt{2}\,\mathrm{Re}(c_0c_1^*)$. Since $\langle\hat{J}_+\rangle$ is a real function, $\langle\hat{J}_y\rangle=0$ and $\langle\hat{J}_x\rangle\neq 0$, which show that the mean spin is always along the x direction. Such a result is valid for arbitrary even N. Equation (7) can be solved exactly; then, one obtains immediately the reduced variance

$$(\Delta\hat{J}_{\mathbf{n}})^2_{\min} = \frac{1}{2} - \frac{\kappa}{S}|\sin St|\sqrt{1-\frac{\kappa^2}{S^2}\sin^2 St} \tag{8}$$

and the squeezing angle

$$\theta_{\min} = \frac{1}{2}\tan^{-1}\left[\frac{S\cos(St)}{\Omega\sin(St)}\right], \tag{9}$$

where $2S=\mathcal{E}_3-\mathcal{E}_1=2\sqrt{\Omega^2+\kappa^2}$ is the level spacing between the second excited state $\mathcal{E}_3$ and the ground state $\mathcal{E}_1$, obtained by solving the eigenvalues of the coefficient matrix of Eq. (7). As shown in Fig. 1(a), we find that at the times $t_k^*=k\pi/S$, ξ revives periodically to its initial value 1. In fact, apart from a globe phase, the states at t_k^*, $|\psi(t_k^*)\rangle=(-1)^k e^{-i\kappa t_k^*}|1,-1\rangle_x$, are just the initial CSS.

From Eq. (9), we find that the vanishing $\theta_{\min}$ occurs at $t_k=(k+1/2)\pi/S$ and the state vector at t_k reads

$$|\psi(t_k)\rangle = (-1)^k i e^{-i\kappa t_k}\{\sin(\eta)|1,-1\rangle_x - \cos(\eta)|1,+1\rangle_x\}, \tag{10}$$

which corresponds to a superposition of two coherent spin states $|1,-1\rangle_x$ and $|1,+1\rangle_x$ with mixing angle $\eta = \tan^{-1}(\Omega/\kappa)$. Obviously, if the coupling is very strong ($\Omega \gg \kappa$), $\sin(\eta)=\Omega/S \to 1$ and $\cos(\eta)=\kappa/S \to 0$, so $|\psi(t_k)\rangle \to |1,-1\rangle_x$, which in turn leads to a very weak squeezing at t_k. On the other hand, if the coupling is very weak ($\Omega \ll \kappa$), $|\psi(t_k)\rangle \to |1,1\rangle_x$, which also results in a weak squeezing at t_k. Therefore, we will study spin squeezing within the intermediate-coupling regime.

In Fig. 1, the time evolutions of ξ and $\theta_{\min}$ are investigated for the coupling $\Omega \geq \kappa$. We observe that local minima of ξ together with $\theta_{\min}=0$ also occur periodically at times t_k. Moreover, with a decrease of Ω, the squeezing parameter at t_k becomes small—i.e., more squeezed. For the coupling $\Omega=\kappa$, the spin system is optimally squeezed at t_k, as shown by the solid black lines in Fig. 1. In this case $\sin(\eta)=\cos(\eta)=1/\sqrt{2}$ and the spin states at t_k are

$$\begin{aligned}|\psi(t_k)\rangle &= (-1)^k \frac{i e^{-i\kappa t_k}}{\sqrt{2}}\{|1,-1\rangle_x - |1,+1\rangle_x\} \\ &= i(-1)^{k+1} e^{-i\kappa t_k}|j \\ &= 1, m=0\rangle. \end{aligned} \tag{11}$$

Here, the state $(|1,-1\rangle_x - |1,+1\rangle_x)/\sqrt{2}$ is a maximally entangled (or Bell) state, while the Dicke state $|j=1,m=0\rangle$ is a maximally squeezed state [2]. For this state, both the mean spin $\langle \hat{J}_x\rangle$ and the variance $(\Delta\hat{J}_{\mathbf{n}})_{\min}$ are equal to zero, which makes it hard to define ξ as Eq. (1). To avoid this problem, Wineland *et al.* proposed another definition of the squeezing parameter—namely, $\xi \to (j/|\langle\hat{J}\rangle|)\xi$, which gives the smallest squeezing $1/\sqrt{2}$ for the $N=2$ case [2].

B. Three-particle case

For the $N=3$ ($j=3/2$) case, we introduce linear combinations of the amplitudes $p_m^{(+)}(t)=c_m(t)+c_{-m}(t)$ with $m=1/2,3/2$. Since $p_{3/2}^{(+)}(0)=p_{1/2}^{(+)}(0)=0$, we get $p_m^{(+)}(t)=0$. Therefore, the amplitudes obey $c_m(t)=-c_{-m}(t)$, from which we can prove that the mean spin is always along the x direction. Such a result remains valid for any odd-N case. From Eq. (3), we obtain a coupled equation for the linear combinations $p_m^{(-)}(t)=c_m(t)-c_{-m}(t)$:

$$i\begin{pmatrix}\dot{p}_{1/2}^{(-)}\\ \dot{p}_{3/2}^{(-)}\end{pmatrix} = \begin{pmatrix}E'_{1/2} & X_{3/2}\\ X_{3/2} & E_{3/2}\end{pmatrix}\begin{pmatrix}p_{1/2}^{(-)}\\ p_{3/2}^{(-)}\end{pmatrix}, \tag{12}$$

where $E'_{1/2}=E_{1/2}-X_{1/2}$. The initial conditions are $p_{3/2}^{(-)}(0)=-1/\sqrt{2}$ and $p_{1/2}^{(-)}(0)=\sqrt{3/2}$. The dynamical evolution of the three-spin system is determined solely by Eq. (12). The analytic expression of the variance is

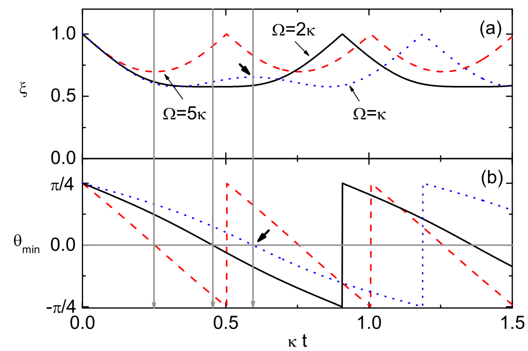

FIG. 2. (Color online) Time evolution of (a) the squeezing parameter and (b) the squeezing angle for the $N=3$ case with various Rabi frequencies: $\Omega=5\kappa$ (dashed red lines), $\Omega=2\kappa$ (solid black lines), and $\Omega=\kappa$ (dotted blue lines).

$$(\Delta\hat{J}_{\mathbf{n}})^2_{\min} = \frac{3}{4} + \frac{3\kappa^2}{S^2}\sin^2 St - \frac{3\kappa}{S}|\sin St|\sqrt{1-\frac{3\kappa^2}{S^2}\sin^2 St}, \tag{13}$$

and the squeezing angle is

$$\theta_{\min} = \frac{1}{2}\tan^{-1}\left[\frac{S\cos(St)}{(\Omega+\kappa)\sin(St)}\right], \tag{14}$$

where $2S=\mathcal{E}_3-\mathcal{E}_1=2\sqrt{\Omega^2+2\kappa\Omega+4\kappa^2}$ is the level spacing for the $N=3$ case and $\mathcal{E}_3$ and $\mathcal{E}_1$ correspond to two eigenvalues of the coefficient matrix of Eq. (12).

In Fig. 2, we investigate time evolution of ξ and $\theta_{\min}$ for the $N=3$ case. Similar to the previous $N=2$ case, our results show that for $\Omega \geq 2\kappa$, local minima of ξ together with $\theta_{\min}=0$ also occur at times $St_k=(k+1/2)\pi$. As shown by the solid black line of Fig. 2, we find that optimal squeezing can be obtained at t_k for the coupling $\Omega=2\kappa$. The maximally squeezed state at t_k reads

$$|\psi(t_k)\rangle = \frac{i(-1)^k}{\sqrt{2}} e^{-3i\kappa t_k/2}\left(\left|\frac{3}{2},\frac{1}{2}\right\rangle - \left|\frac{3}{2},-\frac{1}{2}\right\rangle\right). \tag{15}$$

Such a state gives the smallest squeezing parameter $\xi(t_k)=1/\sqrt{3}$ that the three-particle system can reach. It is worth mentioning that for $\Omega<2\kappa$, the vanishing $\theta_{\min}$ appearing at t_k no longer corresponds to local minima of ξ, as shown by the dotted blue lines of Fig. 2. The time scale t_k is relevant to determine the MST only for Ω equal or larger than the optimal coupling.

In short, we find some basic features for two exactly solvable cases. Local minima of ξ with $\theta_{\min}=0$ occur at the MST t_k. This is no longer true if Ω is smaller than the optimal coupling. The time scale t_k depends on the level spacing $2S$ between the ground state and the second excited state. Due to the symmetric properties of the spin system, the first excited eigenstate is an *idle level* (see below). This is also the reason why we can introduce linear combinations of the amplitudes $p_m^{(\pm)}$, with $m=0,1$ for $N=2$ and $m=1/2,3/2$ for $N=3$. For optimal coupling, the spin system will be evolved into the maximally squeezed state at t_k: $|1,0\rangle$ (for $N=2$) or $(|3/2,1/2\rangle - |3/2,-1/2\rangle)/\sqrt{2}$ (for $N=3$), which is just the

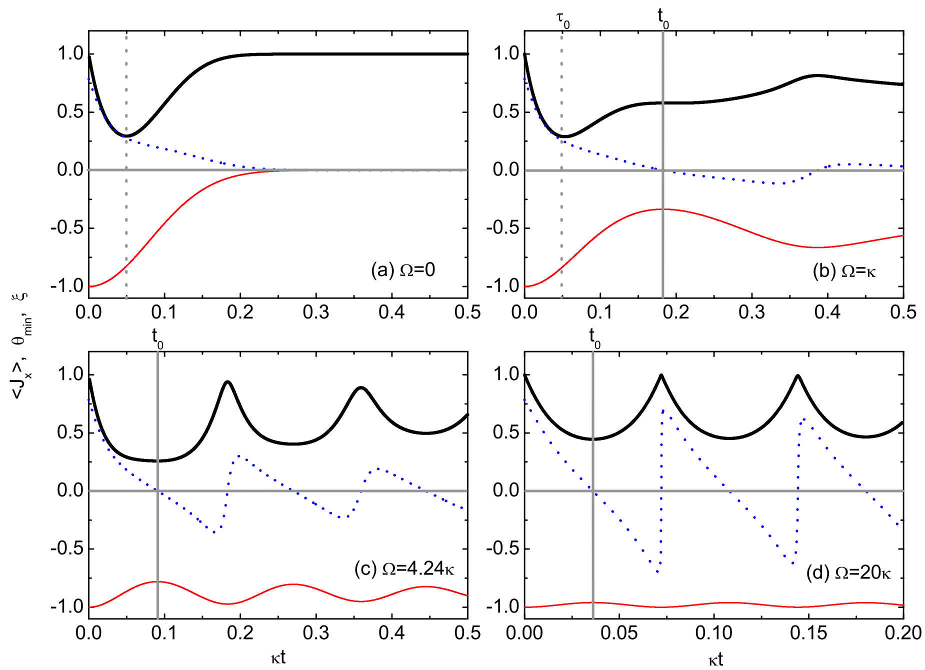

FIG. 3. (Color online) Time evolution of ξ (thick solid lines), $\theta_{\min}$ (dashed blue lines), and $\langle\hat{J}_x\rangle/j$ (thin red lines) for $N=40$ ($j=20$) and various Rabi frequencies: (a) $\Omega=0$, (b) $\Omega=\kappa$, (c) $\Omega=4.24\kappa$ (the optimal coupling), and (d) $\Omega=20\kappa$. The time scale t_0 for different Ω are indicated by the vertical lines.

ground state of $2\kappa\hat{J}_z^2$. We will extend the above results to many-particle cases.

IV. MANY-PARTICLE CASES: THE MAXIMAL-SQUEEZING TIME

In this section, we study spin squeezing for many-particle cases focusing on the time scale of maximal squeezing. For instance, we consider a spin system with particle number $N=40$ [1,16]. The numerical results are shown in Fig. 3. We find that with an increase of Ω, the squeezing ξ and the mean spin $\langle\hat{J}_x\rangle$ show collapsed oscillations [33,38]. Local maxima of the mean spin $\langle\hat{J}_x\rangle$ always appear together with the vanishing $\theta_{\min}$. We can prove this from the Heisenberg equation of $\hat{J}_x$ and Eq. (5): $d\langle\hat{J}_x\rangle/dt \sim \langle\hat{J}_z\hat{J}_y+\hat{J}_y\hat{J}_z\rangle \sim A\tan(2\theta_{\min})$. If the mean spin reaches its local maximum at a certain time t_0, then $d\langle\hat{J}_x\rangle/dt|_{t_0}=0$, which leads to $\theta_{\min}=0$ at t_0 provided that $A\neq 0$.

As shown in Fig. 3(b), for a small coupling with $\Omega=\kappa$, there are two time scales: t_0 for the vanishing $\theta_{\min}$ and τ_0 for the maximal squeezing. Note that the latter time scale τ_0 closes to that of the OAT result ($\Omega=0$ case)—i.e., $\kappa\tau_0 \simeq 0.049\,86$. With an increase of Ω, these two time scales become coincident, as shown in Figs. 3(c) and 3(d). Unlike exactly solvable cases, we find that the optimal coupling for $N=40$ is not a fixed value, but can be arbitrary Ω in a region $4.239 \leq \Omega_R/\kappa \leq 4.242$. Figure 3(c) represents the optimal squeezing case with the coupling $\Omega=4.24\kappa$. Starting from the initial CSS, the spin system evolves into the maximally squeezed state at $\kappa t_0=0.09$.

To investigate the maximally SSS at t_0, we calculate the quasiprobability distribution (QPD, or the Husimi function) $Q(\theta,\phi)$ on the Bloch sphere [3]:

$$Q(\theta,\phi)=|\langle\theta,\phi|\psi(t)\rangle|^2, \tag{16}$$

where $|\theta,\phi\rangle=\exp\{-i\theta(\hat{J}_x\sin\phi-\hat{J}_y\cos\phi)\}|j,-j\rangle$ is the generalized coherent spin state [39,40]. The initial state is a particular case of the CSS—namely, $|j,-j\rangle_x=|\theta=\pi/2,\phi=\pi\rangle$. The QPD can be used to simulate the variation of spin uncertainties. The circle in Fig. 4(a) represents an isotropic spin variance for the initial CSS, while the shaded ellipse parts in Figs. 4(b) and 4(c) are that of the SSS at times about $t_0/2$ and t_0, respectively. Unlike the OAT result [1,16], the maximal variance reduction appears along the z axis with $\theta_{\min}=0$ [18]. In Fig. 4, we also calculate the probability distribution

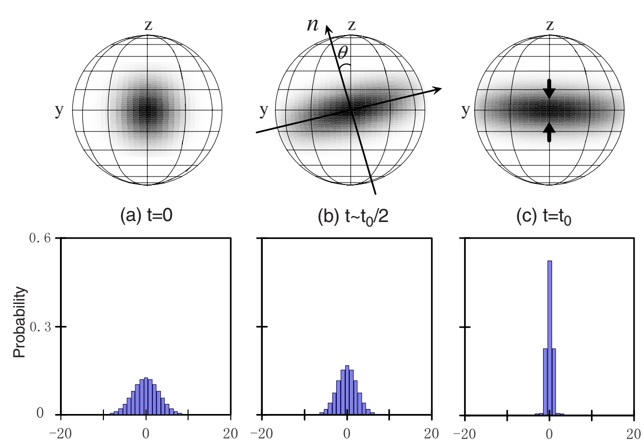

FIG. 4. (Color online) Time evolution of the quasiprobability distribution $Q(\theta,\phi)$ (top) and the probability distribution $|c_m|^2$ (bottom) at times (a) $\kappa t=0$, (b) $\kappa t=0.04$, and (c) $\kappa t=0.09$ (the MST). The QPD is normalized such that $Q(\pi/2,\pi)=1$. In (b), the spin component normal to the x axis is defined as $\hat{J}_{\mathbf{n}}=\hat{\mathbf{J}}\cdot\mathbf{n}$ with $\mathbf{n}=(0,\sin\theta,\cos\theta)$. Other parameters are taken as those of Fig. 3(c).

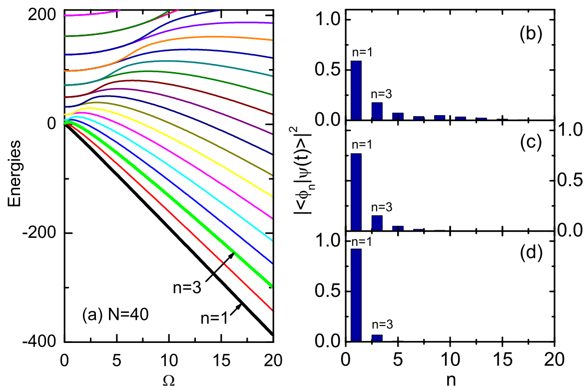

FIG. 5. (Color online) (a) Part of the eigenenergies $\mathcal{E}_n$ as a function of Ω for $N=40$. The spectral distribution $|\langle\phi_n|\psi(t)\rangle|^2$ for (b) $\Omega=\kappa$, (c) $\Omega=4.24\kappa$, and (d) $\Omega=20\kappa$.

$|c_m|^2=|\langle j,m|\psi(t)\rangle|^2$ of the spin state for $N=40$ and the optimal coupling $\Omega=4.24\kappa$. Compared with the initial CSS, we find that the maximally SSS at t_0 has a very sharp probability distribution with a large amplitude of the lowest spin projection—i.e., $m=0$ (for even N) or $m=\pm 1/2$ (for odd N) [41]. Such a sharp probability distribution of the SSS can be explained qualitatively by considering the familiar phase model [42] (see also references therein).

In order to determine the time scale t_0, we employ numerical diagonalization of the Hamiltonian (2) to obtain a set of eigenenergies $\{\mathcal{E}_n,\ n=1,2,3,\ldots\}$, where $n=1$ denotes the ground state, $n=2$ the first excited state, and $n=3$ the second excited state, etc. Thus,

$$\hat{H}|\phi_n\rangle=\mathcal{E}_n|\phi_n\rangle, \tag{17}$$

where $\mathcal{E}_n$ and $|\phi_n\rangle$ with $n=1,2,3,\ldots,(2j+1)$ depend on the parameters N and Ω [43]. As shown in Fig. 5(a), we plot parts of $\mathcal{E}_n$ for $j=20$ ($N=40$) as a function of the coupling Ω. Similar to previous two- and three-particle cases, we suppose that the MST t_0 depends on the level spacing between $n=1$ and $n=3$—namely, $t_0=\pi/(2S)$ with $2S=\mathcal{E}_3-\mathcal{E}_1$. To check it, in Table I, we compare exactly numerical results of time t_0 with $\pi/(2S)$ for various N and Ω. Our diagonalization method gives an accurate prediction of the time t_0. We remark that for the $N=2$ and $N=3$ cases, both results are exactly the same.

To explain the above agreement between the MST t_0 and $\pi/(2S)$, we calculate the spectral distribution of the spin state—i.e., $|\langle\phi_n|\psi(t)\rangle|^2$ in Figs. 5(b)–5(d) for $N=40$ and various Ω. Physically, the spectral distribution measures the population distribution of the state vector $|\psi(t)\rangle$ on the eigenstates $|\phi_n\rangle$ [44]. For fixed parameters N and Ω, the spectral distribution $|\langle\phi_n|\psi(t)\rangle|^2$ is time independent. In fact, one can expand the spin state in terms of $\{|\phi_n\rangle\}$: $|\psi(t)\rangle=\Sigma_n d_n(t)|\phi_n\rangle$ with amplitudes $d_n(t)=\exp[-i\mathcal{E}_n t]d_n(0)$. Here the initial amplitudes $d_n(0)$ depend only on the initial condition (4); therefore, the spectral distribution $|\langle\phi_n|\psi(t)\rangle|^2=|d_n(t)|^2\equiv|d_n(0)|^2$ and is time independent for fixed N and Ω. From our numerical calculations, Figs. 5(b)–5(d), we find that total occupation of the spin state $|\psi(t)\rangle$ on the eigenstates $n=1$ and $n=3$ is over 80%. This is the reason why the MST depends on the level spacing between these two levels. Moreover, we find that the even n eigenstates are in fact the idle levels, just as previous $N=2$ and $N=3$ cases.

Except for $N=2$ and $N=3$, exact solutions for the nonlinear spin system within the small-coupling regime ($1<\Omega/\kappa\ll N$) do not exist [18,33]. In our previous work [41], however, we have obtained an analytic expression of the MST based upon the phase model:

$$\kappa t_0\simeq\frac{\pi}{2}\sqrt{\frac{\kappa}{2\Omega N}}, \tag{18}$$

which is valid for large N ($\geq 10^3$). Our analytic solution of the MST is derived by the prediction $t_0\simeq T/4$, where $T=2\pi/\omega_{\rm eff}$ is the period of the pendulum near the bottom of a periodic potential [41]. In fact, for large N the spin system behaves as a pendulum rotating with oscillating frequency $\omega_{\rm eff}=\sqrt{2\kappa\Omega_R N}$. As shown in Table I, we compare $\pi/(2S)$, the analytic solutions of Eq. (18), and the exact numerical results of the MST for various parameters Ω and N. It is shown that our analytical expression (18) works very well for the large N ($\sim 10^3$), which implies that the oscillating frequency $\omega_{\rm eff}$ has its physical meaning as half the level spacing $S=(\mathcal{E}_3-\mathcal{E}_1)/(2\hbar)$. Note that the phase model or Eq. (18) is valid for large N, while $\pi/(2S)$ is not limited by this. From this sense, we believe that the diagonalization method presented here provides a very comprehensive way to measure the maximal-squeezing time.

V. CONCLUSIONS

In summary, we have studied the maximal-squeezing time of a nonlinear spin system, which can be realized in the two-component BEC or other spin system similar to that of Takeuchi *et al.* [16]. Motivated by two exactly solvable cases

TABLE I. Comparison of exactly numerical t_0, $\pi/(2S)$, and analytic results of Eq. (18) for different N and Ω. The times are in units of $(100\kappa)^{-1}$.

	$N=40$			$N=200$			$N=1000$		
Ω/κ	1	4.24	20	1	6.7	25	1	10.8	50
Exact num.	18.28	9.065	3.604	8.192	3.184	1.549	3.665	1.104	0.4945
$\pi/(2S)$	19.02	8.615	3.573	8.143	3.048	1.533	3.571	1.071	0.4916
Equation (18)	17.56	8.529	3.927	7.854	3.034	1.571	3.512	1.069	0.4967

for $N=2$ and $N=3$, we show that time scale of the maximal squeezing depends on the level spacing between $n=1$ and $n=3$ eigenstates. We explain it by calculating the probability distribution of the spin state on the eigenstates of the Hamiltonian and find that the above two states are occupied predominantly. Such results remain valid for arbitrary N and a wide range of coupling strength.

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