

Modular multiplication operator and quantized baker's maps

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The modular multiplication operator, a central subroutine in Shor's factoring algorithm, is shown to be a coherent superposition of two quantum baker's maps when the multiplier is 2. The classical limit of the maps being completely chaotic, it is shown that there exist perturbations that push the modular multiplication operator into regimes of generic quantum chaos with spectral fluctuations that are those of random matrices. For the initial state of relevance to Shor's algorithm we study fidelity decay due to phase and bit-flip errors in a single qubit and show exponential decay with shoulders at multiples or half-multiples of the order. A simple model is used to gain some understanding of this behavior.

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I. INTRODUCTION

Given an M -dimensional complex Hilbert space, consider an orthonormal basis $|m\rangle$, $m=0, \dots, M-1$. The modular multiplication by a number p coprime to M is the unitary permutation operator U_p :

$$U_p|m\rangle \rightarrow |mp \pmod{M}\rangle. \quad (1)$$

Repeated application of U_p is also the modular exponentiation operator and leads to $U_p^k|m\rangle = |mp^k \pmod{M}\rangle$. U_p is periodic; i.e., there exists k_0 such that $U_p^{k_0} = I$, or $p^{k_0} = 1 \pmod{M}$. This period though is an irregular function of M for a given p and is the "multiplicative order" of $p \pmod{M}$ [1]. This operator is a crucial subroutine of Shor's factoring algorithm [2] in which it is required to perform the following operation on a bipartite Hilbert space:

$$|j\rangle|1\rangle \rightarrow |j\rangle|p^j \pmod{M}\rangle, \quad (2)$$

where $0 \leq j \leq 2^t - 1$ and $M < 2^t$. Typically M is a large number and therefore we need to calculate large powers and their residues mod M . This is considerably simplified by modular arithmetic [3], and the whole modular exponentiation step can be performed with $O((\ln M)^3)$ number of gates [4]. Once this is done, a quantum Fourier transform extracts the period k_0 with a reasonable rate of success. Given the period (order), it is possible to find a factor efficiently, provided that p_{k_0} is even and $p^{k_0/2} \neq -1 \pmod{M}$, by well-known procedures of number theory and using only classical computers [3].

Shor's algorithm exploits the polynomial speed of the quantum Fourier transform to find the order and hence offers a polynomially scaling algorithm for factoring numbers. Shor's algorithm has been implemented experimentally [5], although the number so factored is still very small to excite any practical application. The effects of decoherence and gate errors on Shor's algorithm are important considerations and have been addressed by several authors previously [6–11]. For instance, Ref. [6] discusses the impact of envi-

ronmental decoherence on the algorithm, while in [9] direct detailed simulations have shown that the Shor algorithm is highly sensitive to gate errors, and the effect of static imperfections due to residual interactions between qubits has been studied recently in [11]. In this paper we will not directly simulate Shor's algorithm but look closely at the modular multiplication and exponentiation for the special and simplest case $p=2$. We will call U_2 as S , the shift operator as it performs the simple action of a qubit cyclic shift if M is a power of 2.

Quantum algorithms have been studied earlier with a view to see if they had the properties of quantum chaotic systems [12,13]. Recently it was shown that the spectrum of the unitary part of Shor's algorithm, properly desymmetrized, had typical random matrix fluctuations [13], indicating that the operator itself may be quantum chaotic. However, it was also pointed out that the origin of the chaos is the modular exponentiation part which is akin to nongeneric quantum chaotic systems such as the cat maps [14]. Here we make this connection more precise and show that the classical limit of these subroutines is an admixture of two baker's maps. Baker's maps are paradigms of deterministic classical chaos that are as random as a coin toss [15,17]. The dimensionless inverse Planck constant in the Shor algorithm is the number to be factored, and hence the classical limit is reached through a practically important regime. We show that due to the proximity with such operators, there are perturbations that push the modular exponentiation part (and therefore indeed the whole of Shor's algorithm) into regimes of generic quantum chaos. However, one is not so much interested in stationary-state properties as in time-evolving states, in fact on those states that are used in Shor's algorithm. Therefore a simple model of a static generic gate error on a single qubit is studied below. The fidelity of repeated modular multiplication, or modular exponentiation, shows that decay depends on the classical limit. It is demonstrated that a simple exactly solvable model for the exponential fidelity decay captures the actual behavior reasonably well.

II. BAKER AND SHIFT OPERATORS

The classical baker's map [15–17] B_c is an area-preserving transformation of the unit phase-space square

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$[0,1) \times [0,1)$ onto itself, which takes a phase-space point (q,p) to (q',p') where $(q'=2q, p'=p/2)$ if $0 \leq q < 1/2$ and $(q'=2q-1, p'=(p+1)/2)$ if $1/2 \leq q < 1$. The stretching along the horizontal q direction by a factor of 2 is compensated exactly by a compression in the vertical p direction. This is well known to be a fully chaotic system that in a mathematically precise sense is as random as a coin toss [18]. The area-preserving property makes this map a model of chaotic two-degrees-of-freedom Hamiltonian systems. The lack of a generating Hamiltonian is compensated for by the existence of a classical generating function of the canonical transformation B_c . The chaos is inferred by expressing a phase-space point in the binary representation if $q = 0.a_0a_1a_2\cdots$ and $p = 0.a_{-1}a_{-2}a_{-3}\cdots$, where a_i are either 0 or 1, then $q' = 0.a_1a_2\cdots$ and $p' = 0.a_0a_{-1}a_{-2}a_{-3}\cdots$. Thus the most significant bits of q are lost at the rate of 1 bit per iteration, leading to an exponential increase in any initial error. The Lyapunov exponent is $\ln(2)$ per iteration. This “left shift” is in fact an important mechanism for the generation of Hamiltonian chaos [17], and in more complicated forms arises generically.

This was quantized first by Balazs and Voros [19,20]. Quantization in this context is the construction of an appropriate unitary operator that evolves states over one iteration and has the correct classical limit. Symmetries that may be broken on quantization must be restored in this limit. The Hilbert space of states is finite dimensional and has N position and momentum states, denoted by $|q_n\rangle$ and $|p_m\rangle$. If periodic boundary conditions are assumed, $|q_{n+N}\rangle = |q_n\rangle$, $|p_{m+N}\rangle = |p_m\rangle$, this implies that the transformation function between position and momentum is the discrete Fourier transform $(F_N)_{mn} = \langle p_m | q_n \rangle = \exp[-2\pi i mn/N] / \sqrt{N}$, $m, n = 0, 1, 2, \dots, N-1$. Here N is an effective scaled Planck constant as $N = A/h = 1/h$, where A is the area of the phase space, here unity. Thus the classical limit is the large- N limit. If B is the quantum baker’s map, Balazs and Voros required that $\langle p_m | B | q_n \rangle = \sqrt{2} \langle p_m | q_{2n} \rangle = (F_{N/2})_{mn}$ if n and m are both $\leq N/2 - 1$. This is almost like requiring that B take $|q_n\rangle$ to $|q_{2n}\rangle$, mimicking the classical stretching action, except that the momentum components above $N/2$ are set to zero ($\langle p_m | B | q_n \rangle = 0$ for $p_m \geq N/2$ and $q_n < N/2$). It is also clear from this that B is very close to the action of modular multiplication with $p=2$ [21]. N is throughout assumed to be an even integer. In fact, we will set $N=2L$ and can then consider the quantum baker to act on a Hilbert space of a qubit coupled to an L -dimensional system.

A similar argument is made for the second half of the transformation, and remarkably these conditions are consistent and produce a unitary operator which has a broken parity symmetry [20], the classical symmetry being $(q \rightarrow 1-q, p \rightarrow 1-p)$. Saraceno [22] restored this by imposing antiperiodic boundary conditions, and this leads to the quantum baker’s map

$$B = G_{2L}^{-1} \begin{pmatrix} G_L & 0 \\ 0 & G_L \end{pmatrix}, \quad (3)$$

where $(G_N)_{mn} = \langle p_m | q_n \rangle = \exp[-2\pi i(m+1/2)(n+1/2)/N] / \sqrt{N}$. B is a unitary matrix, whose repeated application is the

quantum version of the full left shift of classical chaos. This quantum map has been continued to be studied as it has many properties of generic quantum chaotic systems, including random-matrix-like spectral fluctuations [20] and eigenfunction scarring [22]. It is also amenable to a simple semiclassical periodic orbit sum and hence has been used in the study of such approximations [23,24]. For N that are powers of 2 it was found that the Hadamard and related transforms highly simplified the eigenstates and some of them are remarkably well described by the Thue-Morse sequence [26] and its Fourier transform [25,27]. It has also been used in the study of entanglement [28] and hypersensitivity of quantum chaos [29]. It is possible to design a quantum circuit for the quantum baker’s map [30], and this has been implemented on a NMR quantum computer experimentally [31].

If one is not mindful of classical symmetries being fully preserved on quantization, there are a large number of possible quantum baker’s maps [20,32]. An important class of such “decorated baker’s” maps [33] are got by embellishing the original baker’s map with relative phases in half-sized Fourier blocks, as well as in the definition of the Fourier transform itself, as done below. In a previous work we have constructed such a decorated quantum baker’s map using the shift operator S [21]. It will be useful to do the converse and construct the shift operator from the quantum baker’s map or similar operators. It is well known that the “hard” part of Shor’s algorithm is the implementation of the modular exponentiation step. On the other hand, the quantum baker’s map is implemented with quantum Fourier transforms (QFTs) and this may make the implementation of the shift operator possible with the QFTs. This is explicitly shown at least for the case $p=2$. More importantly, this will embed the shift operator in a larger family of operators that include maps with well-defined classical limits, thereby making the classical limit of modular multiplication explicit.

The shift operator we have already defined, however, we restate for clarity as

$$S|n\rangle = |2n \pmod{N-1}\rangle, \quad (4)$$

with the caveat that $S|N-1\rangle = |N-1\rangle$. This corresponds to our earlier definition with $M=N-1$ with one more state ($|N-1\rangle$) added to the Hilbert space, but which remains fixed and outside any dynamics we are interested in, but may participate when there are perturbations. Note that since for the baker’s map N is an even integer, S is unitary. A generalized Fourier transform is defined as

$$(F_N(\alpha, \beta))_{nm} = \frac{1}{\sqrt{N}} \exp[-2\pi i(n+\alpha)(m+\beta)/N]. \quad (5)$$

Evaluating the product of the Fourier transform and S we can derive, merely by summing finite geometric series and using elementary properties of exponentials, that

$$S = \frac{1}{\sqrt{2}} F_{2L}^{-1}(\alpha, \alpha) \begin{pmatrix} F_L\left(\alpha, \frac{\alpha}{2}\right) & F_L\left(\alpha, \frac{1+\alpha}{2}\right) \\ e^{-i\pi\alpha} F_L\left(\alpha, \frac{\alpha}{2}\right) & -e^{-i\pi\alpha} F_L\left(\alpha, \frac{\alpha+1}{2}\right) \end{pmatrix}. \tag{6}$$

Note that the operator S does not depend on the phase α that appears on the right-hand side (RHS). This is used to break or keep the parity symmetry. A natural and simple choice is $\alpha=0$, but $\alpha=1/2$ leads to symmetric operators as explained below. The structure of the above identity allows this to be written as

$$S = \frac{1}{\sqrt{2}}(B_{2L} + B'_{2L}), \tag{7}$$

where

$$B_{2L} = F_{2L}^{-1}(\alpha, \alpha) \begin{pmatrix} F_L\left(\alpha, \frac{\alpha}{2}\right) & 0 \\ 0 & -e^{-i\pi\alpha} F_L\left(\alpha, \frac{\alpha+1}{2}\right) \end{pmatrix} \tag{8}$$

and

$$B'_{2L} = F_{2L}^{-1}(\alpha, \alpha) \begin{pmatrix} 0 & F_L\left(\alpha, \frac{1+\alpha}{2}\right) \\ e^{-i\pi\alpha} F_L\left(\alpha, \frac{\alpha}{2}\right) & 0 \end{pmatrix}. \tag{9}$$

Thus remarkably the modular multiplication S can be written as a *sum* of two unitary operators (with a normalization factor). B_{2L} and B'_{2L} are two *quantum baker's maps*, of which the former one is the standard one, which we have discussed above. It is well known that such decorated baker's maps also perform the same classical actions as normal baker's maps [20,32], the classical limit being $L \rightarrow \infty$. The operator B'_{2L} has not been studied nearly as much, but has recently appeared in a work that uses this to study coupled chaotic systems [34]. The classical limit (say, B'_c) as pointed out in this study corresponds to a different stacking order of the vertical partitions of the baker's map after they have been stretched. Instead of the usual left half transiting to the bottom half, it is put in the top half and the right half goes into the bottom half. This fixes the lower right-hand corner of the square. Thus $B'_c(q, p) = (2q, (p+1)/2)$ if $q \leq 1/2$ and $(2q-1, p/2)$ if $q > 1/2$. Again the operator B'_{2L} that appears above differs from the one used earlier in terms of the "decorations." Of course, these decorations are absolutely crucial so that the two unitary evolutions, which are nonperiodic and have random-matrix-like properties, *add* and conspire to produce the simple shift operator that is completely periodic. Previous studies of the classical limit of operators such as S include those of what is called the "extremal quantum baker's map" [35], and it has been suggested that the classical limit corresponds to a "stochastic classical map" [36,37]. In Fig. 1 we have shown a schematic of the classical baker's maps that on quantization and coherent addition yield the

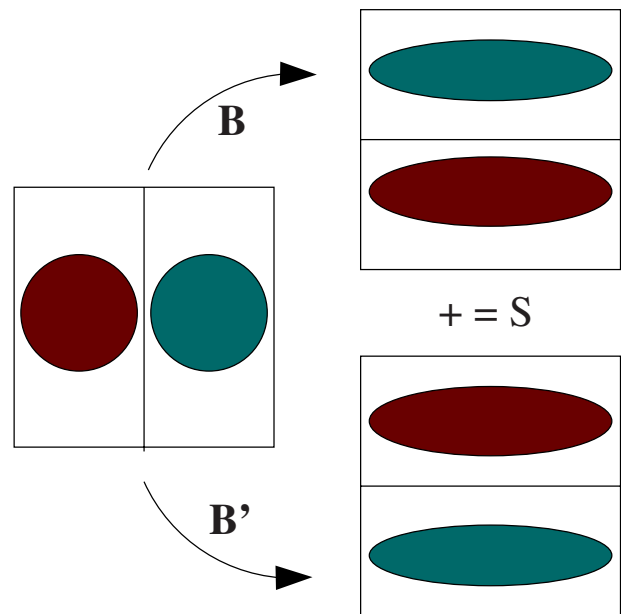


FIG. 1. (Color online) A schematic view of the shift operator as a sum of two baker's maps, one with the usual stacking order of the vertical left half being stretched to the bottom horizontal half (B_c classical, B_{2L} quantum) and the baker's map with a reverse stacking order (B'_c classical, B'_{2L} quantum).

shift operator. Also see Ref. [21] for a description and figure of the action of S on Weyl coherent states.

That the simple shift operator's can be thought of as a coherent superposition of two quantum chaotic evolutions has been demonstrated above in a particularly simple way. This suggests that there maybe perturbations of the operator S that are generic and may possess random-matrix-like [38] and other quantum chaotic properties [39,40]. It is shown below that this is indeed the case. Since quantum chaotic

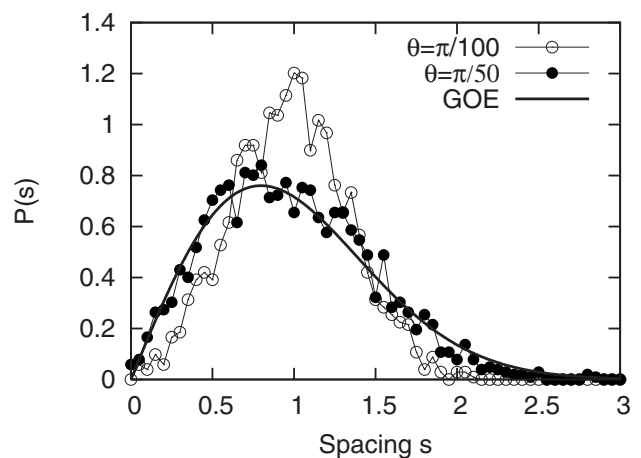


FIG. 2. The nearest-neighbor spacing distribution of even-subspace eigenangles for two perturbations of the shift operator when $N=4094$, the perturbations preserving parity symmetry. The smooth curve shows the corresponding COE result of random matrix theory, the Wigner-Dyson distribution.

operators also are typically sensitive to perturbations [41,42], this may have implications for the operation of the Shor algorithm. This is partly studied by measuring the fidelity of S to small perturbations and showing that the fidelity decays exponentially in time until the order of $N-1$ or half of this. Thereafter it typically shows an enhanced rate of decay at multiples of this time, but could also show strong recurrences. Surprisingly a simple analysis when N is a power of 2 captures many of the qualitative features of the more general case.

III. PERTURBATIONS OF THE SHIFT OPERATOR AND QUANTUM CHAOS

In terms of operations on the Hilbert spaces of the tensor product $\mathcal{H}_2 \otimes \mathcal{H}_L$ we may write the shift operator as

$$S = F_{2L}^{-1}(\alpha, \alpha) \circ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ e^{-i\pi\alpha} & -e^{-i\pi\alpha} \end{pmatrix} \otimes I_L \circ \begin{pmatrix} F_L\left(\alpha, \frac{\alpha}{2}\right) & 0 \\ 0 & F_L\left(\alpha, \frac{1+\alpha}{2}\right) \end{pmatrix}. \quad (10)$$

Thus the modular exponentiation maybe implemented with QFTs that have suitable phases. However, the dimensionalities of the QFTs are not in general powers of 2 and are therefore not the standard ones in use. In this paper we choose to perturb the central operator in the above equation and perturb only the qubit space \mathcal{H}_2 . In particular, consider the smooth embedding of the shift operator in the family

$$S(\theta; \frac{1}{2}, \sigma_x) = F_{2L}^{-1}\left(\frac{1}{2}, \frac{1}{2}\right) \begin{pmatrix} \sin\left(\frac{\pi}{4} - \theta\right) F_L\left(\frac{1}{2}, \frac{1}{4}\right) & \cos\left(\frac{\pi}{4} - \theta\right) F_L\left(\frac{1}{2}, \frac{3}{4}\right) \\ -i \cos\left(\frac{\pi}{4} - \theta\right) F_L\left(\frac{1}{2}, \frac{1}{4}\right) & i \sin\left(\frac{\pi}{4} - \theta\right) F_L\left(\frac{1}{2}, \frac{3}{4}\right) \end{pmatrix}. \quad (12)$$

When $\theta = \pm \pi/4$, the operators correspond to baker's maps of type B'_{2L} and B_{2L} , respectively. For other angles it represents a coherent mixture of the two types of baker's map operator stacking while at $\theta=0$ it is the usual shift operator.

The even subspace of the spectrum of $S(\theta; \frac{1}{2}, \sigma_x)$ is used to show in Fig. 2 the nearest-neighbor spacing statistics for two cases of small angles θ . It is clear that if the perturbation is very small, the rigid, harmonic-oscillator-like spectrum of S widens into one where there is dominant level repulsion

$$\begin{aligned} S(\theta; \alpha, P) &= F_{2L}^{-1}(\alpha, \alpha) \circ \exp(-i\theta P) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ e^{-i\pi\alpha} & -e^{-i\pi\alpha} \end{pmatrix} \\ &\otimes I_L \circ \begin{pmatrix} F_L\left(\alpha, \frac{\alpha}{2}\right) & 0 \\ 0 & F_L\left(\alpha, \frac{1+\alpha}{2}\right) \end{pmatrix} \\ &= F_{2L}^{-1}(\alpha, \alpha) \circ \exp(-i\theta P) \otimes I_L \circ F_{2L}(\alpha, \alpha) S \\ &= V(\theta) S. \end{aligned} \quad (11)$$

Here P is the perturbing Hermitian operator on the qubit space and $V(\theta)$, defined through the last equation, is the Fourier transform of the perturbation generated by P . The operator $S(0; \frac{1}{2}, P)$ is the unperturbed shift operator simply called S so far. The family of operators $S(\theta; \alpha, P)$ now depends on the phase α as well, although $S(0; \alpha, P)$ does not. We display this dependence explicitly as the phase α does play a crucial role. It must be emphasized that the perturbation is chosen as it is a particularly simple one affecting only one qubit. However, the perturbation is also open to interpretation in terms of changes in the classical limits that it engenders. If the modular exponentiation is implemented via QFTs as indicated above, it is conceivable that these types of errors will be important. As the perturbation is a rotation in qubit space, it may, for instance, be the result of stray local magnetic fields in spin realizations.

The operator S has quantum parity symmetry $R: R|n\rangle = |N-n-1\rangle$ —that is, $SR=RS$. If L is a power of 2, then R is simply the product $\otimes^{2L} \sigma_x$. Perturbations of the shift will therefore in general approximately preserve this symmetry. To analyze random matrix properties it is desirable to completely break the symmetry or preserve it and desymmetrize the operators. Since we want to retain the character of a small perturbation, we first choose to preserve the parity symmetry exactly. We can do this by adopting antiperiodic boundary conditions $\alpha=1/2$ and choosing $P=\sigma_x$. This will lead to the family

and for fairly significant perturbations becomes a generic one that belongs to the universality class of the circular orthogonal ensemble (COE) of random matrices [38], well known to apply to quantum chaotic systems that have time-reversal symmetry [40,43]. Previously it was shown that the unitary part of the full Shor algorithm, including the Hadamard and Fourier transforms, had fluctuations that were of the circular unitary ensemble kind [13]. Note that restricting ourselves to the modular multiplication part with a particular perturbation

allows us to preserve the time-reversal symmetry that holds for individual quantum baker's maps [20].

Thus we see that indeed there are perturbations of the shift operator that are quantum chaotic. There is also a crucial dependence on the number N (or the number to be factored $N-1$). If N were a power of 2 such as 4096, instead of 4094 in Fig. 2, there would be much more deviation from the COE distribution, with a large peak near the origin. This anomalous statistic arises from the extreme degeneracy of the eigenangles when N is a power of 2 and is special. A similar situation where there is an extreme dependence of the statistics of the spectrum on the effective Planck constant N arises in the case of the perturbed cat maps [44], and presumably for similar reasons. Earlier it was also pointed out that perturbing the cat maps slightly so that the sawtooth map arises leads to a rapid restoration of the generic fluctuation characteristics of quantum chaotic systems [45]. Thus the similarities of the shift map to the quantum chaotic cat maps with their special arithmetic properties is further highlighted here.

IV. FIDELITY DECAY

Nonstationary properties are now studied, as indeed the Shor algorithm is the result of the time evolution of a particular initial state which corresponds to the state $|1\rangle$. The algorithm requires finding the states $|x^j \bmod(N-1)\rangle$ for x coprime to $N-1$. As stated previously we take $x=2$ throughout and we now study how gate errors would proliferate in time. In particular, we study the fidelity

$$f(t) = |\langle 1|S^{-t}S^t(\theta; \alpha, P)|1\rangle|^2 = |\langle 2^t \bmod(N-1)|S^t(\theta; \alpha, P)|1\rangle|^2. \quad (13)$$

In this section choose $\alpha=0$ for simplicity and note that there is a weakly broken parity symmetry as a result of this.

A. Case $P=\sigma_x$

The first case we take will be a rather special one wherein the perturbation is ineffective: the bit flip $P=\sigma_x$. In this case we have that

$$S(\theta; 0, \sigma_x) = F_{2L}^{-1}(0,0) \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-i\theta}F_L(0,0) & \frac{1}{\sqrt{2}}e^{i\theta}F_L\left(0, \frac{1}{2}\right) \\ \frac{1}{\sqrt{2}}e^{-i\theta}F_L(0,0) & \frac{1}{\sqrt{2}}e^{i\theta}F_L\left(0, \frac{1}{2}\right) \end{pmatrix} \quad (14)$$

and $f(t)=1$ for all time t . Note that there are only phases multiplying the Fourier blocks, and therefore the classical limit of this family of operators is the same as that of the simple shift operator: a coherent sum of two baker's maps. However, the reason the fidelity is unity for all times is due to a rather intriguing if simply verifiable identity. Below F_{2L} is used for $F_{2L}(0,0)$. The perturbation operator is

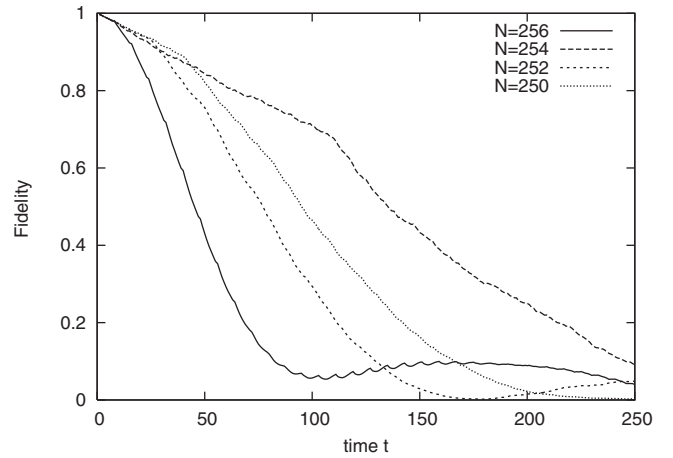


FIG. 3. The fidelity decay for four neighboring values of N . The perturbation is $P=\sigma_y$, the phase $\alpha=0$, and $\theta=0.05$.

$$V(\theta) = F_{2L}^{-1}(\exp(-i\theta\sigma_x) \otimes I_L)F_{2L} = I_L \otimes \exp(-i\theta\sigma_z). \quad (15)$$

The last equality is an identity valid for all integer L . This in turn simply follows from the identity

$$F_{2L}^{-1}(\sigma_x \otimes I_L)F_{2L} = I_L \otimes \sigma_z, \quad (16)$$

which may be directly verified. Note that since $\sigma_x \otimes I_L$ is a circulant matrix, it has to be diagonalized by the Fourier transform, and since the eigenvalues are ± 1 , these are the only possible diagonal entries. It is also easily verified that this has the structure of L repetitions of $(1, -1)$ pairs which are the diagonal entries of σ_z . In some sense the Fourier transform is simultaneously performing a bit reversal and a 90° rotation in qubit space. However, note that for this identity to be true we do not require that L be a power of 2. Similar identities do *not* hold for the other two Pauli matrices, but there are approximations that we will state further ahead. Since σ_z merely changes the phase of the state (in standard basis), it follows that the fidelity $f(t)=1$ always.

B. Case $P=\sigma_y$

In this case we see that the classical limit is altered by the perturbation. The Hadamard transform in qubit space is further rotated around the y axis in spin space and the final operator is similar to that used in Eq. (12), which we recall is for the case when the phase $\alpha=1/2$ and for a σ_x perturbation.

$$S(\theta; 0, \sigma_y) = F_{2L}^{-1} \begin{pmatrix} \sin\left(\frac{\pi}{4} - \theta\right)F_L(0,0) & \cos\left(\frac{\pi}{4} - \theta\right)F_L\left(0, \frac{1}{2}\right) \\ \cos\left(\frac{\pi}{4} - \theta\right)F_L(0,0) & -\sin\left(\frac{\pi}{4} - \theta\right)F_L\left(0, \frac{1}{2}\right) \end{pmatrix}. \quad (17)$$

Therefore this case is the closest to the parity-preserving case we have already discussed and shows sharp transitions to features of a quantum chaotic spectrum. In Fig. 3 we plot the

fidelity $f(t)$ for a set of N values that are close to 256. Notice that although the N values are as close as can be (N must be even), the fidelity decays in qualitatively different manners. Except for a very-short-time scale, the decays are different and one sees a prominent “shoulder” in each of the curves at which the fidelity starts to decay even faster.

The easiest case to discern this in the figure is for $N = 254$ when the shoulder occurs at $t = 110$. It is quite easy to numerically relate the time at which this occurs to the multiplicative order of 2 modulo $N - 1$, referred to henceforth loosely as simply the order of $N - 1$. Recall that this is the smallest number $k_0(N - 1)$ such that $2^{k_0(N - 1)} = 1 \pmod{N - 1}$. One is guaranteed that such a number exists because $N - 1$ is an odd integer. Indeed $k_0(253) = 110$, while $k_0(255) = 8$, $k_0(251) = 50$, and $k_0(249) = 82$. Thus the fidelity shows a shoulder either exactly at $t = k_0(N - 1)$ or at $t = k_0(N - 1)/2$, the first case being observed for $N = 256$ and $N = 254$ while the latter case for $N = 252$ and 250. It is significant then that for numbers of larger orders the fidelity can decay considerably even for small perturbations. Note that $\theta = 0.05$ in the figure, which roughly translates to a 0.51:0.49 mixture of the two types of baker’s maps, while a 0.5:0.5 “mixture” will be the unperturbed shift operator. The larger the order is, the higher powers of S must be calculated and the higher chance of the fidelity to be lowered. It is interesting that the objective of the Shor algorithm—namely, finding the order—already appears in the fidelity as a critical time.

A qualitative understanding of these behaviors is reached with a surprisingly simple model. Consider the case when N is a power of 2 (say, $N = 2^M$), and let the perturbation be

$$V(\theta) = I_L \otimes \exp(-i\theta\sigma_x). \tag{18}$$

Note that this has the same structure as the perturbation from the previous case, but is not the true perturbation in this one. Then the initial state is $|0 \cdots 01\rangle$ and it is clear that for $t \leq M$

$$\begin{aligned} [V(\theta)S]^t |0 \cdots 01\rangle &= I_{2^{M-t}} \otimes \exp(-i\theta\sigma_x) \otimes \exp(-i\theta\sigma_x) \\ &\otimes \cdots \otimes \exp(-i\theta\sigma_x) |0 \cdots 01\rangle \end{aligned} \tag{19}$$

and $f(t) = |\cos^2(\theta)|^t$. Thus the initial fidelity decay is in fact exponential with a rate $-\ln[|\cos^2(\theta)|]$. However, beyond $t = M$ there is an additional error adding up and so $f(t) = |\cos^2(\theta)|^{2^{M-t}} |\cos^2(2\theta)|^{t-M}$ for $M < t \leq 2M$. Thus in this range of time the fidelity decays about 4 times as fast. One can write in general for this model that

$$f(t) = |\cos^2(r\theta)|^{(r+1)M-t} |\cos^2[(r+1)\theta]|^{t-rM}, \tag{20}$$

where $r = [t/M]$ and $[x]$ is the integer part of x . In Fig. 4 is shown how good an approximation this can be for the case B situation when N is indeed a power of 2 (or the number we want to factor is 1 less than a power of 2). There is even good quantitative agreement. On the other hand, when $N = 254$ (253 has a high order of 110), the approximate formula is only qualitatively correct as seen in Fig. 4.

The model works reasonably well because

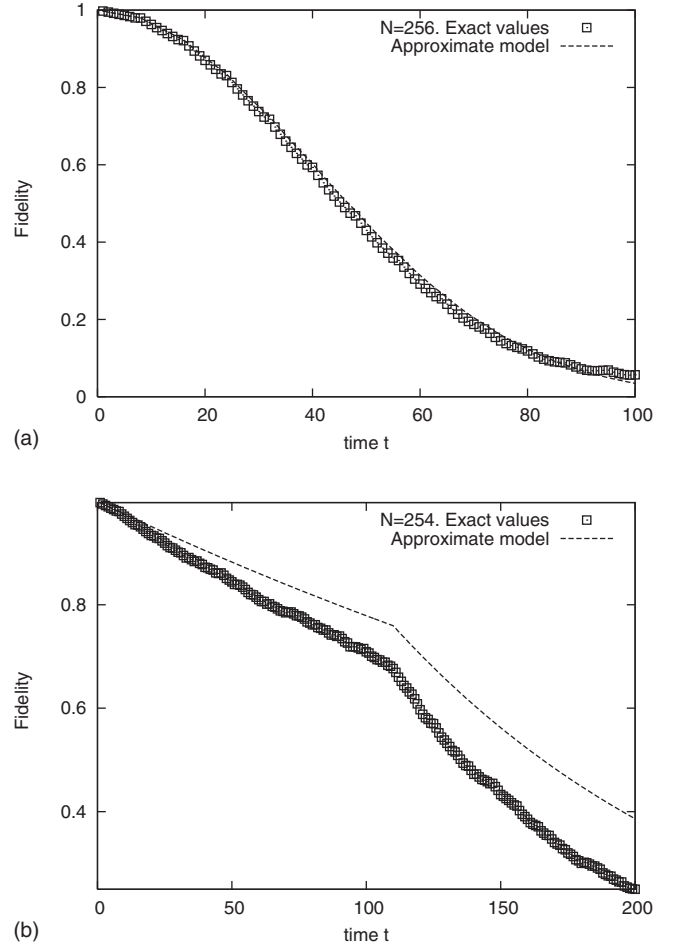


FIG. 4. The comparison of the fidelity decay for the σ_y perturbation ($\alpha = 0$ and $\theta = 0.05$) with the analytical estimate in Eq. (20) from an approximate model for the two cases of N .

$$F_{2L}^{-1}(\exp(-i\theta\sigma_y) \otimes I_L)F_{2L} \approx I_L \otimes \exp(-i\theta\sigma_x), \tag{21}$$

which follows from

$$F_{2L}^{-1}(\sigma_y \otimes I_L)F_{2L} \approx I_L \otimes \sigma_x, \tag{22}$$

so that the exact perturbation operator which would have been the LHS of Eq. (21) is approximated by its RHS, which we have used above. This is the counterpart of Eq. (16); however, here this is only an approximation.

This “model” or approximation does not explain the appearance of half of the periods for some values of N , such as for 252 and 250 above. Indeed when N is a power of 2 we will always observe the first shoulder at the order of $N - 1$. This is in fact the result of the possibility that there exists an integer k' such that $2^{k'} \pmod{N - 1} = -1$, which implies that $k' = k_0/2$. If there exists such an integer, then we must, according to Shor’s algorithm, choose a different integer (other than 2) to find its order of. That is, we cannot use the order of 2 to find a factor of $N - 1$, which is the ultimate objective. For our analysis, this situation implies that

$$S^{k_0(N-1)/2} \equiv R' = \begin{pmatrix} 1 & & \\ & R_{N-2} & \\ & & 1 \end{pmatrix}, \quad (23)$$

where R_{N-2} is the parity operator with 1 along its secondary diagonal and zero elsewhere. That is, $S^{k_0(N-1)/2}$ is almost the parity operator except that 0 and $N-1$ instead of being interchanged are fixed by S and hence all its powers. To clarify Eq. (23) may or may not hold depending on if $2^{k_0(N-1)/2} \equiv -1 \pmod{N-1}$ or not. For instance, this is never the case if N is a power of 2. This simplified model has also been considered by [29] recently to show that exponential fidelity decay does not necessarily mean a hypersensitivity to perturbations. However, in the context of this paper it is interesting that the model works approximately even when N is not a power of 2 and preliminary results indicate that there is hypersensitivity to perturbations as well [46].

In general, for an initial state $|\psi_0\rangle$ we have that

$$f(t) = |\langle \psi_0 | V_t V_{t-1} \cdots V_1 | \psi_0 \rangle|^2, \quad (24)$$

where V_l is the perturbation in the interaction picture: $V_l = S^{-l} V S^l$. Thus always $V_{k_0(N-1)} = V$ but in the situation where Eq. (23) holds we have that $V_{k_0(N-1)/2} = R'^{-1} V R'$. Thus it is clear that these times are special for the fidelity as seen in the numerical calculations as well. While these arguments along with the approximate model give a fair understanding of the decay, it is not complete and a more detailed analysis of the product above must be carried out, which the author is unable to provide. There is a rather large literature surrounding the so-called Loschmidt echo [47], or fidelity, wherein quantum chaotic systems have been subjected to a small perturbation on reversal. The current discussion is in fact closely related, but previous work has naturally concentrated on the generic case of a nondegenerate operator that is perturbed. In the case of the shift operator, it can be highly degenerate, as well as completely periodic, thereby making it “nongeneric.” It can be compared again to the quantum-chaotic cat maps that are also periodic and degenerate in general. Smooth perturbations of this, for instance, of the type that has been studied before [49] could produce fidelity decays of a similar character, which we have noted here.

C. Case $P = \sigma_z$

In this case,

$$S(\theta; 0, \sigma_z) = F_{2L}^{-1}(0, 0) \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\theta} F_L(0, 0) & \frac{1}{\sqrt{2}} e^{-i\theta} F_L\left(0, \frac{1}{2}\right) \\ \frac{1}{\sqrt{2}} e^{i\theta} F_L(0, 0) & \frac{1}{\sqrt{2}} e^{i\theta} F_L\left(0, \frac{1}{2}\right) \end{pmatrix}. \quad (25)$$

Note that there seems to be only a minor change—namely, those of signs of phases in the Fourier blocks—compared to the first case, and also the classical limit still remains unaltered by the phase-flip perturbation. However, the fidelity does decay even due to the “quantum perturbation” and seems to be of a similar character to that observed when P

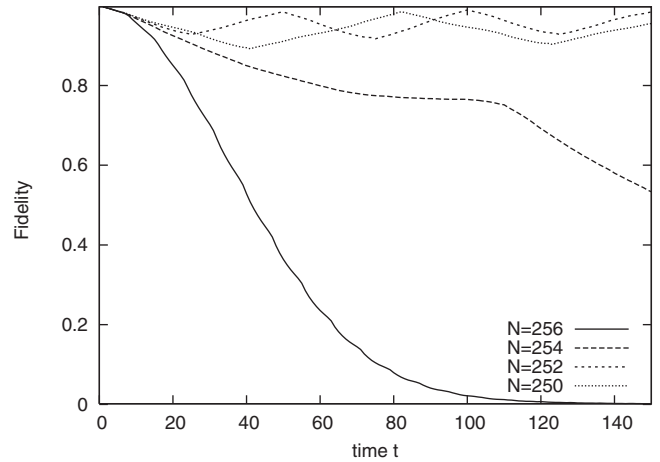


FIG. 5. The fidelity decay for four neighboring values of N . The perturbation is $P = \sigma_z$, the phase $\alpha = 0$, and $\theta = 0.05$.

$= \sigma_y$ —namely, the previous case. The differences start showing up sharply in the case when Eq. (23) holds—namely, when $k_0(N-1)$ is such that $2^{k_0(N-1)} \equiv -1 \pmod{N-1}$. It appears to be generically the case that beyond this time there are large oscillations reminiscent of fidelity decay in near-integrable systems [48]. This is illustrated in Fig. 5 for values of N . In the cases when $N = 252$ and 250 , Eq. (23) holds and we see that beyond a time of half the order there are regular oscillations with this period.

That we must expect that fidelity decay is due to a counterpart of the approximation used in the previous case—namely,

$$F_{2L}^{-1}(\exp(-i\theta\sigma_z) \otimes I_L) F_{2L} \approx I_L \otimes \exp(-i\theta\sigma_y), \quad (26)$$

which follows from

$$F_{2L}^{-1}(\sigma_z \otimes I_L) F_{2L} \approx I_L \otimes \sigma_y, \quad (27)$$

which is a result of combining the identity in Eq. (16) and the approximation in Eq. (22). Thus this final case of perturbation we consider is sort of intermediate between cases A and B; however, for practical purposes it is closer to case B, as the time behavior beyond the time of the order or half the order is not likely to be of interest from the point of view of the Shor algorithm. The approximate formula in Eq. (20) continues to be approximately good for those N for which the condition in Eq. (23) does not hold, and for those for which it does, it is approximately good until a time of half the order at which the oscillations begin.

The fidelity study above is done for one particular model of errors which is a static gate error on only one qubit. There are many other forms of errors in a quantum computation as mentioned earlier, and a detailed study of these is beyond the scope of the present paper. For instance, it is of interest to look at nonstatic errors, as may be easily done with the model above for a gate error. On the other hand, other types of dynamical errors are possible as in random (time-dependent) couplings between the qubits used. In one study

the effects of dynamical versus static errors have been pointed out in the context of fidelity decay in a system of qubits on a two-dimensional lattice [50].

V. DISCUSSION

Three archetypal perturbations (phase-flip, bit-flip, and a combination thereof), which are possible in the critical part of Shor's algorithm—namely, the modular multiplication or exponentiation part—have been studied. Confining ourselves to the simplest possible case when the multiplier is 2, these perturbations can be interpreted in terms of coherent superpositions of quantum baker's maps, whose classical limits are completely chaotic and are models of randomness. Thus it has been shown that there are generic perturbations of the modular exponentiation operator that will qualify as “quantum chaotic.” This has been demonstrated by computing the nearest-neighbor spacing statistics and seeing that it is of the type expected of random matrices. More pertinent to the algorithm itself we have studied the fidelity decay that occurs with the relevant initial state and shown that for the three types of perturbations there are three possible fidelity decay behaviors. This can be interpreted in terms of the fact that some perturbations alter the classical limit while some do not, as well as in arising from some identities (one exact and one approximate) that involve the Pauli spin matrices and the Fourier transform, which, while the author has not seen before, are completely elementary and likely to be known and useful already. A simple model of the fidelity decay is afforded by these identities that describes surprisingly well the

exponential decay in time punctuated by shoulders at times related to the order. An exact solution of the problem seems unlikely, and semiclassical analysis is cumbersome due to the fact that the modular exponentiation (when the multiplier is 2) is essentially the *sum* of two unitary operators with well-defined classical limits.

The precise impact of the exponential fidelity decay on the functioning of the algorithm remains to be seen. Such a study for the case of static imperfections was recently carried out [11]. We have been primarily interested in pointing to the deep and exact relationship between the modular exponentiation part of the Shor algorithm and the quantization of an archetypal model of classical deterministic chaos—namely, the baker's map. If we had larger multipliers than 2, as will generally be the case, it is reasonable to expect that these will be related to generalized baker's maps with more than two partitions [20]. The number of possible stacking are also more, but it is completely conceivable that once again there are perturbations to the modular exponentiation operator that are quantum chaotic and close to the quantization of such baker's maps. That such baker's maps will have larger Lyapunov exponents and have greater classical randomness may make the quantum operators even more susceptible to such gate errors; however, this is at the moment mere speculation.

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