Implementation of quantum operations on single-photon qudits

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We show that a general linear transformation from one single photon qudit to another, the dimension of which can be either equal or unequal to that of the first one, can be implemented by linear optics. As an application of the scheme we elaborate a method to deterministically realize any finite-element positive operator-valued measure on single photon signals, which is also generalizable to any quantum system in principle.

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I. INTRODUCTION

Linear optics is considered as one of the promising candidates for quantum computing (for a recent overview see, e.g., Ref. [1]) and can be also applied to many other areas such as quantum cryptography (for a review see, e.g., Ref. [2]). In these applications an essential technique is the implementation of all possible operations, including generalized quantum measurements in the form of positive operator- valued measures (POVMs), on the signals encoded as photon states by practical linear optics circuits.

A typical and important case of the signal states is single photon qudits, i.e., the linear combinations of the modes $a_k^\dagger|0\rangle$, $k=1,\ldots,N$ (multiple-rail encoding). It was proposed by Reck *et al.* [3] that any unitary operator $\mathcal{U}\in \mathrm{U}(N)$ on the N-dimensional qudits, $\sum_{i=1}^{N} c_i a_i^\dagger|0\rangle$, can be realized by an N-port interferometer, which is an array of beam splitters and phase shifters performing SU(2) elements, because this unitary operator can be decomposed into the product of these SU(2) elements (see Fig. 1). This scheme was further studied in Ref. [4] and has been applied to a variety of research fields in quantum information theory and experiment.

The generalization of the scheme is the implementation of all possible linear maps on single photon qudits, which is a fundamental task in processing quantum information. It is intimately related to the realization of POVMs that are at the heart of many quantum information processing protocols. A finite-element POVM is a set of non-negative operators $\{\Pi_i\}$, where Π_i are its elements, satisfing

$$\sum_{i=1}^{n} \Pi_i = I, \tag{1.1}$$

with I being the identity operator. It has been proved that any rank-one POVM in the form of $\Pi_i = k_i^2|\phi_i\rangle\langle\phi_i|$, where $\langle\phi_i|\phi_j\rangle \neq \delta_{ij}$ and $|k_i|\leqslant 1$, can be realized by the Neumark extension [5], which extends the POVM elements to the orthogonal projectors in a larger space. For the input signals prepared with single photons, such a POVM can be implemented with linear optics circuits, performing unitary transformations, and photon detectors only [6]. The realization of POVMs with arbitrary rank is, however, much more difficult. Since $\Pi_i \geqslant 0$, it can be decomposed into $\Pi_i = A_i^\dagger A_i$ [7]. The general POVM will be implemented if we simultaneously realize the maps

$$\rho_{\mathrm{in}} \longrightarrow \rho_{\mathrm{out},i} = \frac{A_i \rho_{\mathrm{in}} A_i^\dagger}{\mathrm{Tr}(A_i \rho_{\mathrm{in}} A_i^\dagger)}, \tag{1.2}$$

and successfully detect these outputs. The detection operators A_i (and their transforms by an arbitrary unitary operator $A_i' = U_i A_i$) can be any allowed linear map in quantum mechanics with equal dimensional input and output signal space, i.e., an $N\times N$ square matrix.

The detection operators of a POVM are, however, only a special case of more general maps called quantum operations (QOs). A QO connects pairs of input and output states via the map

$$\rho_{\mathrm{in}} \longrightarrow \rho_{\mathrm{out}} = \frac{\mathcal{E}(\rho_{\mathrm{in}})}{\mathrm{Tr}[\mathcal{E}(\rho_{\mathrm{in}})]}. \tag{1.3}$$

$\mathcal{E}$ is a linear, trace-decreasing map that preserves the complete positivity (CP), and generally occurs with nonunit probability $\mathrm{Tr}[\mathcal{E}(\rho_{\mathrm{in}})]\leqslant 1$. The general form of $\mathcal{E}$ is given as [8]

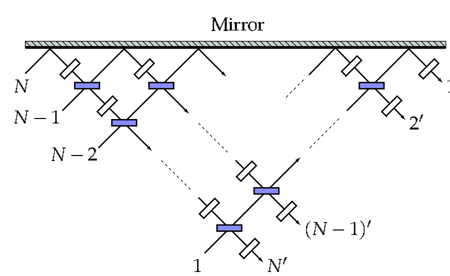

FIG. 1. (Color online) The unitary transformation module constructed with beam splitters (dark rectangles) and phase shifters (white recangles). Any unitary operator U can be decomposed into the product of SU(2) elements implemented by the beam splitters and the phase shifters, and the maximum number of beam splitters needed is $N(N-1)/2$. The input ports are with unprimed numbers and output ports with primed numbers.

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$$\mathcal{E}(\rho_{\rm in}) = \sum_i K_i \rho_{\rm in} K_i^\dagger, \tag{1.4}$$

with the independent Kraus operators K_i satisfying the bound $\Sigma_i K_i^\dagger K_i \leqslant I$. If a QO transforms pure states to pure it is called a pure map. In this case there is only one term $K\rho_{\rm in}K^\dagger$ in the above equation with K being a contraction, i.e., $\|K\| \leqslant 1$. For an isolated pure state input $\rho_{\rm in}=|\psi_{\rm in}\rangle\langle\psi_{\rm in}|$ [9], the QOs in Eq. (1.3) can be therefore written in the form [10]

$$|\psi_{\rm in}\rangle \to |\psi_{\rm out}\rangle = \frac{K|\psi_{\rm in}\rangle}{\|K|\psi_{\rm in}\rangle\|}. \tag{1.5}$$

The output signal of such a linear map, with K as contraction, is detected with a probability $\langle\psi_{\rm in}|K^\dagger K|\psi_{\rm in}\rangle \leqslant 1$. These contractions can be more general than the detection operators of a POVM because the input space dimension N_1 and the output space dimension N_2 of K can be different, so K is an $N_2 \times N_1$ matrix whose entries are complex numbers. The detection operators A_i of a POVM correspond to a special case of contractions when $N_1 = N_2 \equiv N$.

II. IMPLEMENTATION OF NONUNITARY LINEAR MAPS

We now address the problem of how to realize any possible linear map K on single photon qudits with only three unitary operator modules of the kind shown in Fig. 1. We realize K by its unitary dilation $\mathcal{U}$, the unitary operator constructed from K in a larger space, which we obtain by using the direct sum extension of the system with an ancilla $\mathcal{H}_S \oplus \mathcal{H}_A$. In terms of Hilbert space dimensionality, this scheme minimizes the physical resources needed to realize a QO [11]. We embed the state vector $(c_1, c_2, \ldots, c_{N_1})^T$ (T stands for transpose) of the input signal $|\psi_{\rm in}\rangle = \Sigma_{i=1}^{N_1} c_i a_i^\dagger |0\rangle$, into a larger space and map it by $\mathcal{U}$ to a vector containing the state vector of the output $|\psi_{\rm out}\rangle = \Sigma_{i=1}^{N_2} c_i' a_i'^\dagger |0\rangle$ (unnormalized) of K:

$$\begin{pmatrix} c_1' \\ \vdots \\ c_{N_2}' \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathcal{U}_{1,1} & \mathcal{U}_{1,2} & \mathcal{U}_{1,3} & \cdots \\ \mathcal{U}_{2,1} & \mathcal{U}_{2,2} & \mathcal{U}_{2,3} & \cdots \\ \mathcal{U}_{3,1} & \mathcal{U}_{3,2} & \mathcal{U}_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{N_1} \\ \vdots \\ 0 \end{pmatrix}. \tag{2.1}$$

It should be noted that we realize this unitary dilation always with a vacuum state ancilla $(0,0,\cdots,0)^T$ added to the input state vector as a direct sum. Next, we will derive the algorithm to generate the unitary dilation $\mathcal{U}$ of linear map K.

$K^\dagger K$ and $KK^\dagger$ are positive matrices of $N_1 \times N_1$ and $N_2 \times N_2$, respectively. Suppose $N_1 \geqslant N_2$, we choose to diagonalize $KK^\dagger$ by a unitary operator U. We can also construct an $N_1 \times N_1$ unitary matrix V to obtain the singular value decomposition (SVD), $K = U\Sigma V^\dagger$, with the uniquely determined singular values on the diagonal of the $N_2 \times N_1$ matrix Σ. If $N_1 < N_2$, on the other hand, we will choose to diagonalize $K^\dagger K$ to get a similar result.

Then we extend the rectangular matrix Σ to the following $\max(N_1,N_2) \times \max(N_1,N_2)$ square matrix:

$$\Sigma' = \begin{pmatrix} |\sigma_1| & & & & & \\ & \ddots & & & & \\ & & |\sigma_{N_2}| & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \tag{2.2}$$

where the singular values σ_i satisfy $|\sigma_i| \leqslant 1$ since K is a contraction. The extension of Σ in the case of $N_1 < N_2$ also takes the above form except that $|\sigma_{N_2}|$ is replaced by $|\sigma_{N_1}|$. Using the fact that Σ' is still a contraction, we can obtain a $\max(2N_1, 2N_2) \times \max(2N_1, 2N_2)$ unitary dilation

$$G = \begin{pmatrix} \Sigma' & (I - \Sigma'^2)^{1/2} \\ (I - \Sigma'^2)^{1/2} & -\Sigma' \end{pmatrix} \tag{2.3}$$

of it (see, e.g., Ex. I.3.6 in Ref. [7]), which acts on a space $\mathcal{H} \oplus \mathcal{H}$ with $\mathcal{H}$ being $\max(N_1, N_2)$ dimensional. We also extend U and V to $\max(2N_1, 2N_2)$ by $\max(2N_1, 2N_2)$ matrices by adding the identity matrix I in the diagonal and zero matrices off the diagonal. A general linear map K is therefore realized by the following unitary dilation

$$\mathcal{U} = UGV^\dagger. \tag{2.4}$$

In our setup, we perform its equivalence by acting $\mathcal{U}^\dagger$ on the spatial mode vector $(a_1^\dagger, \ldots, a_{N_1}^\dagger)$. The circuits to implement V and $U^\dagger$ are the corresponding N_1-port and N_2-port modules. After the input spatial mode vector is processed by V, we redirect the output to a $\max(2N_1, 2N_2)$-port module of G with the input ports numbered from N_1+1 to $\max(2N_1, 2N_2)$ in Fig. 1 black or a vacuum state. Here is some detail about the step to implement Σ through its unitary dilation G. Picking out the entries containing only one of the singular values σ_i from the matrix of G, we form a 2×2 submatrix, which can be transformed by a rotation $T_{i,i+\max(N_1,N_2)}$ to a diagonal one

$$\begin{pmatrix} |\sigma_i| & (1-\sigma_i^2)^{1/2} \\ (1-\sigma_i^2)^{1/2} & -|\sigma_i| \end{pmatrix} \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

With a series of rotations in the form of $T_{i,i+\max(N_1,N_2)} \otimes I_{\rm rest}$, for $i = 1, \ldots, \min(N_1, N_2)$, G can be realized by $\min(N_1, N_2)$ beam splitters with the reflection coefficients $R = 1 - \sigma_i^2$ and phase shifters giving rise to $e^{i\pi}$. Therefore, the upper bound of the total number of the beam splitters required in the scheme is

$$N_{\max} = \frac{N_1^2}{2} + \frac{N_2^2}{2} - \left|\frac{N_1}{2} - \frac{N_2}{2}\right|, \tag{2.6}$$

which is determined by the dimensions of the input and output Hilbert spaces. In the whole extended space, we will obtain two outputs after the action of the three unitary operator modules: one is the exact output $(a_1'^\dagger, \ldots, a_{N_2}'^\dagger)$ of the linear map K from the output ports of $U^\dagger$, and the other is an extra output $(a_{N_2+1}'^\dagger, \ldots, a_{\max(2N_1,2N_2)}'^\dagger)$ from the output ports of G numbered from $(N_2+1)'$ to $[\max(2N_1, 2N_2)]'$. Figure 2

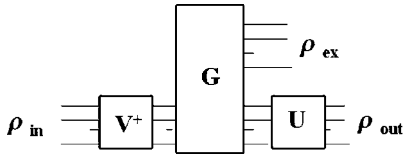

FIG. 2. The circuit to perform the unitary dilation $\mathcal{U}$ of K, the general linear transformation on a pure state input $\rho_{\rm in}=|\psi_{\rm in}\rangle\langle\psi_{\rm in}|$. We will obtain two outputs, $\rho_{\rm out}=|\psi_{\rm out}\rangle\langle\psi_{\rm out}|$ of K and an extra $\rho_{\rm ex}=|\psi_{\rm ex}\rangle\langle\psi_{\rm ex}|$, from the corresponding terminals. Because the linear map K is not generally inversable, we need to add an ancilla $|\psi_{\rm ex}\rangle$ to $|\psi_{\rm out}\rangle$, i.e., $|\psi'_{\rm in}\rangle=|\psi_{\rm out}\rangle\oplus|\psi_{\rm ex}\rangle$, if we are to convert it back to the original $|\psi_{\rm in}\rangle$ by using the same circuit from the inverse direction.

displays the scheme that realizes the effect of K on the input state $\rho_{\rm in}$.

This linear optics scheme can be directly applied to where we need nonunitary transformation on photon states, e.g., in the production of single photon qudits in any form of $\Sigma_i c_i a_i^\dagger|0\rangle$, where $\Sigma_i|c_i|^2\leqslant 1$ (possibly unnormalized), by multiple-rail encoding, and in the enhancement of the entanglement of a pair of partially entangled photons such as $\Sigma_i c_i a_i^\dagger|0\rangle_1 a_i^\dagger|0\rangle_2$, with different $|c_i|$ unequal, by one party operation.

III. IMPLEMENTATION OF A GENERAL POVM

Now we look in some detail at the realization of POVMs as one important application of our scheme. We start with the simplest situation of $n=2$, where the two POVM elements are always commutative, $[\Pi_1,\Pi_2]=0$. Suppose that the dimension of the signal space is N, and the $N\times N$ detection operators A_i of the POVM can be factorized by SVD as $A_i=V_i\Sigma_i U_i$ with U_i, V_i unitary and Σ_i diagonal. We first set up an N-port module for U_1 and, after the signal leaving the U_1 module, we process it with a $2N$-port module to implement the unitary dilation of Σ_1. From its output ports numbered from $1'$ to N' we get an output $|\psi^1_{\rm mid}\rangle\sim\Sigma_1 U_1|\psi_{\rm in}\rangle$, while from the ports numbered from $(N+1)'$ to $2N'$ another output $|\psi^2_{\rm mid}\rangle\sim\Sigma_{1C}U_1|\psi_{\rm in}\rangle$, with $\Sigma^2_{1C}=I-\Sigma^2_1$. Then we will just redirect them to modules of V_1 and V_2 and finally obtain the outputs $A_1|\psi_{\rm in}\rangle/\|A_1|\psi_{\rm in}\rangle\|$ or $A_2|\psi_{\rm in}\rangle/\|A_2|\psi_{\rm in}\rangle\|$ from the corresponding terminals.

For a POVM with the number of elements $n\geqslant 3$, the situation is much trickier. Instead of Π_2, what we realize from the corresponding output ports of $|\psi^2_{\rm mid}\rangle$ is the operator $I-\Pi_1$. By the diagonalization, all elements of a general POVM can be factorized into $\Pi_i=U_i^\dagger\Sigma_i^2 U_i$, where the different U_i do not generally commute, i.e., $[U_i,U_j]\neq 0$ for $i\neq j$. In the realization of Π_2, therefore, we need to consider two different situations: (1) If $\|\Pi_1\|<1$ [12], because $I-\Pi_1>\Pi_2$ when $n\geqslant 3$, we can find a diagonal matrix Σ_2^* with $\|\Sigma_2^*\|\leqslant 1$ [13] and a unitary operator U_{2L} such that

$$\Pi_2=U_1^\dagger\Sigma_{1C}U_{2L}^\dagger\Sigma_2^{*2}U_{2L}\Sigma_{1C}U_1=U_2^\dagger\Sigma_2^2U_2. \tag{3.1}$$

These two matrices Σ_2^* and U_{2L} are obtained by a standard diagonalization procedure following the above equation.

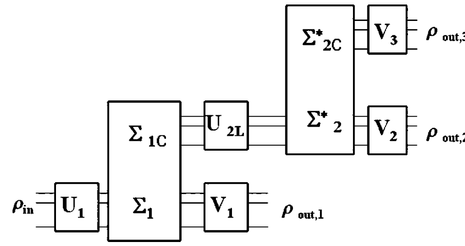

FIG. 3. The setup to perform any POVM with three elements. The seven unitary operator modules, two of which perform the pairs of operators Σ_1 and $\Sigma_{1C}=(I-\Sigma_1^2)^{1/2}$, Σ_2^* and $\Sigma_{2C}^*=(I-\Sigma_2^{*2})^{1/2}$, respectively, are designed with the POVM elements. The detectors at the terminals effect a dephasing to eliminate the interference between $A_i|\psi_{\rm in}\rangle$ [16], and capture the outputs $\rho_{{\rm out},i}=A_i\rho_{\rm in}A_i^\dagger/{\rm Tr}(A_i\rho_{\rm in}A_i^\dagger)$, for $i=1,2,3$, with the probabilities $p_i={\rm Tr}(A_i\rho_{\rm in}A_i^\dagger)$. The outputs redirected to only one set of detectors form a probability distribution as the mixture [18], $\Sigma_i p_i\rho_{{\rm out},i}=\Sigma_i A_i\rho_{\rm in}A_i^\dagger$, there.

Then, after performing U_{2L} with an N-port module, Σ_2^* as a contraction map can be implemented by a $2N$-port linear optics module with at most N beam splitters. To realize A_2 completely, we add one more module of a proper V_2. (2) If $\|\Pi_1\|=1$, after the signal goes through the part of the circuit implementing $I-\Pi_1$, some of the output ports will be black because the corresponding components have been projected out by Π_1 [14]. Then we will just inverse the remaining $(N-D)\times(N-D)$ nonzero part of the Σ_{1C} matrix, where D is the multiplicity of the unit eigenvalue of Π_1, in finding U_{2L} and Σ_2^* of this size in Eq. (3.1).

Repeating the above procedure from the output ports where the operator $I-\Pi_1-\Pi_2$ is realized, we add all the corresponding modules performing U_{nL}, Σ_n^*, etc., for $n\geqslant 3$, to implement the remaining A_3, $A_4,\ldots$, A_n, respectively. The total number of modules of Fig. 1 needed in our scheme to realize a general POVM with n elements is $3n-2$. As an illustration of this general scheme, Fig. 3 shows the setup to perform any POVM with three elements.

We have reduced the problem of realizing a POVM to that of finding a sequence of unitary operators with the POVM elements and realizing them with the ancilla states of vacuum and then detecting the outputs with the standard projective measurements on the extended space. The algorithm to obtain these unitary operators is given, and the implementation of any POVM, with elements of arbitrary rank, can be therefore realized for single photon input signals. Using this method, we will directly obtain the output states of a POVM, which can be tailored by choosing the appropriate V_i modules, from the corresponding terminals where the signal detectors are placed. If our signals are just single photon polarization qubits, $|\psi_{\rm in}\rangle=c_1|H\rangle+c_2|V\rangle$ (polarization modes H and V), we can use much a simpler circuit to implement any POVM on them as in Ref. [15], where a POVM is realized as the decomposition of an identity operator but the necessary algorithm to obtain, e.g., Σ_n^*, U_{nL} for the implementation of all specified Π_i is not given. Given the beyond-linear-optics methods to implement unitary operations on more complicated quantum systems than single photon states, this scheme can be applied to more general situations of photonic states as well as the signals of any other type of radiation.

IV. IMPLEMENTATION OF A GENERAL QUANTUM OPERATION OR CP MAP

In the whole extended output space $\mathcal{H}_T$, the overall output of the unitary map $\mathcal{U}$ implemented by the POVM circuit is $|\psi'_{out}\rangle=((A_1|\psi_{in}\rangle)^T,\ldots,(A_n|\psi_{in}\rangle)^T)^T$. Applying a proper dephasing map [16] to this output as $\mathcal{D}(\rho'_{out})$, we will obtain a direct sum of $A_i\rho_{in}A_i^\dagger$ in $\mathcal{H}_T$, which can be transformed by a contraction map $\mathcal{L}$ [17] to $\mathcal{W}(\rho_{in})=(\Sigma_{i=1}^n A_i\rho_{in}A_i^\dagger)/n$ in one subspace of $\mathcal{H}_T$. The other $n-1$ pieces of such outputs will be obtained in some other subspaces if we extend $\mathcal{L}$ to a unitary operator. The output state of the total map $\mathcal{W}$ in one of the subspaces, normalized as in Eq. (1.3), is $\rho_{out}=\Sigma_i A_i\rho_{in}A_i^\dagger$ of a QO mapping $|\psi_{in}\rangle$ from a pure to mixed state. A general QO can be therefore realized by a corresponding combined map $\mathcal{W}$. To a single set of detectors that effect a dephasing map by detecting $A_i|\psi_{in}\rangle$ coming from different terminals, what is being measured is effectively the output of a general QO (see the caption of Fig. 3).

V. CONCLUSION

In summary, we have presented the linear optics schemes (including the photon detection) to realize all QOs and POVMs on a single photon qudit. The circuits to perform all the relevant tasks are only the combinations of some scalable unitary operator modules which have been widely applied in quantum information processing. Given current technologies, our schemes can realize all linear transformations and POVMs on single photon signals in a deterministic way.

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[9] The input does not couple to the environment ρ_E or any other system to evolve as the tensor product $\rho_{in}\otimes\rho_E$. A photonic state, whose coupling to the environment is negligibly weak, can be regarded as an isolated system.

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[12] For the Hermitian operators Π_i, we simply have $\|\Pi_i\|=\max\{|\lambda_k|:\Pi_i|\phi_k\rangle=\lambda_k|\phi_k\rangle,k=1,\ldots,N\}$.

[13] Since $\|UAV\|=\|A\|$ for an arbitrary linear operator A and all unitary matrices U and V, we obtain the following from Eq. (1.2):

$$\|\Sigma_2^*\|=\|\Sigma_2^*U_{2L}U_1\|=\|\Pi_2^{1/2}(I-\Pi_1)^{-1/2}\|\leq 1,$$

here we have used Lemma V.1.7 in Ref. [7] with the existence of $(I-\Pi_1)^{-1}$ due to the fact that $\|\Pi_1\|<1$.

[14] Suppose one of the POVM elements, Π_j, has a unit norm, so its diagonalized form Σ_j^2 has some entries 1. From Eq. (1.1), on the other hand, we have

$$\Sigma_j^2+\sum_{i\neq j}U_jU_i^\dagger\Sigma_i^2U_iU_j^\dagger=I,$$

and then we find that the corresponding entries of Σ_i^2, for all $i\neq j$, are 0.

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$$\mathcal{L}=\frac{1}{\sqrt{2}}\begin{pmatrix} O & O \\ I & I \end{pmatrix},$$

with $\|\mathcal{L}\|=1$. The combined map,

$$\mathcal{W}(\rho_{in})=\mathcal{P}\mathcal{L}\mathcal{D}(\mathcal{U}\rho_{in}\mathcal{U}^\dagger)\mathcal{L}^\dagger\mathcal{P}=\frac{1}{2}(A_1\rho_{in}A_1^\dagger+A_2\rho_{in}A_2^\dagger),$$

with $\mathcal{P}$ being the projection onto a subspace, outputs a mixed state.

[18] See, e.g., J. Watrous, CPSC 701 Lecture Notes: Theory of Quantum Information, University of Waterloo (unpublished internet notes, 2004).