Entanglement invariant for the double Jaynes-Cummings model

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(Received 26 June 2007; published 8 October 2007)

We study entanglement dynamics between four qubits interacting through two isolated Jaynes-Cummings Hamiltonians, via an entanglement measure based on the wedge product. We compare the results with similar results obtained using bipartite concurrence resulting in what is referred to as "entanglement sudden death." We find a natural entanglement invariant under evolution, demonstrating that entanglement spreads out over all of the system's degrees of freedom that become entangled through the interaction. We also provide an analysis of why certain initial states lose all their entanglement in a finite time, although their excitation and coherence vanish only asymptotically with time.

DOI: 10.1103/PhysRevA.76.042313 PACS number(s): 03.67.Mn, 03.65.Ud

I. INTRODUCTION

Entanglement plays a key role in quantum-information processes [1] and therefore it is important to study entanglement dynamics in different scenarios. The simplest situation, two-qubit entanglement dynamics, has been extensively studied in different contexts [2–12].

When interacting with a reservoir, one would naively expect the entanglement between the two qubits to vanish asymptotically. However, for certain initial entangled states, the entanglement can vanish completely in a finite time. This is often referred to as "entanglement sudden death;" see, for example, [3–9] and the references therein. A recent experimental demonstration was presented in Ref. [10] and an open-systems analysis was done in Refs. [3–7]. Nevertheless, if one were to include the "reservoirs" in the studied system and consider the full entanglement between two noninteracting partitions of the system, one would expect the entanglement to be preserved, and therefore an associated entanglement invariant should exist. In addition, considering the reservoir as part of closed system until the moment it is ignored (and in mathematical terms "traced over"), one can reap insights into both the qualitative and quantitative transfer of excitation (and associated entanglement) from the atoms to the reservoir.

An important bipartite interaction is described by the Jaynes-Cummings (JC) model [13] that describes, in a concise and elegant way, the near-resonant interaction between a single two-level atom and a single-mode quantized field. If the initial atom-field system contains a single excitation, the system is a model of a two-qubit system. Since the JC model is excitation number preserving, the system will always stay within the two-qubit Hilbert space (but of course such a JC model spans only the one-excitation subspace of the full two-qubit space). However, since this model is one of the few exactly solvable models in quantum physics it has been exploited for studying the dynamics of entanglement [14].

Recently, a double JC model has been proposed in this context [8,9]. The model consists of two separate JC-model systems (atom A interacting only with the cavity field a and similarly for the atom B and the field b), where it has been assumed that the systems are identical. (Note that this model is applicable to any one-excitation, two-qubit system that is linearly coupled, e.g., to the experiment in Ref. [10] where fields couple pairwise to each other rather than atoms to fields.) A major reason this particular interaction has been chosen is because it is local to subsystems Aa and Bb. If the atoms, or the fields, couple to each other, the coupling will alter the entanglement between the system partitions Aa and Bb in general, and subsequently, if the fields are traced over, between A and B. The whole point with these studies, however, is to study the entanglement dynamics between A and B in absence of any coupling, direct or indirect, between them.

The focus of interest has been the pairwise entanglement dynamics in terms of the concurrence [15] between the initially entangled atoms. Through the JC interaction they may become unentangled through the excitation transfer to the initially unexcited fields which are traced over after the interaction. In Ref. [8], in particular, the authors study entanglement between the two atoms, and they find that for the initial state $|\psi(0)\rangle=\cos\alpha|\uparrow\downarrow\rangle+\sin\alpha|\downarrow\uparrow\rangle$, where we have used the notation $|\uparrow\rangle_A\otimes|\downarrow\rangle_B=|\uparrow\downarrow\rangle$, etc., and $|\uparrow\rangle$ ($|\downarrow\rangle$) denotes the atom's excited (ground) state, the concurrence $\mathcal{C}_{AB}$ behaves in a harmonic oscillatory manner. Translating this into a dissipation language, the entanglement vanishes asymptotically with increasing coupling to the reservoir. In contrast, the concurrence $\mathcal{C}_{AB}$ of the state $|\phi(0)\rangle=\cos\alpha|\uparrow\uparrow\rangle+\sin\alpha|\downarrow\downarrow\rangle$ for certain values of α, specifically $|\tan\alpha|<1$, falls rapidly and nonsinusoidally to zero in a finite time and remains zero for some time. In a dissipation language this means that the entanglement will vanish in a finite time although both the atomic excitation and the atomic coherence decay asymptotically.

In order to study the transfer of entanglement between the atom and the reservoir the authors extend the work in Ref. [8] and study all the six concurrences $\mathcal{C}_{AB}$, $\mathcal{C}_{Aa}$, $\mathcal{C}_{Bb}$, $\mathcal{C}_{ab}$, $\mathcal{C}_{Ab}$, and $\mathcal{C}_{Ba}$ for the four-qubit system in Ref. [9]. They find that for the state $|\psi(0)\rangle$ the sum of the concurrences between the atoms and the fields, $\mathcal{C}_{AB}+\mathcal{C}_{ab}$, is constant under the JC evolution. This is not so for the state $|\phi(0)\rangle$, but another function of the six pairwise concurrences and the initial state (parametrized by α), $\mathcal{C}_{AB}+\mathcal{C}_{ab}+(\mathcal{C}_{Aa}+\mathcal{C}_{Bb})|\tan\alpha|-(\mathcal{C}_{Ab}+\mathcal{C}_{Ba})$, is an entanglement invariant.

In Ref. [12] the dynamics for the initial state $|\psi(0)\rangle$ in an equivalent model, two separately systems composed by two two-level systems in a dipolelike interaction, is considered.

Here, the authors consider the effect of different coupling constants in the separately systems. However, the pairwise entanglement dynamics is expressed in terms of the negativity [16] and its relation to with the energy transfer.

Motivated by these studies, and in the hope that a more general entanglement invariant can be found to illuminate the transfer of entanglement to the system's different parts, we study the same four-qubit system, where we do not assume that the systems are identical. To be able to consider all possible bipartite entanglement we use an entanglement measure introduced by Heydari in Ref. [17], which is based on the wedge product. We take all the different possible partitions of the four-qubit system into account, and we find an entanglement invariant which does not depend on the system parameters or on the initial state, provided that it belongs to a class of pure states denoted X states in Ref. [5], including both the states $|\psi(0)\rangle$ and $|\phi(0)\rangle$. The invariant shows that the fields become entangled with all other parts of the system, all of which is "destroyed" (or rather ignored) when treating the fields as reservoirs.

II. THE MODEL

Consider a model consisting of two two-level atoms A and B, each interacting with a single-mode near-resonant cavity field, denoted a and b, respectively. Following Refs. [8,9], we will assume that each atom-cavity system is isolated and that the cavities are initially in the unexcited state while the atoms are initially in an entangled state. The dynamics of this model is given by the double JC Hamiltonian

$$\hat{H}_{\text{tot}} = \hat{H}_A + \hat{H}_B, \tag{1}$$

where the Hamiltonians (under the rotating-wave approximation and setting $\hbar=1$) are [13]

$$\hat{H}_k = \nu_k(\hat{a}_k^\dagger \hat{a}_k + 1/2) + \frac{\omega_k}{2}\hat{\sigma}_z^k + g_k(\hat{a}_k^\dagger \hat{\sigma}_-^k + \hat{a}_k \hat{\sigma}_+^k), \tag{2}$$

where $k=A,B$ (where the letter case is to be interpreted as appropriate), ν_k is the field frequency, ω_k is the transition frequency between the atomic excited and ground states, and g_k is the coupling constant between the cavity field and the atom. The field annihilation operators are $\hat{a}_k$, and $\hat{\sigma}_\pm^k$ are the spin-flip operators defined by $\hat{\sigma}_-^k|\uparrow\rangle_k=|\downarrow\rangle_k$, $\hat{\sigma}_-^k|\downarrow\rangle_k=\hat{\sigma}_+^k|\uparrow\rangle_k=0$, $\hat{\sigma}_+^k|\downarrow\rangle_k=|\uparrow\rangle_k$, and $\hat{\sigma}_z^k$ is the atomic inversion operator, viz., $\hat{\sigma}_z^k|\uparrow\rangle_k=|\uparrow\rangle_k$ and $\hat{\sigma}_z^k|\downarrow\rangle_k=-|\downarrow\rangle_k$. As mentioned in the Introduction, if we have at most one excitation in each atom-cavity system, each such system will stay within a two-qubit space. Hence, since the two atom-cavity systems do not interact, the double JC model will result in a four-qubit state (but again, spanning only a subspace of the whole four-qubit Hilbert space).

The corresponding evolution operator for the Hamiltonian (2) is

$$\hat{U}_k = e^{-it\hat{H}_k^0}\left[\cos(\hat{\Omega}_k t) - it\,\text{sinc}(\hat{\Omega}_k t)\left(\frac{\Delta_k}{2}\hat{\sigma}_z^k + g_k(\hat{a}_k^\dagger\hat{\sigma}_-^k + \hat{a}_k\hat{\sigma}_+^k)\right)\right], \tag{3}$$

where $\hat{H}_k^0=\nu_k[\hat{a}_k^\dagger\hat{a}_k+((1+\hat{\sigma}_z^k)/2)]$ is a constant of motion, proportional to the total number of excitations of system k, $\Delta_k=\omega_k-\nu_k$ is the detuning between the atom and the cavity for each system, $\text{sinc}(x)\equiv x^{-1}\sin(x)$, and

$$\hat{\Omega}_k = \{g_k^2[a_k^\dagger a_k + (1+\hat{\sigma}_z^k)/2] + \Delta_k^2/4\}^{1/2}.$$

We will consider that the atoms are initially in an X state, characterized by a (reduced) atom density operator whose nonzero elements are found only in the main diagonal and antidiagonal in the basis $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$. This class of atom states has the property that the corresponding two-qubit density matrix preserves the X form when evolving under the action of certain system dynamics [5,7–9]. In this case [8,9], the reason is simple. When an atom transfers its excitation to the initially empty field, it leaves a signature in terms of the excitation in the field. Therefore, there cannot exist any coherence between the states $|\uparrow\uparrow\rangle$ and the states $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ unless such coherence existed initially. This is not the case, by definition, for the X states, and therefore they will retain their X form under the assumed evolution whose form was motivated in the Introduction.

As pointed out in Ref. [5], the X class of states include the Bell states and the Werner states. Following Refs. [8,9] we will focus on the Bell-like pure states

$$|\phi(0)\rangle = \cos\alpha|\uparrow\uparrow\rangle + \sin\alpha e^{i\beta}|\downarrow\downarrow\rangle,$$

$$|\psi(0)\rangle = \cos\alpha|\uparrow\downarrow\rangle + \sin\alpha e^{i\beta}|\downarrow\uparrow\rangle,$$

where $0\leq\alpha\leq\pi/2$, $0\leq\beta\leq\pi$. (In Sec. V we will consider more general states.) As motivated above, we will assume that the initial state for the four qubit model is

$$|\Phi(0)\rangle = |\phi(0)\rangle\otimes|00\rangle, \quad |\Psi(0)\rangle = |\psi(0)\rangle\otimes|00\rangle, \tag{4}$$

respectively, where the abbreviated notation $|0\rangle_a\otimes|0\rangle_b=|00\rangle$ has been used. Notice that Bell states can be recovered by setting $\alpha=\pi/4$ and $\beta=0,\pi/2$. The initial states (4) under the action of the operator $\hat{U}_A\otimes\hat{U}_B$ evolve as

$$|\Phi(t)\rangle = x_1|\uparrow\uparrow 00\rangle + x_2|\uparrow\downarrow 01\rangle + x_3|\downarrow\uparrow 10\rangle + x_4|\downarrow\downarrow 11\rangle + x_5|\downarrow\downarrow 00\rangle, \tag{5}$$

$$|\Psi(t)\rangle = y_1|\uparrow\downarrow 00\rangle + y_2|\downarrow\uparrow 00\rangle + y_3|\downarrow\downarrow 10\rangle + y_4|\downarrow\downarrow 01\rangle, \tag{6}$$

where the coefficients for the state (5) are given by

$$x_1 = f_A(t)f_B(t)\cos\alpha, \tag{7}$$

$$x_2 = f_A(t)g_B(t)\cos\alpha, \tag{8}$$

$$x_3 = g_A(t)f_B(t)\cos\alpha, \tag{9}$$

$$x_4 = g_A(t)g_B(t)\cos\alpha, \tag{10}$$

$$x_5 = h_A(t)h_B(t)e^{i\beta}\sin\alpha. \tag{11}$$

The functions $f_k(t)$, $g_k(t)$, and $h_k(t)$ are given by

$$f_k(t) = e^{-i\nu_k t}\left(\cos(\Omega_k t) - i\frac{\Delta_k}{2\Omega_k}\sin(\Omega_k t)\right), \tag{12}$$

$$g_k(t) = -i\frac{g_k}{\Omega_k}e^{-i\nu_k t}\sin(\Omega_k t), \tag{13}$$

$$h_k(t) = e^{i\Delta_k t/2}, \tag{14}$$

where the Rabi frequencies are $\Omega_k=(g_k^2+\Delta_k^2/4)^{1/2}$.

Similarly, the state (6) will have the the coefficients

$$y_1 = f_A(t)h_B(t)\cos\alpha, \tag{15}$$

$$y_2 = h_A(t)f_B(t)e^{i\beta}\sin\alpha, \tag{16}$$

$$y_3 = g_A(t)h_B(t)\cos\alpha, \tag{17}$$

$$y_4 = h_A(t)g_B(t)e^{i\beta}\sin\alpha. \tag{18}$$

III. ENTANGLEMENT DYNAMICS

In this section we will analyze the entanglement evolution in the double JC model. As an entanglement measure we will use a wedge-product-based measure introduced in Ref. [17], which, for the two-qubit case, coincides with the well-known concurrence [15], and in the multiqubit case with the entanglement monotones [18]. This measure is defined for any number of subsystems, each having an arbitrary, but finite, dimension.

By partitioning the total system into two we can compute the entanglement between these partitions. Consider some partition composed by P_1 with dimension M and P_2 with dimension N (note that each partition could contain more than one physical subsystem). Assume a pure system defined by

$$|\psi\rangle = \sum_{m=1}^{M}\sum_{n=1}^{N}\alpha_{mn}|m\rangle\otimes|n\rangle, \tag{19}$$

where $\{|m\rangle\}$ and $\{|n\rangle\}$ are orthonormal bases. In order to estimate the entanglement between partitions P_1 and P_2, we project $|\psi\rangle$ onto the basis states of one of the partitions. To this end we define the unnormalized state

$$|\psi_m\rangle = \langle m|\psi\rangle.$$

If the system can be written as a tensor product between a pure state in each partition, then all states $|\psi_m\rangle$ are parallel. That is, $|\psi_m\rangle=c_m|\psi_1\rangle$ for all $m=1,\ldots,M$, where c_m denotes a c number. If, on the other hand, the pure state (19) is entangled, then at least two of the vectors, say $|\psi_m\rangle$ and $|\psi_l\rangle$, are not parallel, and the degree to which they are not parallel is characterized by the "area" the vectors span. This area is given by the wedge product between the vectors, but as the wedge product, in general, is signed and complex, we take the absolute square of the area as a measure of the nonseparability between these two vectors. The square of the measure introduced in Ref. [17] can hence be written as the determinant

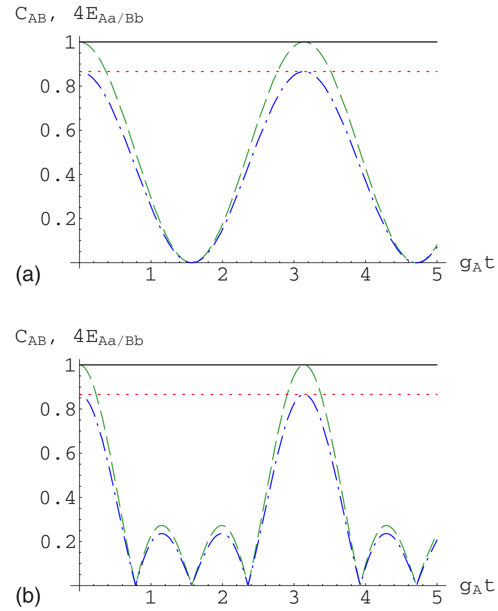

FIG. 1. (Color online) (a) The concurrence $\mathcal{C}_{AB}$ between the atoms in state $|\Psi(0)\rangle$ as a function of the coupling parameter $\Omega_A t$ when $g_A/g_B=1$ for $\alpha=\pi/4$ (green, dashed line), $\pi/6$ (blue, dash-dotted line); and $4E_{Aa\text{-}Bb}$ for $\alpha=\pi/4$ (black, solid line) and $\pi/6$ (red, dotted line). (b) The concurrence when $g_A/g_B=2$ for the parameter values $\alpha=\pi/4$ (green, dashed line) and $\pi/6$ (blue, dash-dotted line); and $4E_{Aa\text{-}Bb}$ for $\alpha=\pi/4$ (black, solid line) and $\pi/6$ (red, dotted line).

$$\mathcal{A}^2(m,l) = \begin{vmatrix} \langle\psi_m|\psi_m\rangle & \langle\psi_m|\psi_l\rangle \\ \langle\psi_l|\psi_m\rangle & \langle\psi_l|\psi_l\rangle \end{vmatrix}.$$

Summing all contributions and using symmetry and the fact that the wedge product between a vector and itself vanishes, the entanglement between P_1 and P_2 can finally be defined

$$E_{P_1\text{-}P_2} = \frac{1}{2}\sum_{m=1}^{M}\sum_{l=1}^{M}\mathcal{A}^2(m,l). \tag{20}$$

In the case of the double JC model, the possible partitions are (a) one qubit–three qubit partitions A-Bab, B-Aab, a-ABb, and b-ABa; (b) two qubit–two qubit partitions Aa-Bb, Ab-Ba, and AB-ab.

In Figs. 1 and 2 we plot the evolution of the concurrence $\mathcal{C}_{AB}$ between the two atoms A and B, and the evolution of $4E_{Aa\text{-}Bb}$ (since $0\leqslant\mathcal{C}_{AB}\leqslant 1$, we use $4E_{Aa\text{-}Bb}$ to scale it to essentially the same range of values as $\mathcal{C}_{AB}$ may obtain) in the case of exact resonance ($\Delta_A=\Delta_B=0$). Note that, while $\mathcal{C}_{AB}$ represents only the remaining entanglement between the two atoms after the field states have been traced out, $E_{Aa\text{-}Bb}$ represents the entire bipartite entanglement between the atom-field systems Aa and Bb.

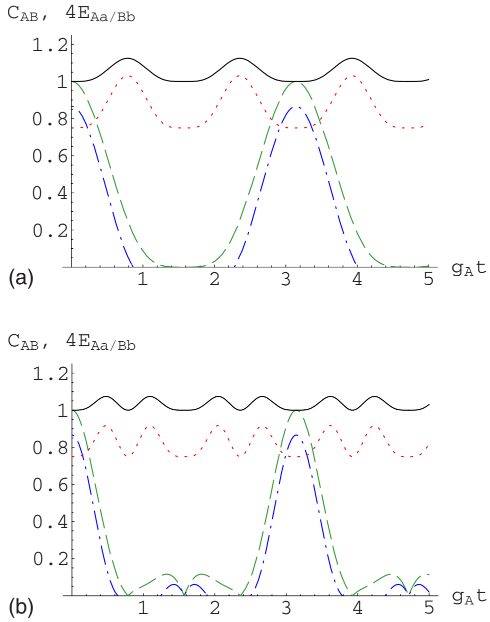

FIG. 2. (Color online) (a) The concurrence $\mathcal{C}_{AB}$ between the atoms in state $|\Phi(0)\rangle$ as a function of the coupling parameter $\Omega_A t$ when $g_A/g_B=1$ for $\alpha=\pi/4$ (green, dashed line), $\pi/6$ (blue, dash-dotted line); and $4E_{Aa\text{-}Bb}$ for $\alpha=\pi/4$ (black, solid line) and $\pi/6$ (red, dotted line). (b) The concurrence when $g_A/g_B=2$ for the parameter values $\alpha=\pi/4$ (green, dashed line) and $\pi/6$ (blue, dash-dotted line); and $4E_{Aa\text{-}Bb}$ for $\alpha=\pi/4$ (black, solid line) and $\pi/6$ (red, dotted line).

First we consider different values of α. In Fig. 1(a), corresponding to $g_A=g_B$, we can observe that concurrence for the initial state $|\Psi(0)\rangle$ evolves in a typical oscillating way between 0 and 1 [8,9,12]; meanwhile $E_{Aa\text{-}Bb}=\sin^2\alpha\cos^2\alpha$ is invariant and equals 1/4 of the initial concurrence squared. In Fig. 1(b) we show the time evolution of the concurrence between the atoms, and that of $4E_{Aa\text{-}Bb}$, for $g_A/g_B=2$. When $g_A\neq g_B$ the concurrence is not evolving in a typical oscillatory manner as was pointed out in Ref. [12]. Meanwhile, $E_{Aa\text{-}Bb}$ depends only on the initial state, namely, on α, and not on the ratio g_A/g_B.

Figure 2 shows the evolution of the concurrence and $4E_{Aa\text{-}Bb}$ for the state $|\Phi(0)\rangle$ when both cavity-atom systems are in exact resonance. In Fig. 2(a) we can observe the so-called entanglement sudden death for different values of α. (In the next section we shall discuss this phenomenon in more detail.) We also see that $E_{Aa\text{-}Bb}$ is no longer constant but increases initially and oscillates with time. However, the oscillation amplitude decreases as $\alpha\rightarrow\pi/2$. In Fig. 2(b) we show the effect of a different ratio between g_A and g_B, and we notice that $E_{Aa\text{-}Bb}$ evolves in a different way for different ratios.

The difference between the evolution of the two states arises because state $|\Psi(0)\rangle$ is evolving simultaneously in two closed manifolds, one consisting of the one-excitation, subsystem Aa manifold (consisting of states $|\uparrow\downarrow 00\rangle$ and $|\downarrow\downarrow 10\rangle$), the other consisting of the one-excitation, subsystem Bb manifold (consisting of states $|\downarrow\uparrow 00\rangle$ and $|\downarrow\downarrow 01\rangle$). On the other hand, the state $|\Phi(0)\rangle$ is evolving only in one manifold, consisting of the states $|\uparrow\uparrow 00\rangle$, $|\uparrow\downarrow 01\rangle$, $|\downarrow\uparrow 10\rangle$, and $|\downarrow\downarrow 11\rangle$. State $|\Phi(0)\rangle$ also has a ground-state component $|\downarrow\downarrow 00\rangle$ but this state does not evolve.

IV. DISSIPATIVE DYNAMICS

While the model presented above is closed and does not involve any dissipation, the atoms' evolution under dissipation can still be described. As can be seen from Eqs. (7)–(10), (12), (13), and (15)–(18), the excitation of the atoms is transferred to the fields in a monotonic fashion during the time interval where $0\leq\Omega_k t\leq\pi/2$. In the resonant case ($\Delta_A=\Delta_B=0\Rightarrow\Omega_k=g_k$), all the excitation will be transferred from the atoms to the cavity fields. Formally, during this time interval, one can then see the fields as a dissipative channel for the atoms' excitation. On can subsequently map a dissipative evolution, e.g., spontaneous emission of the atom's excitation obeying an exponential decay proportional to $\exp(-\gamma_k t')$, onto the JC dynamics by the identification between the times t and t': $\exp(-\gamma_k t')=\cos^2(\Omega_k t)$. Quite obviously, $\Omega_k t\rightarrow\pi/2$ correspond to $t'\rightarrow\infty$. Hence, if the entanglement between the atoms (after the fields are traced out) become zero in a time $\tau<\pi/(2\Omega_k)$, then in the dissipative picture it vanishes in a finite time $t'=\gamma^{-1}\ln[\cos^{-2}(\Omega_k\tau)]$. This is indeed what happens for the state $|\phi(0)\rangle$ as it evolves.

One may then ask why one state's entanglement vanishes in finite time while the other's does not. The reason is the fundamentally different way the states decay. The state $|\psi(0)\rangle$ decays directly into the ground state $|\downarrow\downarrow\rangle$. Whatever excitation is left in the atoms will still be in a superposition state, and such a statistical mixture between a Bell state and the ground state cannot be written as a convex sum of any separable states, no matter to what extent the state has decayed. The state $|\phi(0)\rangle$ decays to the ground state via the intermediate states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$. As the decay leaves different "signatures" in the reservoirs (the states $|01\rangle$ and $|10\rangle$, respectively), no coherence between these states is established. When the excitation of these intermediate states is large compared to the remaining coherence between the states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, the state can be written as a convex combination of separable states so the state is no longer entangled. This happens when

$$\tan\alpha<\sin^2(\Omega_A t)=1-\exp(-\gamma_k t') \tag{21}$$

as pointed out in Refs. [6,10]. If $\cos\alpha<\sin\alpha$, the atomic excitation is insufficient to excite the intermediate states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ to the extent that the entanglement between the atoms vanishes in a finite time.

If the coupling constants g_A and g_B are different, say, $g_A>g_B$ as in Figs. 1(b) and 2(b), the dissipative picture just presented is valid only as long as $0\leq\Omega_A t\leq\pi/2$, where $\Omega_k=g_k$ when the atoms and cavities are resonant. For times longer than $\pi/(2g_A)$ the excitation of atom A starts to revive again in the JC model, a phenomenon not compatible with dissipation. However, as seen from the figures, the behavior of the states for times $\Omega_A t<\pi/2\Leftrightarrow\Omega_B t<\pi/4\approx 0.79$ is qualitatively the same as in the symmetric ($g_A=g_B$) case.

V. AN ENTANGLEMENT INVARIANT

The form of the chosen interaction Hamiltonian, which does not include any interaction between subsystems Aa and Bb, ensures that no entanglement is formed between these subsystems that was not already present in the initial state. Therefore, it is reasonable to expect an entanglement invariant to exist that measures the net entanglement between these subsystems. (Should the Hamiltonian include coupling between any of A and a and any of B or b, such an expectation would of course be unfounded since, with regards to the partition Aa-Bb, the Hamiltonian would be nonlocal.) However, in general, $E_{Aa\text{-}Bb}$ represents only some of the entanglement present in the system because, due to excitation transfer, some of the initial Aa-Bb entanglement will, e.g., be manifested as a-ABb entanglement as soon as the evolution starts. If we include the entanglement in the other partitions we can find the invariant

$$\begin{aligned}\mathcal{E} = 2E_{Aa\text{-}Bb} + E_{AB\text{-}ab} + E_{Ab\text{-}Ba} \\ - (E_{A\text{-}Bab} + E_{B\text{-}Aab} + E_{a\text{-}ABb} + E_{b\text{-}ABa}),\end{aligned} \tag{22}$$

valid for all parameter values, that is, even for nonresonant coupling and different atom-cavity coupling ratios. We see that the invariant in not symmetric in all bipartitions of the system, reflecting the fact that the interaction has specifically been chosen not to alter the entanglement between the partitions Aa and Bb. The presence of the negative terms in the invariant stems from the fact that, e.g., the entanglement in partitions Aa-Bb and A-Bab are not independent and neither is the entanglement between, e.g., the partitions Aa-Bb and Ab-Ba. Hence, the terms apart from $E_{Aa\text{-}Bb}$ in $\mathcal{E}$ can be seen as a means of not "counting the same entanglement twice." However, even in this simple model the interdependence of the various kinds of entanglement of the state is not very intuitive, so a detailed explanation of the form of $\mathcal{E}$ is probably rather difficult and we will not attempt to provide one here.

The introduced measure (22) does not depend on time for the initial pure X states (5) and (6). The value of $\mathcal{E}$ in both cases is

$$\mathcal{E} = \sin^2\alpha \cos^2\alpha. \tag{23}$$

This value is proportional to the square of the two atoms' initial concurrence C [15]:

$$4\mathcal{E} = C^2.$$

The result is expected, because, as the Hamiltonian is chosen not to change the entanglement between Aa-Bb, it must remain equal to its initial value at all times. Hence, the oscillations and different behavior for different ratios g_A/g_B present in $E_{Aa/Bb}$ for the state (5) (see Fig. 2) are compensated by the rest of the terms in Eq. (22). What is more significant is that for the generic class of states

$$|\Phi_g\rangle = c_1|\uparrow\uparrow 00\rangle + c_2|\downarrow\uparrow 10\rangle + c_3|\uparrow\downarrow 01\rangle + c_4|\downarrow\downarrow 11\rangle + c_5|\downarrow\downarrow 00\rangle, \tag{24}$$

$$|\Psi_g\rangle = d_1|\uparrow\downarrow 00\rangle + d_2|\downarrow\uparrow 00\rangle + d_3|\downarrow\downarrow 10\rangle + d_4|\downarrow\downarrow 01\rangle, \tag{25}$$

we find that Eq. (22) is still invariant under evolution under $\hat{U}_A \otimes \hat{U}_B$, where

$$\mathcal{E}(|\Phi_g\rangle) = |c_5|^2(|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2), \tag{26}$$

$$\mathcal{E}(|\Psi_g\rangle) = (|d_1|^2 + |d_3|^2)(|d_2|^2 + |d_4|^2). \tag{27}$$

As we have discussed in Sec. III, both states $|\Phi_g\rangle$ and $|\Psi_g\rangle$ are composed by states belonging to two different evolution manifolds. The corresponding entanglement measures in Eqs. (26) and (27) are just products of the total excitation probabilities of the two manifolds. As expected, $\mathcal{E}$ is not invariant for a more general state, e.g., a superposition of the states (24) and (25), since it would then consist of states associated with four evolution manifolds. (Such states would in general not lead to states of the X class after tracing out the fields, whereas $|\Phi_g\rangle$ and $|\Psi_g\rangle$ do, for reasons explained in Sec. IV.) However, we believe that similar invariants to that expressed in Eq. (22) could be found, even for states more general than $|\Phi_g\rangle$ and $|\Psi_g\rangle$, but it is then likely that they should be written as functions of higher powers of the four different excitation probabilities while the measure (20) contains only direct products of probabilities.

The generic class of states $|\Phi_g\rangle$ and $|\Psi_g\rangle$ are the most general class of states that do not involve more than two evolution manifolds, and that simultaneously stay in the corresponding two-qubit subspaces. In contrast, e.g., the two-excitation state $|\uparrow\downarrow 10\rangle$ will excite the state $|\downarrow\downarrow 20\rangle$ under the JC Hamiltonian. The latter state cannot be described in the four-qubit context and must therefore be excluded from the generic states if one has such a context in mind. It should be noted that in the dissipative picture, the states $|\Phi_g\rangle$ and $|\Psi_g\rangle$ model coupling to excited reservoirs, and in general the states cannot, at any time, be written as a product state between the atoms and the fields. Hence, some entanglement between atom and fields is already there at the start of the evolution. The states $|\Phi(0)\rangle$ and $|\Psi(0)\rangle$ are special cases of $|\Phi_g\rangle$ and $|\Psi_g\rangle$ where the atoms couple to initially empty reservoirs, a relevant but special case.

When we study the terms in Eq. (22), we have seen that, for both states, any single term is zero only at discrete times. This means that at all times, except a discrete set of times of zero measure, all parts of the system become entangled in some degree through excitation transfer. This phenomenon is generic for entangled systems and has, e.g., been used to entangle subsystems that have never interacted through so-called entanglement swapping [19]. In a dissipative system, this entanglement spread is of course detrimental and may lead to complete elimination of entanglement.

VI. DISCUSSION

In this work we have discussed the entanglement dynamics for two excited atoms coupled to cavity field modes through a Jaynes-Cummings Hamiltonian. We have discussed both the closed-system dynamics and the dynamics if the cavities are viewed as reservoirs. We have discussed why different initial atomic states face different fates (asymptotic vs sudden decay) with respect to their entanglement under dissipation. We have also shown that, as expected, there exists an entanglement invariant valid for a large class of so-called *X* states in the closed system. The invariant shows that the entanglement spreads out over all the degrees of freedom of the whole system. When treating the fields as reservoirs (i.e., when tracing over the fields), some of the entanglement transferred to the cavities is ignored. The state $|\phi(0)\rangle$ transfers its excitation over a larger set of distinguishable field states ($|01\rangle$, $|10\rangle$, and $|11\rangle$) than the state $|\psi(0)\rangle$ (that excites only the field states $|01\rangle$ and $|10\rangle$), and therefore it is not so surprising that the former state may lose all of its entanglement through even a finite dissipation.

It would be interesting to find an invariant for every state in the double JC system Hilbert space. We think that with a similar entanglement measure, but including higher powers of the excitation manifold probabilities, such an invariant could be found.

ACKNOWLEDGMENTS

This work was supported by the Swedish Foundation for International Cooperation in Research and Higher Education (STINT), the Swedish Research Council (VR), and the Swedish Foundation for Strategic Research (SSF).

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