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How to detect a possible correlation from the information of a subsystem in quantum-mechanical systems

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The possibility of detecting correlations between two quantum-mechanical systems from only the information of a subsystem is investigated. For generic cases, we prove that there exist correlations between two quantum systems if the time derivative of the reduced purity is not zero. Therefore, an experimentalist can conclude that correlations between the system and some environment are nonzero if the time derivative of the reduced purity is found not to be zero. A quantitative estimation of the time derivative of the reduced purity with respect to correlations is also given. This clarifies the role of correlations in the mechanism of decoherence in open quantum systems.

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I. INTRODUCTION

In many contexts in physics, it is important to know the existence (or absence) of correlations [1] of a system of interest S and its environment E (an unknown system). For example, in order to achieve successful quantum-information processing, quantum communication, or quantum-mechanical control, one has to manage system-environment correlations, which may enhance the decoherence of the states of the system. However, in many cases, we know neither the structure of the environment nor the nature of the interaction with the system. Under these circumstances, one has to detect possible correlations between S and E only from the measurements of the system S, not from those of the total system $S+E$ (Fig. 1). To do this, if an ensemble of independently identical systems is available, the following well-known criterion [2] for quantum systems can be applied.

(A) If the system S is in a pure state, then S has no correlations with any other environment E.

From this statement, an experimentalist can safely conclude that there are no correlations with any environment if the (reduced) state is found to be a pure state. Indeed, some of the unconditional security proofs of quantum cryptography partially rely on this fact [3], where an unknown eavesdropper is assumed to prepare any environment and do anything that is physically allowed.

Unfortunately, statement A is not applicable when the reduced state is in a mixed state. Indeed, in this case no static properties of a subsystem can provide information on the correlations, since the same reduced mixed states can be generated from total states with and without system-environment correlations [4]. Therefore, in such cases, we need to use dynamical information as well. Here what we would like to discuss and try to show is the following statement.

(B) If the time derivative of the purity of S is not zero at time $t=t_0$, then there exist nonzero correlations with a certain environment at that time.

If this statement is universally true, then it enables an experimentalist to confirm nonzero correlations with some environment if the time derivative of the purity is found not to be zero [5]. The purpose of this paper is to investigate statement B for arbitrary quantum-mechanical systems [6] under the usual postulates for (open) quantum mechanics (see, for instance, [7–9]), which include the following.

(i) *State space*. For any quantum-mechanical system S, there exists a separable Hilbert space $\mathcal{H}_S$. Any state of S is represented by a density operator ρ_S—a positive trace class operator on $\mathcal{H}_S$ with unit trace.

The purity P_S for ρ_S is defined by

$$P_S = \mathrm{Tr}_S\{\rho_S^2\}. \tag{1}$$

(ii) *Composite system*. Let S and E be quantum-mechanical systems with Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_E$. The composite system $S+E$ is associated with the tensor product Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$.

For a total density operator ρ_{tot} on $\mathcal{H}_S \otimes \mathcal{H}_E$, the reduced states ρ_S and ρ_E for S and E are given by $\rho_S=\mathrm{Tr}_E\{\rho_{tot}\}$ and $\rho_E=\mathrm{Tr}_S\{\rho_{tot}\}$, where Tr_S and Tr_E are the partial traces with respect to S and E, respectively. (In the following, ρ_S and ρ_E always represent the reduced density operators on S and E from the total density operator ρ_{tot}.) No correlations in the density operator ρ_{tot} on $S+E$ equivalently means that ρ_{tot} is given by a tensor product of the reduced density operators of the two subsystems:

$$\rho_{tot} = \rho_S \otimes \rho_E. \tag{2}$$

(iii) *Evolution*. A quantum system S is dynamically isolated or open, and, without or with a certain environment E, the dynamics of S is eventually described by the von Neumann equation (Schrödinger equation) on the total system. Namely, there exists a self-adjoint Hamiltonian H on $\mathcal{H}_S \otimes \mathcal{H}_E$ with which the von Neumann equation holds:

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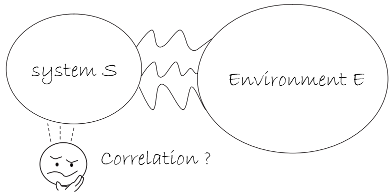

FIG. 1. How to detect possible correlations between your quantum system S and its environment E.

$$i\hbar\frac{d}{dt}\rho_{tot}(t)=[H,\rho_{tot}(t)], \tag{3}$$

where $\rho_{tot}(t)$ is a density operator on $\mathcal{H}_S\otimes\mathcal{H}_E$ at time t. (In the following, we set Planck's constant $\hbar$ to be 1.)

Notice, however, that there appears a problem of domain when H is an unbounded operator [10]. To avoid this problem, it is generally adopted in the axiomatic approach of quantum mechanics that the dynamics is governed by a unitary time evolution:

$$\rho_{tot}(t)=U_t\rho_{tot}U_t^{\dagger}, \tag{4}$$

where ρ_{tot} is the initial density operator at $t=0$ and U_t is a unitary operator given by $U_t=e^{-iHt}$ (for a time-independent Hamiltonian H). Then, for any density operator ρ_{tot}, the dynamics (4) is applied without any problem. In this paper, we assume a unitary dynamics (4) for an isolated quantum system, from which the von Neumann equation (3) holds under appropriate conditions.

In a formal analysis, statement B for quantum-mechanical systems can be proved in the following way. Let the time derivative of the purity of a quantum system S at $t=t_0$ be nonzero. Since the purity does not change in an isolated system, S should be an open system interacting with some environment E. Let H be a self-adjoint Hamiltonian on $\mathcal{H}_S\otimes\mathcal{H}_E$ for which the von Neumann equation (3) holds. Assume that there are no correlations at $t=t_0$, namely, the initial density operator takes the product form $\rho_{tot}=\rho_S\otimes\rho_E$. Then from the von Neumann equation we obtain

$$\begin{aligned}P_S'(t_0)\equiv \left.\frac{d}{dt}P_S(t)\right|_{t=t_0} &= 2\mathrm{Tr}_S\left\{\left.\rho_S(t)\frac{d}{dt}\rho_S(t)\right|_{t=t_0}\right\}\\ &=-2i\mathrm{Tr}_S\{\rho_S\mathrm{Tr}_E[H,\rho_S\otimes\rho_E]\}\\ &=-2i\mathrm{Tr}_{SE}\{\rho_S\otimes\mathbb{I}_E[H,\rho_S\otimes\rho_E]\}\\ &=0,\end{aligned} \tag{5}$$

where the cyclic property [11] of the trace Tr_{SE} and $[\rho_S\otimes\mathbb{I}_E,\rho_S\otimes\rho_E]=0$ have been used to estimate the last equality. Therefore, by contradiction, we conclude that ρ_{tot} has nonzero correlations at $t=t_0$. It is worth noticing that, although use has been made of a Hamiltonian in the proof, experimentalists do not have to know anything about environments, including how they are interacting with the system. The only things they have to believe are postulates (i), (ii), and (iii) of quantum mechanics.

The above analysis, however, is still rough, without sufficient mathematical rigor, especially for the case of infinite-dimensional Hilbert spaces. Moreover, if the Hamiltonian is described by an unbounded operator, we have to deal with the domain carefully, which makes the statement quite nontrivial. In the following, we discuss the validity of statement B, including infinite-dimensional Hilbert spaces, in a careful manner. In Sec. II, we provide a rigorous version of statement B and show a more general statement (Theorem 1) in the case of bounded Hamiltonians, which quantitatively generalizes statement B. This shows how purity changes in the presence of correlations, and hence clarifies the role of correlations in the mechanism of decoherence in open quantum systems. In Sec. III, we discuss statement B in the case of unbounded Hamiltonians and show a certain counterexample. Finally, we slightly modify statement B to be correct (Theorem 3) for the case of unbounded Hamiltonians. This is done by assuming that the total energy has finite variance, and hence we conclude that statement B is universally valid for all the generic cases. Section IV closes the paper with some concluding remarks and discussion.

II. THE CASE OF BOUNDED HAMILTONIANS: QUANTITATIVE ESTIMATION OF STATEMENT B

In this section, we discuss statement B, including infinite-dimensional cases with mathematical rigor, but for the case of Hamiltonians described by bounded operators. We obtain a useful theorem which generalizes statement B in a quantitative manner (Theorem 1). As usual when discussing open quantum systems [8], we shall divide the total Hamiltonian H into the sum of free Hamiltonians H_S and H_E for systems S and E and an interaction Hamiltonian H_{int}:

$$H=H_S\otimes\mathbb{I}_E+H_{int}+\mathbb{I}_S\otimes H_E. \tag{6}$$

We assume that H_S, H_E, and H_{int} are bounded self-adjoint operators on $\mathcal{H}_S$, $\mathcal{H}_E$, and $\mathcal{H}_S\otimes\mathcal{H}_E$, respectively, and hence H is also a bounded self-adjoint operator on $\mathcal{H}_S\otimes\mathcal{H}_E$.

In order to quantify correlations between S and E in a state ρ_{tot}, we use the quantum mutual information [12,13]:

$$I(\rho_{tot})\equiv\mathrm{Tr}_{SE}\{\rho_{\mathrm{tot}}\log\rho_{tot}-\rho_{tot}\log\rho_S\otimes\rho_E\},$$

where ρ_S and ρ_E are the reduced density operators on S and E, respectively, and log denotes a logarithm with base two. Notice that $I(\rho_{tot})\geqslant 0$, and $I(\rho_{tot})=0$ if and only if ρ_{tot} has no correlations. Notice also that [15]

$$\|\rho_{tot}-\rho_S\otimes\rho_E\|_1^2\leqslant 2I(\rho_{tot}), \tag{7}$$

where $\|\cdot\|_1$ is the trace norm $\|W\|_1\equiv\mathrm{Tr}_{SE}\{\sqrt{W^{\dagger}W}\}$ [11].

For any density operator ρ_{tot} on $\mathcal{H}_S\otimes\mathcal{H}_E$, we define the correlation operator ρ_{cor} [16] by

$$\rho_{cor}\equiv\rho_{tot}-\rho_S\otimes\rho_E, \tag{8}$$

which is a trace class operator on $\mathcal{H}_S\otimes\mathcal{H}_E$. By definition, it holds that $\rho_{cor}=0$ if and only if ρ_{tot} has no correlations. Since $\mathrm{Tr}_E\{\rho_S\otimes\rho_E\}=\rho_S$, it follows that

$$\mathrm{Tr}_E\{\rho_{cor}\}=0. \tag{9}$$

We have the following quantitative estimation of the time derivative of the reduced purity.

Theorem 1. Let S and E be quantum-mechanical systems with total Hamiltonian H of the form (6), and let ρ_{tot} be the density operator at $t=t_0$. Assume that the total system $S+E$ is closed and H is bounded. Then the reduced purity $P_S(t)$ is time differentiable at $t=t_0$ and

$$P'_S(t_0) = -2i\mathrm{Tr}_{SE}\{\rho_S \otimes \mathbb{I}_E[H_{int},\rho_{cor}]\}. \qquad (10)$$

The absolute value of the time derivative is bounded from above by

$$|P'_S(t_0)| \leq 2\|\rho_S\|\,\|[H_{int},\rho_{cor}]\|_1, \qquad (11a)$$

$$\leq 4\|H_{int}\|\,\|\rho_{cor}\|_1, \qquad (11b)$$

$$\leq 4\sqrt{2I(\rho_{tot})}\|H_{int}\| \qquad (11c)$$

where $\|\cdot\|$ denotes the operator norm [11].

Proof. We shall use the notations $\mathcal{B}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ to denote the sets of all the bounded operators and trace class operators on the Hilbert space $\mathcal{H}$, respectively. Since $H \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_E)$ and $\rho_{tot}(t) \in \mathcal{T}(\mathcal{H}_S \otimes \mathcal{H}_E)$, it follows that $[H,\rho_{tot}(t)] \in \mathcal{T}(\mathcal{H}_S \otimes \mathcal{H}_E)$ due to the ideal property of trace class operators [17]. In this case, the von Neumann equation (3) holds [9] for any density operator, where the time derivative is defined with respect to the trace norm. Therefore, by observing the inequalities [11]

$$|\mathrm{Tr}\{A\rho\}| \leq \|A\rho\|_1 \leq \|A\|\,\|\rho\|_1 \quad [\forall\ A \in \mathcal{B}(\mathcal{H}),\ \rho \in \mathcal{T}(\mathcal{H})] \qquad (12)$$

and $\|\rho_S(t) \otimes \mathbb{I}_E\| \leq 1$ [18], we get that $P_S(t)$ is differentiable for any time t and

$$P'_S(t_0) = -2i\mathrm{Tr}_{SE}\{\rho_S \otimes \mathbb{I}_E[H,\rho_{tot}]\}.$$

By the cyclic property of the trace [19], it follows that $\mathrm{Tr}_{SE}\{\rho_S \otimes \mathbb{I}_E[H,\rho_S \otimes \rho_E]\} = \mathrm{Tr}_{SE}\{[\rho_S \otimes \rho_E,\rho_S \otimes \mathbb{I}_E]H\} = 0$, and therefore we have

$$P'_S(t_0) = -2i\mathrm{Tr}_{SE}\{\rho_S \otimes \mathbb{I}_E[H,\rho_{cor}]\}.$$

Moreover, since $\mathrm{Tr}_{SE}\{\rho_S \otimes \mathbb{I}_E[H_S \otimes \mathbb{I}_E,\rho_{cor}]\} = \mathrm{Tr}_S\{\rho_S[H_S,\mathrm{Tr}_E\rho_{cor}]\} = 0$ from (9), and $\mathrm{Tr}_{SE}\{\rho_S \otimes \mathbb{I}_E[\mathbb{I}_S \otimes H_E,\rho_{cor}]\} = \mathrm{Tr}_{SE}\{[\rho_S \otimes \mathbb{I}_E,\mathbb{I}_S \otimes H_E]\rho_{cor}\} = 0$ again by the cyclic property of the trace, we obtain (10). From (12), $[H_{int},\rho_{tot}] \in \mathcal{T}(\mathcal{H}_S \otimes \mathcal{H}_E)$ and $\|\rho_S \otimes \mathbb{I}_E\| = \|\rho_S\|$, we have

$$|P'_S(t_0)| \leq 2\|\rho_S\|\,\|[H_{int},\rho_{cor}]\|_1.$$

The second inequality (11b) follows from the triangle inequality for the trace norm, $\|\rho_S\| \leq 1$, and again (12). The third inequality (11c) follows from (7). ■

Theorem 1 provides a quantitative estimation of the time derivative of the reduced purity in terms of the amount of correlations $I(\rho_{tot})$ and the strength of interaction $\|H_{int}\|$ [20]. It is worth noticing that the inequalities (11a)–(11c) include the following well-known fact [5]: the purity of a system does not change without interaction with an environment. Indeed, experimentalists usually confirm the existence of an interaction between the system and some environment if they find that the reduced purity is not constant. However, not only that, Eqs. (11) imply that correlations play an essential role in changing the purity even with the existence of an interaction. Moreover, Eq. (10) implies that the commutator between the interaction Hamiltonian and the correlation operator is essential for changes of purity, or decoherence.

From Theorem 1, we obtain a rigorous version of statement B.

Theorem 2. With the same assumptions as in Theorem 1, if there are no correlations at $t=t_0$ (i.e., $\rho_{tot}=\rho_S \otimes \rho_E$ at $t=t_0$) then $P_S(t)$ is time differentiable at $t=t_0$ and $P'_S(t_0)=0$. In other words, if the time derivative of the reduced purity is not zero, then there exist nonzero correlations between S and E at that time.

Proof. Since $\rho_{tot}=\rho_S \otimes \rho_E$ implies $\rho_{cor}=0$, we have $P'_S(t_0) = 0$ from inequality (11a). ■

Note that the opposite statement is not generally true. [For instance, if $H_{int}=0$, we have $P'_S(t_0)=0$ even in the presence of correlations.] Therefore, it is incorrect to infer that there are no correlations when the time derivative of the reduced purity is zero. Notice also that the above theorems do not contradict the results in Ref. [21], where we showed that the effect of an initial correlation does not appear in van Hove's limit (the weak-coupling limit) and therefore system S behaves as if the total system started from the factorized initial state. Indeed, this is true only for the van Hove time scale $\tau=\lambda^2 t$, where $\lambda \ll 1$ is the coupling constant, and on much shorter time scales than τ we can find a difference between the cases of no correlations and nonzero correlations, as we have seen in the above theorems. (See also [16] for the effect of an initial correlation.)

III. THE CASE OF UNBOUNDED HAMILTONIANS: A COUNTEREXAMPLE TO STATEMENT B

In the previous section, we confirmed that statement B is universally true for any bounded Hamiltonian. However, Hamiltonians are generally unbounded, especially from above, like that of the harmonic oscillator. Although the quantitative estimation (11) in Theorem 1 turns out to be trivial when $\|H_{int}\|=\infty$, we may still expect the validity of Theorem 2, i.e., statement B. In this section, we discuss statement B in the case of unbounded Hamiltonians, and provide an explicit counterexample to show that statement B itself may fail to hold in certain cases. Finally, we slightly modify statement B to be universally correct for all of the generic cases, by assuming that the total energy has a finite variance (theorem 3).

Counterexample to statement B

Let our system be described by $\mathcal{H}_S=\mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2}$ where $\mathcal{H}_{S_1}$ is an infinite-dimensional separable Hilbert space, and $\mathcal{H}_{S_2}$ is a two-dimensional Hilbert space; $\mathcal{H}_{S_2} \simeq \mathbb{C}^2$. (For instance, S can be a system of a nonrelativistic electron with spin 1/2.) In order to provide a counterexample to statement B, it is enough to consider the simplest possible environment with a two-dimensional Hilbert space $\mathcal{H}_E \simeq \mathbb{C}^2$. Assume that initially the total system is in a state $\rho_{tot}=\rho_S \otimes \rho_E$ which has no correlations, where

$$\rho_S=\sum_{n=1}^{\infty} p_n|\phi_n\rangle\langle\phi_n|\otimes|s_1\rangle\langle s_1|,\quad \rho_E=|e_1\rangle\langle e_1|, \tag{13}$$

with $p_n\geqslant 0$, $\Sigma_{n=1}^{\infty}p_n=1$, and $\{|\phi_n\rangle\}_{n=1}^{\infty}$, $\{|s_n\rangle\}_{n=1}^{2}$, and $\{|e_n\rangle\}_{n=1}^{2}$ are the orthonormal bases of $\mathcal{H}_{S_1}$, $\mathcal{H}_{S_2}$, and $\mathcal{H}_E$, respectively.

We use the Hamiltonian H, given by the spectral decomposition

$$H=\sum_{n=1}^{\infty}\sum_{k=1}^{4} h_{nk}|\phi_n\otimes\chi_k\rangle\langle\phi_n\otimes\chi_k|,$$

with eigenvalues (point spectra) $h_{n1}=0$, $h_{n2}=h_{n3}=h_n$, $h_{n4}=2h_n$ with $h_n\geqslant 0$ ($n\in\mathbb{N}$), where $\{|\chi_k\rangle\}_{k=1}^{4}$ is an orthonormal basis of $\mathcal{H}_{S_2}\otimes\mathcal{H}_E$ given by

$$|\chi_1\rangle\equiv\frac{1}{\sqrt{2}}(|s_1\otimes e_1\rangle+i|s_2\otimes e_2\rangle),$$

$$|\chi_2\rangle\equiv|s_2\otimes e_1\rangle,|\chi_3\rangle\equiv|s_1\otimes e_2\rangle,$$

$$|\chi_4\rangle\equiv\frac{1}{\sqrt{2}}(|s_1\otimes e_1\rangle-i|s_2\otimes e_2\rangle).$$

Obviously H is a positive self-adjoint operator on $\mathcal{H}_S\otimes\mathcal{H}_E$ and it is unbounded whenever the sequence $\{h_n\}$ is not bounded from above. The time evolution map $U_t=\exp(-iHt)$ is given by

$$U_t=\sum_{n=1}^{\infty}|\phi_n\rangle\langle\phi_n|\otimes X_t^n,$$

where $X_t^n\equiv|\chi_1\rangle\langle\chi_1|+e^{-ih_nt}(|\chi_2\rangle\langle\chi_2|+|\chi_3\rangle\langle\chi_3|)+e^{-i2h_nt}|\chi_4\rangle\langle\chi_4|$. From (13) we have

$$\rho_{tot}(t)=U_t\rho_{tot}U_t^{\dagger}=\sum_{n=1}^{\infty}p_n|\phi_n\rangle\langle\phi_n|\otimes|X_t^ns_1\otimes e_1\rangle\langle X_t^ns_1\otimes e_1|,$$

where $|X_t^ns_1\otimes e_1\rangle=e^{-ih_nt}[\cos(h_nt)|s_1\otimes e_1\rangle-\sin(h_nt)|s_2\otimes e_2\rangle]$. By taking the partial trace over E, we have $\rho_S(t)=\Sigma_{n=1}^{\infty}p_n|\phi_n\rangle\langle\phi_n|[\cos^2(h_nt)|s_1\rangle\langle s_1|+\sin^2(h_nt)|s_2\rangle\langle s_2|]$. From this, we obtain an analytical form for the reduced purity:

$$\begin{aligned}P_S(t)&=\sum_{n=1}^{\infty}p_n^2[\cos^4(h_nt)+\sin^4(h_nt)]\\&=P_S(0)-\frac{1}{2}\sum_{n=1}^{\infty}[p_n\sin(2h_nt)]^2\\&=\frac{3}{4}P_S(0)+\frac{1}{4}\sum_{n=1}^{\infty}p_n^2\cos(4h_nt),\end{aligned}\tag{14}$$

where $P_S(0)=\Sigma_{n=1}^{\infty}p_n^2$. Therefore, if the time derivative in the infinite sum in (14) is the sum of the derivatives of the individual terms, we obtain $P_S'(0)=0$ and statement B holds. For instance, let $p_n=1/2^n$ and $h_n=nE_0/4$ with the unit of energy E_0. Then, since $|(d/dt)p_n^2\cos(4h_nt)|=|[nE_0\sin(nE_0t)]/4^n|\leqslant nE_0/4^n$ and $\Sigma_{n=1}^{\infty}(nE_0/4^n)<\infty$, it follows that $\Sigma_{n=1}^{\infty}p_n^2\cos(4h_nt)$ is differentiable with respect to t, and we have $(d/dt)\Sigma_{n=1}^{\infty}p_n^2\cos(4h_nt)=\Sigma_{n=1}^{\infty}4p_n^2h_n\sin(4h_nt)$. Hence, this example satisfies statement B even though the Hamiltonian is unbounded [see Fig. 2(a)]. (In the following, we set E_0 to be 1.)

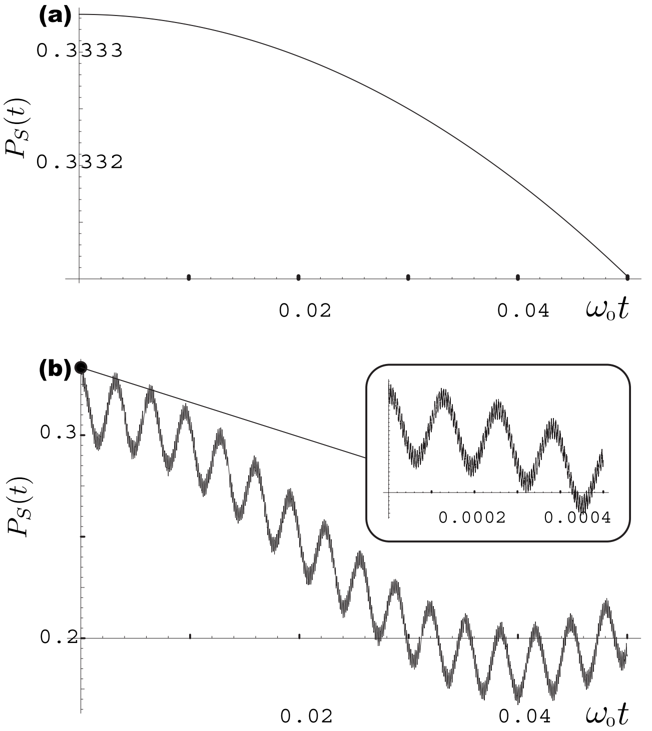

FIG. 2. Time evolution of the reduced purity (14) for (a) $p_n=1/2^n$, $h_n=n/4$ and (b) $p_n=1/2^n$, $h_n=25^n\pi/4$, with the unit of time $\omega_0\equiv\hbar/E_0$. Notice that in both cases the Hamiltonians are unbounded from above. One sees the flat time derivative at $t=0$ in (a), which makes statement B true, while one sees nondifferentiability in (b), which breaks down statement B.

However, we can construct a counterexample to statement B in the sense that $P_S(t)$ is not differentiable with respect to t at $t=0$ even though the initial state is given in a product form. We provide an interesting example where $P_S(t)$ is continuous but not differentiable at any time t by connecting the reduced purity to the so-called Weierstrass function $f(t;a,b)$ [22], defined by

$$f(t;a,b)=\sum_{n=0}^{\infty}a^n\cos(b^n\pi t),$$

with two parameters $0<a<1$ and positive odd integer b satisfying $ab>1+3\pi/2$. It is known that the function is continuous everywhere but differentiable nowhere with respect to t. From the form of (14), a proper choice of p_n and h_n, for instance, $p_n=1/2^n$, $h_n=25^n\pi/4$, makes $P_S(t)$ an essentially Weierstrass function:

$$P_S(t)=\frac{1}{4}\left[1-\cos(\pi t)+f\left(t;\frac{1}{4},25\right)\right] \tag{15}$$

[see Fig. 2(b)]. This provides a counterexample to statement B; namely, even with a product initial state, the time derivative of the purity is not necessarily zero; though this case just provides a case of nonexistence of the time derivative.

Therefore, in the case of unbounded Hamiltonians, we need to modify our statement B. Indeed, the following weaker statement can be proved to be true.

Theorem 3. Let H be a self-adjoint Hamiltonian bounded from below, but not necessarily bounded from above, and let ρ_{tot} be the density operator at $t=t_0$. If the variance of H with respect to ρ_{tot} is finite then

$$\rho_{tot}=\rho_S\otimes\rho_B\Rightarrow P_S'(t_0)=0.$$

The assumption that the Hamiltonian is bounded from below is physically required for the system to be stable. Hence, even when the Hamiltonian H is unbounded, statement B is correct provided that H has a finite variance with respect to the total state. In fact, it is easy to see that the variance of H is infinite for the initial state used in the above counterexample leading to (15).

To avoid technical difficulties when dealing with unbounded Hamiltonians, in the present paper we do not give a proof of Theorem 3. Instead, we just notice the following. First, finiteness of the variance of H with respect to a pure state $\rho_{tot}=|\psi\rangle\langle\psi|$ is equivalent to the statement that $|\psi\rangle$ is in the domain of H. Therefore, from the mathematical point of view, the assumption of finiteness of the variance of H allows us to avoid domain problems for unbounded operators. Second, the von Neumann equation holds when the variance of H is finite, which is the essential reason for Theorem 3 to be correct [23]. We plan to discuss and provide a systematic investigation for the case of unbounded Hamiltonians in a forthcoming presentation, including a complete proof of Theorem 3.

IV. CONCLUDING REMARKS AND DISCUSSION

We have discussed the problem of how to detect possible correlations between a system of interest S and some environment, from knowledge of observations on the system S only. We conjectured statement B, from which one can conclude that there are nonzero correlations with some environment when the time derivative of the reduced purity is not zero. In some sense, it is a counterpart of statement A; one can conclude that there are no correlations when the reduced purity is 1 using statement A, while one can conclude that there are correlations when the time derivative of the reduced purity is not zero. For instance, an experimentalist can first use statement A, and, if the state is pure, can conclude that there are no correlations. If the state is mixed, then statement B can be used. If the time derivative of the purity is not zero, correlations exist, provided that statement B is universally true. In this paper, we have investigated the validity of statement B for arbitrary quantum-mechanical systems. When the total Hamiltonian is bounded, we proved it to be universally correct (Theorem 2), by giving a more general statement (Theorem 1) that quantitatively implies statement B. Theorem 1 also clarifies the cause of purity change (decoherence or purification) due to interaction and correlations. However, when the total Hamiltonian is unbounded, we have also shown a counterexample to statement B. In the example, the reduced purity evolves essentially as a Weierstrass function even with a product initial state, whence the reduced purity fails to be differentiable. Therefore, a certain modification is necessary in statement B. If one considers a state with a finite variance of energy as a natural realization in nature, one can conclude that statement B is universal for all generic states in that sense. However, considering our original goal of estimating possible correlations, especially for the situation where we do not know anything about the environment (other than our theoretical knowledge of quantum theory), it is preferable to assume nothing additional about the environment [24]. In order to do this, another plausible conjecture is the following.

Conjecture 1. $\exists P_S'(0)$ and $P_S'(0)\neq 0\Rightarrow\rho_{tot}\neq\rho_S\otimes\rho_B$. If this is correct, it turns out that one can conclude that the correlations are nonzero if one finds a nonzero time derivative (including differentiability) of the reduced purity. In this direction, in a forthcoming presentation, we will discuss statement B including a complete proof of Theorem 3 and an investigation of the above conjecture. The case of a quantum field will also be presented elsewhere, using an algebraic formalism of quantum fields [25].

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[1] In this paper, by correlations we mean statistical correlations in the state of the composite physical system $S+E$; i.e., we say there are no correlations between S and E (statistical independence) if the joint probability distribution of any two observables O_S from S and O_E from E is the product of the probability distributions for O_S and O_E. Otherwise, we say that there exist nonzero correlations.

[2] See, for instance, B. d'Espagnat, *Conceptual Foundations of Quantum Mechanics* (Benjamin, Reading, MA, 1976); S. C. Italo, Lett. Nuovo Cimento Soc. Ital. Fis. **2**, 823 (1971). Although the proof there is given only for finite-dimensional

cases, we notice that statement A is true even for quantum fields, where quantum states are treated as positive linear functionals on the algebras of observables.

[3] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991); D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, *ibid.* **77**, 2818 (1996); **80**, 2022 (1998); M. Koashi and J. Preskill, *ibid.* **90**, 057902 (2003); M. Koashi, e-print arXiv:quant-ph/0505108.

[4] Some researchers might take the stance that all the mixedness originates from correlations with some environment, and eventually the total system should always be described by a pure state. If this is universally true, then statement A is enough to conclude the existence of correlations when system S is in a mixed state. However, the scope of this paper includes more general situations, and statement B is still useful even if there exists mixedness not originating from correlations.

[5] Notice that an experimentalist would usually conclude that nonzero interaction exists with some environment when the time derivative of the purity of S is not zero, since the purity of system S does not change if S is isolated. Compared to this, statement B allows experimentalists to confirm nonzero correlations with some environment. (See Theorem 1 below.)

[6] In a rigorous sense, in this paper, we treat only an environment that is a quantum-mechanical system with a fixed separable Hilbert space. Therefore, quantum fields with infinite degrees of freedom are not included. These cases will be investigated in a forthcoming presentation.

[7] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932) [*Mathematical Foundations of Quantum Mechanics*, translated by E. T. Beyer (Princeton University Press, Princeton, NJ, 1955)].

[8] C. W. Gardiner and P. Zoller, *Quantum Noise*, 2nd ed. (Springer, Berlin, 2000); H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).

[9] E. B. Davies, *Quantum Theory of Open Systems* (Academic, New York, 1976).

[10] M. Reed and B. Simon, *Methods of Modern Mathematical Physics 1. Functional Analysis* (Academic, San Diego, 1980).

[11] R. Schatten, *Norm Ideals of Completely Continuous Operators* (Springer, Berlin, 1960).

[12] G. Lindblad, Commun. Math. Phys. **33**, 305 (1973).

[13] In finite-dimensional cases, it has been shown that this quantity has an operational meaning as the amount of work needed to erase both the quantum and classical correlations [14].

[14] B. Groisman, S. Popescu, and A. Winter, Phys. Rev. A **72**, 032317 (2005).

[15] M. Ohya and D. Petz, *Quantum Entropy and Its Use* (Springer, Berlin, 1993).

[16] H. Hayashi, G. Kimura, and Y. Ota, Phys. Rev. A **67**, 062109 (2003).

[17] $\mathcal{T}(\mathcal{H})$ has the ideal property in $\mathcal{B}(\mathcal{H})$; namely, it holds that $A\rho$, $\rho A \in \mathcal{T}(\mathcal{H})$ for any $A \in \mathcal{B}(\mathcal{H})$, $\rho \in \mathcal{T}(\mathcal{H})$ [11].

[18] Note that $\|\rho\| \leqslant \|\rho\|_1$ for any $\rho \in \mathcal{T}(\mathcal{H})$ [11]. Since $\mathcal{T}(\mathcal{H}_S) \ni \rho_S(t) \geqslant 0$ and $\|\mathbb{I}_E\| = 1$, we have $\|\rho_S \otimes \mathbb{I}_E\| = \|\rho_S\| \, \|\mathbb{I}_E\| \leqslant \|\rho_S\|_1 = \mathrm{Tr}\{\rho_S\} = 1$.

[19] For any $A \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{T}(\mathcal{H})$, we have $\mathrm{Tr}\{AT\} = \mathrm{Tr}\{TA\}$ [11].

[20] T. Miyadera and H. Imai, Phys. Rev. A **74**, 064302 (2006).

[21] S. Tasaki, K. Yuasa, P. Facchi, G. Kimura, H. Nakazato, I. Ohba, and S. Pascazio, Ann. Phys. (N.Y.) **322**, 631 (2007); K. Yuasa, S. Tasaki, P. Facchi, G. Kimura, H. Nakazato, I. Ohba, and S. Pascazio, *ibid.* **322**, 657 (2007).

[22] K. Weierstrass, in *Classics on Fractals*, edited by G. A. Edgar (Addison-Wesley, Reading, MA, 1993), pp. 3–9.

[23] Indeed, the nondifferentiability of (15) is essentially due to the fact that the von Neumann equation does not hold in this example. However, it is worth noticing that we can provide an example that satisfies statement B even when the von Neumann equation does not hold.

[24] Moreover, in order to include the case where the environment can be a quantum field with infinite degrees of freedom, the assumption that the environment is to be described by a particular fixed Hilbert space should also be removed.

[25] R. Haag, *Local Quantum Physics* (Springer, Berlin, 1992); O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, 2nd ed. (Springer, Berlin, 1987, 1996), Vols. 1 and 2.