Quantum limits, nonseparable transformations, and nonlinear optics

Alfredo Luis

Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain
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We examine the relationship between nonseparable transformations and nonlinear optical processes, surpassing the Heisenberg limit in precision measurements.

I. INTRODUCTION

Quantum metrology deals with the fundamental measuring capabilities allowed by the quantum theory. For all measuring schemes the physical variable x to be measured (such as a length, time, force, temperature, etc.) is encoded in a transformation U_x experienced by a series of probes. Measurements performed in the final states of the probes allow to infer the value of x.

In this context the so-called Heisenberg limit plays a relevant position. It states that the uncertainty Δx of the inferred value of x scales as the inverse of the number of particles experiencing the same transformation. Using different approaches it has been argued that this is the ultimate bound to the precision of quantum measurements [1,2]. Then, most works in quantum metrology address the probe states, the final measurements, and the proper data analysis required to reach the Heisenberg limit [3].

Furthermore, it is also legitimate to compare the performance of different transformations U_x that can be used for the measurement of one and the same physical variable x. In some recent works we have shown that the Heisenberg limit applies exclusively to linear transformations and can be surpassed by nonlinear ones [4]. More recent works have shown that entangling transformations may surpass also the Heisenberg limit [5,6].

In this work we compare both approaches to precision measurements surpassing the Heisenberg limit. We show that nonlinear optical transformations provide natural practical realizations of abstract entangling transformations. Moreover, the optical realizations reveal that, under equivalent conditions, nonlinear separable transformations may achieve better precision than entangling ones.

II. NONSEPARABLE AND NONLINEAR TRANSFORMATIONS

For the sake of completeness in this section we recall the demonstration that nonseparable transformations allow to beat the Heisenberg limit [5,6]. We will consider transformations $U_x = \exp[-i\chi(x)H]$, where $\chi(x)$ is a function of x, which we will refer to as susceptibility, that depends on the physical processes producing the transformation. The idea is to consider transformations U_x that do not factorize as the product of individual probe transformations $U_x \neq \Pi_{j=1}^{N} U_x^{(j)}$, where $U_x^{(j)}$ acts exclusively on the probe j and N is the number of probes.

From very general arguments, the uncertainty of x, estimated via a linear propagation of uncertainties for simplicity

(other strategies would lead to similar conclusions [3,4]), is given by [1,2,7,8]

$$\Delta x = \frac{1}{2\left|\dfrac{\partial \chi}{\partial x}\right|\sqrt{\nu}\Delta H}, \tag{1}$$

where $(\Delta H)^2 = \langle H^2 \rangle - \langle H \rangle^2$ is the variance of H on the collective state of the N probes and ν is the number of repetitions of the measurement.

The capabilities of nonseparable transformations is demonstrated by the case where the generator H is the product of identical generators h_j for the individual probes,

$$H_{ns} = \prod_{j=1}^{N} h_j, \tag{2}$$

which can be compared with the more standard case of separable transformations generated by $H_\ell = \Sigma_j h_j$. Throughout we refer to the transformation generated by H_ℓ as linear. On the one hand, H_ℓ is a linear combination of h_j. On the other hand, h_j are usually number operators so that the transformation generated by H_ℓ produces a linear relation between input and output basic variables, such as position and momentum in mechanics or quadratures in optics. Analogously, H_{ns} is a nonlinear function of h_j and generates nonlinear transformations.

Optimum Δx implies maximum ΔH_ℓ and ΔH_{ns}, which are

$$\Delta H_\ell = \frac{N}{2}(\lambda_M - \lambda_m), \quad \Delta H_{ns} = \frac{1}{2}(\lambda_M^N - \lambda_m^N), \tag{3}$$

where λ_M and λ_m are the maximum and minimum eigenvalues of h_j, respectively, which are reached in both cases by states of the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(\prod_{j=1}^{N}|\lambda_M\rangle_j + \prod_{j=1}^{N}|\lambda_m\rangle_j\right), \tag{4}$$

which are the analog of the quantum superposition of states (Schrödinger cat states) or maximally entangled (NOON) states [9].

For definiteness let us consider $\lambda_m = 0$, denoting λ_M simply by λ, so that

$$\Delta x_\ell = \frac{1}{\sqrt{\nu} N \lambda \left| \dfrac{\partial \chi_1}{\partial x} \right|}, \quad \Delta x_{ns} = \frac{1}{\sqrt{\nu} \lambda^N \left| \dfrac{\partial \chi_N}{\partial x} \right|}, \quad (5)$$

where we have taken into account that the susceptibilities are in general different for different generators (i.e., for different N) since the underlying physical processes will be different.

The exponential versus linear behavior as functions of N in Eqs. (5) suggests that the uncertainty Δx_{ns} may be lesser than Δx_ℓ. However, conclusions based on the dependence on N are not valid because of the unknown dependence on N of χ_N.

Nevertheless, this lack of knowledge is avoided by focusing on the dependence of the uncertainty in the total number of particles n employed in the measurement for a fixed number of probes N. Denoting by k the number of particles in each probe, the total number of particles is $n = Nk$. Moreover, it usually holds that $\lambda \propto k = n/N$, since h_j are very often number operators. Thus the scalings of ΔH and Δx with the total number of particles n for fixed N are

$$\Delta H_\ell \propto n, \quad \Delta H_{ns} \propto n^N \qquad (6)$$

and

$$\Delta x_\ell \propto n^{-1}, \quad \Delta x_{ns} \propto n^{-N}. \qquad (7)$$

We stress that we are comparing the performances of the measurement of the same variable x carried out with two different measuring schemes with fixed N by examining the dependence on the number of particles n conveyed by the input state.

We can appreciate that the accuracy provided by linear transformations is bounded by the Heisenberg limit, while nonseparable ones surpass it. The scaling of Δx_{ns} is essentially the same found in Ref. [6]. Moreover, it is worth noting that nonlinear susceptibilities need not be small as demonstrated by the giant nonlinearities arising in the context of coherent multilevel optics [10].

Finally, we point out that the Heisenberg limit can be beaten even by factorized preparations of the probes $\rho = \Pi_j \rho_j$. If all the probes are prepared in the state with maximum Δh—i.e., $\Delta h = \langle h \rangle = \lambda/2$—we get

$$\Delta H_{ns} = \left(\frac{\lambda}{\sqrt{2}} \right)^N \sqrt{1 - 2^{-N/2}} \simeq \left(\frac{\lambda}{\sqrt{2}} \right)^N, \qquad (8)$$

where the last approximation holds for $N \gg 1$.

III. NONLINEAR OPTICS

In principle, product Hamiltonians of the form (2) might be regarded as somewhat exotic. However, they do actually arise in the realm of nonlinear optics, so that the two schemes [4] and [5,6] surpassing the Heisenberg limit can be compared on the same grounds.

More explicitly, nonseparable product Hamiltonians (2) arise in light propagation through nonlinear media in the form (for simplicity we embody the x derivative of the susceptibilities χ' with the generators)

$$H_{ns} = \chi'_{ns} \prod_{j=1}^{N} a_j^\dagger a_j, \qquad (9)$$

where a_j is the complex-amplitude operator of mode j. This can be compared with the nonlinear single-mode Hamiltonian (and thus trivially separable or nonentangling) [4]

$$H_{n\ell} = \chi'_{n\ell} (a^\dagger a)^N. \qquad (10)$$

The simplest case $N=2$ of the Hamiltonians (9) and (10) are the cross-Kerr (or cross-phase modulation) and Kerr effects, respectively, which are well known in quantum optics [11].

If we consider the same mean number of photons for the two arrangements, $n = \Sigma_{j=1}^{N} \langle a_j^\dagger a_j \rangle = \langle a^\dagger a \rangle$, the maxima of ΔH_{ns} and $\Delta H_{n\ell}$ are

$$\Delta H_{ns} = \frac{\chi'_{ns}}{2} \left(\frac{2n}{N} \right)^N, \quad \Delta H_{n\ell} = \frac{\chi'_{n\ell}}{2} (2n)^N, \qquad (11)$$

which both surpass the Heisenberg limit in agreement with Eqs. (6) and (7). The maximum precision is reached by the analogs of the states (4) in the photon-number basis,

$$|\Psi\rangle_{ns} = \frac{1}{\sqrt{2}} \left(\prod_{j=1}^{N} |0\rangle_j + \prod_{j=1}^{N} |2n/N\rangle_j \right),$$

$$|\Psi\rangle_{n\ell} = \frac{1}{\sqrt{2}} (|0\rangle + |2n\rangle). \qquad (12)$$

From Eq. (11) we can appreciate that, for the same set of parameters n and N, the nonlinear separable transformation provides better resolution than the nonlinear nonseparable.

This is also true in the more practical case of considering factorized semiclassical quadrature coherent states as the states experiencing the transformation $|\Psi\rangle = \Pi_{j=1}^{N} |\beta\rangle_j$ and $|\Psi\rangle = |\alpha\rangle$ with $n = N|\beta|^2 = |\alpha|^2$, where $|\beta\rangle$ and $|\alpha\rangle$ are coherent states. In such a case we get

$$\Delta H_{ns} = \sqrt{N} \chi'_{ns} \left(\frac{n}{N} \right)^{N-1/2}, \quad \Delta H_{n\ell} = \chi'_{n\ell} n^{N-1/2}. \qquad (13)$$

Both schemes surpass the Heisenberg limit and here again the nonlinear separable scheme presents a more favorable behavior.

IV. CONCLUSIONS

Both nonseparable and nonlinear transformations allow us to perform more accurate measurements than separable linear transformations. A suitable arena to implement both of them is nonlinear optics. In this context we have shown that, under equivalent conditions, nonlinear separable transformations present a better performance than nonlinear entangling ones. This result suggests that the key point for surpassing the Heisenberg limit is nonlinearity rather than nonseparability.

These conclusions are significant since they imply a *volte-*

face in quantum metrology. This is because they show that it is advantageous to use nonlinear processes to encode signals, instead of using them to generate fragile nonclassical entangled states as input states for linear detection schemes. This idea is accessible in practice via nonlinear optical processes.

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