Quantum teleportation of the angular spectrum of a single-photon field

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We propose a quantum teleportation scheme for the angular spectrum of a single-photon field, which allows for the transmission of a large amount of information. Our proposal also provides a method to tune the frequencies of spatially entangled fields, which is useful for interactions with stationary qubits.

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I. INTRODUCTION

Quantum teleportation allows for the transfer of quantum information from one location to another, without direct transmission of the physical system containing the information. Teleportation plays an important role in quantum-information processing since it is the basis for quantum repeaters [1] and nonlocal interactions between light and matter [2], as well as linear optics quantum computing [3]. The original proposal [4] considered discrete d-level systems, and has been demonstrated in several experiments for the case $d=2$ [5–8]. Further proposals generalized it to continuous variables [9–11]. A quantum teleportation protocol which uses transverse spatial properties of intense light fields has also been proposed [12,13].

Spatial properties of single-photon fields have attracted recent interest in quantum information, since they can be employed as continuous or d-dimensional carriers of quantum information [14,15]. Here we describe a scheme for teleporting the angular spectrum of a single-photon field. Our results are general, and could thus be used to transmit information encoded in discrete transverse spatial degrees of freedom such as orbital angular momentum [16], Hermite-Gaussian modes [17], or continuous degrees of freedom such as position (near field) and linear momentum (far field) [14,15]. Additionally, our scheme allows one to adjust the frequency of one or more spatially entangled fields, which could be interesting for atom-photon interactions, where it is necessary to tune the optical frequency to the appropriate atomic transition.

II. THE PROTOCOL

Consider the setup illustrated in Fig. 1. Alice receives a polarized monochromatic single-photon field which has quantum information encoded into spatial properties. Without loss of generality, the quantum state of this field is given by

$$|\phi\rangle_1 = \int d\mathbf{q}\, u(\mathbf{q})|\mathbf{q}\rangle_1, \tag{1}$$

where $u(\mathbf{q})$ is the normalized angular spectrum of the field in the $z=0$ plane, and is related to the field profile by an inverse Fourier transform

$$u(\mathbf{q}) = \int d\boldsymbol{\rho}\, e^{-i\mathbf{q}\cdot\boldsymbol{\rho}}\mathcal{U}(\boldsymbol{\rho}). \tag{2}$$

Here and below, $\mathbf{q}=k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}}$ is the transverse component of the wave vector $\mathbf{k}$, and we are considering a paraxial field, such that $|\mathbf{q}|\ll|\mathbf{k}|$. The ket $|\mathbf{q}\rangle$ represents a single photon with well-defined polarization, well-defined frequency ω, and wave vector $\mathbf{k}=\mathbf{q}+k(1-q^2/2k^2)\hat{\mathbf{z}}$.

Alice would like to use quantum teleportation to send the angular spectrum $u(\mathbf{q})$ of the single-photon field to Bob. In order to do so, Alice and Bob share a biphoton entangled field, produced by degenerate spontaneous parametric down-conversion (SPDC). In the paraxial approximation, the two-photon component of the quantum state produced by SPDC is [18,19]

$$|\psi\rangle_{23} = \iint d\mathbf{q}\, d\mathbf{q}'\Phi(\mathbf{q},\mathbf{q}')|\mathbf{q}\rangle_2|\mathbf{q}'\rangle_3. \tag{3}$$

The normalized function $\Phi(\mathbf{q},\mathbf{q}')$ is given by [19]

$$\Phi(\mathbf{q},\mathbf{q}') = v(\mathbf{q}+\mathbf{q}',Z)\gamma(\mathbf{q}-\mathbf{q}'), \tag{4}$$

where $v(\mathbf{q},Z)$ is the normalized angular spectrum of the pump beam at the SPDC crystal plane Z, $\gamma(\mathbf{q}) = \sqrt{2L/\pi^2K}\,\mathrm{sinc}(Lq^2/4K)$ is the phase-matching function, L is the longitudinal length of the SPDC crystal, and K is the pump beam wave number.

The initial three-photon quantum state is then given by

$$|\psi_i\rangle = \iint\int d\mathbf{q}\, d\mathbf{q}'\mathbf{q}''u(\mathbf{q})\Phi(\mathbf{q}',\mathbf{q}'')|\mathbf{q}\rangle_1|\mathbf{q}'\rangle_2|\mathbf{q}''\rangle_3. \tag{5}$$

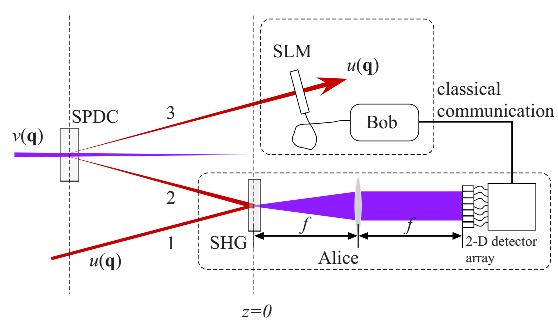

FIG. 1. (Color online) Illustration of the proposed scheme. SLM is a spatial light modulator.

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Let us assume that Alice is in possession of fields 1 and 2 and Bob of field 3. Following the usual recipe for quantum teleportation [4], it is necessary for Alice to perform a joint measurement on her fields, and send the measurement result to Bob using classical communication. Depending on the outcome, Bob performs a unitary operation on his field, and then recovers the initial quantum state of field 1.

As part of her measurement, Alice injects fields 1 and 2 into a nonlinear up-conversion crystal cut for second-harmonic generation (SHG). The full multimode Hamiltonian operator for SHG was described in Ref. [20], and can be approximately written as

$$\mathbf{H} \propto \iiint d\mathbf{q}\, d\mathbf{q}' \mathbf{a}_1(\mathbf{q})\, \mathbf{a}_2(\mathbf{q}')\mathbf{a}_4^\dagger(\mathbf{q}+\mathbf{q}') + \text{H.c.}, \tag{6}$$

where $\mathbf{a}^\dagger(\mathbf{q})$ and $\mathbf{a}(\mathbf{q})$ are the usual photonic creation and destruction operators for paraxial fields with transverse wave vector $\mathbf{q}$ and well-defined polarization and frequency. Index 4 labels the second-harmonic field, which is initially in the vacuum state. After the nonlinear interaction, the state $|\psi\rangle_i|0\rangle_4$ then becomes

$$|\psi\rangle \propto \iiint d\mathbf{q}\, d\mathbf{q}' d\mathbf{q}'' u(\mathbf{q})\Phi(\mathbf{q}',\mathbf{q}'')|0\rangle_1|0\rangle_2|\mathbf{q}''\rangle_3|\mathbf{q}+\mathbf{q}'\rangle_4. \tag{7}$$

Alice will now measure the transverse wave vector $\mathbf{q}+\mathbf{q}'$ of the up-converted field, and send the measurement result to Bob. This procedure is achieved by using a lens to map the momentum distribution of the field at the crystal face onto the position distribution in the detection plane, and measuring the position with an array of pointlike detectors, as shown in Fig. 1. The paraxial field operator that describes photodetection at position $\boldsymbol{\rho}_D$ in the detection plane is [15]

$$\mathbf{E}^{(+)}(\boldsymbol{\rho}_D) \propto \mathbf{a}\left(\frac{\kappa}{f}\boldsymbol{\rho}_D\right), \tag{8}$$

where κ is the magnitude of the up-converted field wave vector, and f is the focal length of the lens. Applying this operator to the state (7), detection of a photon at the position $\boldsymbol{\rho}_D$ selects the state

$$|\psi_f\rangle \propto \iint d\mathbf{q}\, d\mathbf{q}' u(\mathbf{q})\Phi(\mathbf{q}_D-\mathbf{q},\mathbf{q}')|\mathbf{q}'\rangle_3, \tag{9}$$

where $\mathbf{q}_D \equiv \kappa\boldsymbol{\rho}_D/f$, and the modes 1, 2, and 4 were excluded for simplicity, as all of them are in the vacuum state. In order to complete the teleportation protocol, Alice must send the value of the vector $\mathbf{q}_D$ to Bob by classical communication.

III. EXAMPLES AND DISCUSSION

A. Ideal and nonideal cases

Let us first consider the limiting case in which the angular spectrum of the SPDC pump field is a plane wave, $v(\mathbf{q}) \propto \delta(\mathbf{q})$, and the phase-matching function has a large width so that $\gamma(\mathbf{q}) \propto 1$. Using (4), we see that these conditions lead to a initial state (3) equivalent to the usual Einstein-Podolsky-Rosen (EPR) state [21]. As a result, the quantum state (9) is

$$|\psi_f^{EPR}\rangle = \int d\mathbf{q}\, u(\mathbf{q}+\mathbf{q}_D)|\mathbf{q}\rangle_3, \tag{10}$$

which is the original state (1) up to a translation in $\mathbf{q}$. Applying the momentum translation operator, defined by

$$\mathbf{P}(\boldsymbol{\alpha})|\mathbf{q}\rangle = |\mathbf{q}+\boldsymbol{\alpha}\rangle, \tag{11}$$

Bob will then have $\mathbf{P}(\mathbf{q}_D)|\psi_f^{EPR}\rangle = |\phi\rangle_3$, which corresponds to perfect teleportation of the initial state $|\phi\rangle_1$.

Let us now consider the more general case. For simplicity, we will also assume that the pump beam waist is located at the $z=0$ position, corresponding to the center of the up-conversion crystal. Then, after the transformation (11), the final state is

$$|\psi_f\rangle = C \iint d\mathbf{q}\, d\mathbf{q}' u(\mathbf{q})v(\mathbf{q}'-\mathbf{q})\gamma(2\mathbf{q}_D-\mathbf{q}'-\mathbf{q})|\mathbf{q}'\rangle_3, \tag{12}$$

where C is a normalization constant. The standard figure of merit of the success of the teleportation protocol is the fidelity [22], defined as $\mathcal{F}=|\langle\phi|\psi_f\rangle|$. For the output state given in Eq. (12), we have

$$\mathcal{F}(\mathbf{q}_D) = \left| C \iint d\mathbf{q}\, d\mathbf{q}' u^*(\mathbf{q}')u(\mathbf{q})v(\mathbf{q}'-\mathbf{q})\gamma(2\mathbf{q}_D-\mathbf{q}'-\mathbf{q}) \right|, \tag{13}$$

which depends on the widths ε and g of the angular spectrum $v(\mathbf{q})$ and phase-matching function $\gamma(\mathbf{q})$, respectively. For classical fields, the widths of these functions must obey the inequality $\varepsilon/g \geq 1$ [23]. Quantum fields, such as those composed by photon pairs generated by SPDC, may violate this inequality [23], and in practice one can make ε small by manipulating the pump beam, while g can be made large by using a thin nonlinear crystal, so that $\varepsilon/g < 1$. Finally, as the fidelity also depends on $\mathbf{q}_D = \kappa\boldsymbol{\rho}_D/f$, we will consider the average fidelity $\bar{\mathcal{F}}=(1/Q)\int_Q d\mathbf{q}_D \mathcal{F}(\mathbf{q}_D)$, integrated over the region Q, which is related to the detection area.

As an example, let us assume that both $v(\mathbf{q})$ and $\gamma(\mathbf{q})$ are given by Gaussian functions,

$$v(\mathbf{q}) = \frac{1}{\varepsilon\sqrt{\pi}} \exp\left(-\frac{q^2}{2\varepsilon^2}\right), \tag{14}$$

$$\gamma(\mathbf{q}) = \frac{1}{g\sqrt{\pi}} \exp\left(-\frac{q^2}{2g^2}\right), \tag{15}$$

where $g \approx \sqrt{2K/L}$. Furthermore, suppose that $u(\mathbf{q})$ is given by a Hermite-Gaussian (HG) mode

$$u_{nm}(\mathbf{q}) = D_{nm} H_n\left(\frac{q_x}{\sigma}\right) H_m\left(\frac{q_y}{\sigma}\right) \exp\left(-\frac{(q_x^2+q_y^2)}{2\sigma^2}\right), \tag{16}$$

where $H_n(x)$ is the nth-order Hermite polynomial and D_{nm} is a constant. These modes form a complete basis, so any angular spectrum $u(\mathbf{q})$ can be expressed as a linear combination of $u_{nm}(\mathbf{q})$. Figure 2 shows some numerical results for the

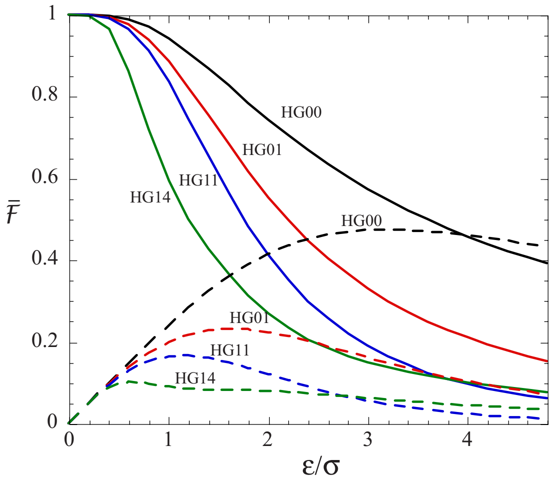

FIG. 2. (Color online) Average fidelities $\bar{\mathcal{F}}$ for several HG modes (16) as a function of the ε/σ for $g/\sigma=200$ (solid lines), and for a region $Q=3\sigma\times3\sigma$ (dashed lines). Average fidelities in the classical limit $g=\varepsilon$.

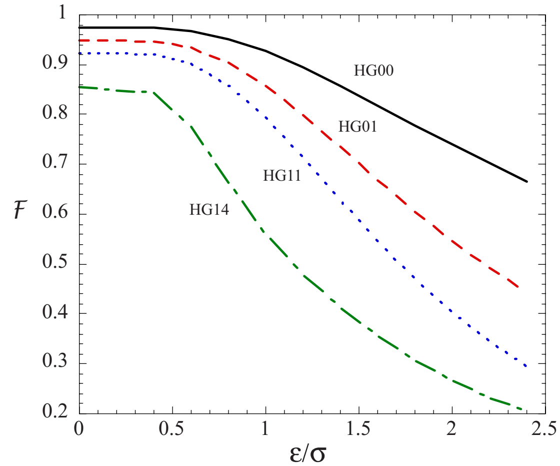

FIG. 3. (Color online) Fidelity $\mathcal{F}$ for several HG modes as a function of ε/σ for $g/\sigma=200$, and a momentum uncertainty function of width $a=0.8\sigma$.

average fidelities $\bar{\mathcal{F}}$ of several HG modes as a function of the parameter ε/σ, in the case of a large phase-matching width $g/\sigma=200$ [24], and a region $Q=3\sigma\times3\sigma$. In all cases, when $\varepsilon/\sigma<0.4$, one can achieve average fidelities greater than 0.95. Also shown are the corresponding average fidelities in the classical limit $g=\varepsilon$, which are considerably small when $\varepsilon/\sigma<1$. We note that the use of spatial degrees of freedom allows one to encode an arbitrary amount of quantum information in the angular spectrum of a single-photon field, which, in turn, can be transmitted reliably through the above teleportation scheme. For example, considering the set of HG modes from zeroth to Nth order, there are a total of $M=\Sigma_{j=0}^{N+1}j=(N+1)(N+2)/2$ modes available. Using parameters $\varepsilon/\sigma=0.3$ and $g/\sigma=200$, the average fidelity is above 0.97 for modes up to $N=7$ order, which is equivalent to sending $\log_2 M>5$ qubits.

B. Polychromatic fields and nonpoint detectors

Up until now we have considered monochromatic fields and point detectors. In practice, teleportation is realized with pulsed laser beams, which greatly increases the efficiency of the nonlinear processes involved [5–7]. Consider that the SPDC pump beam and the field 1 are spectrally described by Gaussian pulses centered at ω_0 and ω_1, with widths $\Delta\omega_0$ and $\Delta\omega_1$, respectively. Provided that $\Delta\omega_0\ll\omega_0$ and $\Delta\omega_1\ll\omega_1$, each field can be described by the product of its spectral and spatial functions [25]. Assuming that the SHG field 4 is also described by a Gaussian pulse centered at ω_4 and with width $\Delta\omega_4\ll\omega_4$, the final state of field 3 is given by (9); however, the momentum $\mathbf{q}_D$ is no longer well defined, since $\Delta\kappa=\Delta\omega_4/c$. Thus, under the above conditions the effect of pulsed fields is to introduce an uncertainty in the momentum shift. The same is true for nonpoint detectors, in which case Alice can determine the detection position $\boldsymbol{\rho}_D$ only to within the area defined by the detector. Thus, we can take into account the error caused by nonmonochromatic fields and nonpoint detectors by integrating the density operator ϱ_f corresponding to state (9) over a weighting function $\mathcal{D}$ which describes the uncertainty in the overall momentum shift $\mathbf{q}_D$:

$$\varrho=\int d\mathbf{q}_D\,\mathcal{D}(\mathbf{q}_D-\mathbf{q}_0)|\psi_f\rangle\langle\psi_f|. \tag{17}$$

The Hermitian and normalization conditions of the above state are guaranteed provided that $\mathcal{D}(\mathbf{q}_D)$ is a real function and $\int d\mathbf{q}_D\,\mathcal{D}(\mathbf{q}_D-\mathbf{q}_0)=1$, respectively. The vector $\mathbf{q}_0=\kappa_0\boldsymbol{\rho}_0/f$ depends on the central wave number κ_0 and the geometric center $\boldsymbol{\rho}_0$ of the detector. The question arises as to which momentum shift should be applied by Bob, given that the detector has fired. One possibility is to apply $\mathbf{P}(\mathbf{q}_0)$, as defined in (11), though this may not be the optimal procedure. After this correction, the final state is

$$\varrho_c=\mathbf{P}(\mathbf{q}_0)\varrho\mathbf{P}^\dagger(\mathbf{q}_0). \tag{18}$$

In this case the fidelity, given by $\mathcal{F}=\sqrt{\mathrm{tr}(|\phi\rangle\langle\phi|\varrho_c)}$ [22], is

$$\mathcal{F}=\left(|C|^2\int d\mathbf{q}_D\mathcal{D}(\mathbf{q}_D-\mathbf{q}_0)|T(\mathbf{q}_D,\mathbf{q}_0)|^2\right)^{1/2}, \tag{19}$$

where

$$T(\mathbf{q}_D,\mathbf{q}_0)=\iint dq\,dq' u(\mathbf{q})u^*(\mathbf{q}'-\mathbf{q}_D-\mathbf{q}_0)\times v(\mathbf{q}'-\mathbf{q})\gamma(2\mathbf{q}_D-\mathbf{q}'-\mathbf{q}). \tag{20}$$

In Fig. 3 we show some numerical calculations for the fidelities $\mathcal{F}$ as a function of the parameter ε/σ, for the case in which $g/\sigma=200$ and the momentum uncertainty function is a square of width $a=0.8\sigma$ centered at the origin, so that $\mathcal{D}(\mathbf{q}_D)=$const inside the square, and zero outside. Comparing these results with the corresponding monochromatic and point-detector cases, we see that all the fidelities suffer a small decrease. Despite this fact, these fidelities can still be higher than the best classical results in the best-case scenario of pointlike detectors and monochromatic fields (limiting case). It is important to note the difference between the average fidelities $\bar{\mathcal{F}}$ for the point-detector case, shown in Fig. 2, and the fidelity $\mathcal{F}$, shown in Fig. 3. While in the former the exact detection position and the second-harmonic wave number are exactly known, so the fidelity is averaged over all possibles momenta $\mathbf{q}_D$, in the second case this information is no longer precisely available, which enables us to perform only an average momentum shift.

C. Frequency tuning

Another interesting feature of our proposal is the possibility of transfer of spatial entanglement between single-photon fields of different frequencies, which is the spatial analog of the time-bin entanglement experimentally demonstrated in [26]. Returning to Eq. (7), we see that, provided $\Phi(\mathbf{q},\mathbf{q}')$ is a nonseparable function, fields 3 and 4, which may have different frequencies, are entangled. In particular, if $u(\mathbf{q}) \propto \delta(\mathbf{q})$, then

$$|\psi\rangle \propto \iint d\mathbf{q}\, d\mathbf{q}'\Phi(\mathbf{q}',\mathbf{q})|\mathbf{q}\rangle_3|\mathbf{q}'\rangle_4, \tag{21}$$

which is the original state (3) generated by SPDC now encoded in fields 3 and 4. As a result, the frequency of field 4 can be tuned by controlling the frequency of field 1 and adjusting the phase-matching conditions in the up-conversion crystal. This control is important in quantum-information processing since it is necessary to tune the field frequencies to those of transitions of stationary quantum systems such as atoms or ions.

IV. CONCLUSION

We have proposed a quantum teleportation scheme for teleporting the angular spectrum of a single-photon field. We note that our procedure works equally well with a weak coherent field. The efficiency of our scheme depends upon the probability of generation and detection of the up-converted field, as well as the detector arrangement. For single-photon fields, SHG efficiencies up to 10^{-7} have already been achieved [27]. One might also imagine a scheme to implement the nonlinear interaction through linear-optics quantum gates and ancilla photons, as in Ref. [28]. Finally, our proposal permits spatial entanglement between fields of different frequencies, which can be the control and tuning for quantum-information applications envolving atom-photon interactions.

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