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Quantum rearrangement and self-consistent BCS-BEC crossover thermodynamics

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Based on previous works, analytical calculational procedures for dealing with the strongly interacting fermion ground state are further developed through a medium-dependent potential in terms of the Bethe-Peierls contact interaction model. The methods are exact in the unitary limit regime and they lead to self-consistent equations analogous to the Hartree ones. The single particle energy spectrum rearrangement effects on the thermodynamics due to the Hugenholtz–van Hove theorem constraint are addressed. These effects lead to an additional instantaneous correlation potential contribution to the system physical chemical potential and pressure, i.e., equation of state, in order to enforce the classical thermodynamic consistency. The Dyson-Schwinger equations represent implicitly the various Bethe-Goldstone expansion ones. In a thermodynamically self-consistent way, the universal dimensionless factor is analytically calculated to be $\xi=\frac{4}{9}$, which defines the ratio of the unitary fermions energy density to that of the ideal noninteracting ones at $T=0$.

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I. INTRODUCTION

The strongly interacting matter physics serves currently as a pivotal topic in the fundamental many-body-theory–statistical-mechanics context. An interesting observable that characterizes this kind of physics is the unitary limit thermodynamics.

Taking the Feshbach resonance techniques, it is now possible to magnetically tune the two-body interparticle interaction strength. Increasing the scattering length of fermions from $-\infty$ to $+\infty$ resulting in bound boson systems can exhibit the crossover from Bardeen-Cooper-Schrieffer (BCS) to Bose-Einstein condensation (BEC). This will lead to the universal thermodynamics of the Bethe-Peierls contact interaction fermion system with a zero-range, infinite scattering length interaction [1–11]. To solve the universal resonant superfluidity thermodynamic problem, at least in principle, more theoretical efforts are urgent to understand the detailed dynamics by going beyond the mean field theory [12].

At the crossover point, the dimensionless coefficient ξ relates the total energy $E/N=\xi\frac{3}{5}\epsilon_f$ with the Fermi kinetic energy $\epsilon_f=k_f^2/(2m)$. Here, m is the bare fermion mass while k_f is the Fermi momentum. The determination of the fundamental constant ξ as a challenging many-body topic attracts much attention theoretically or experimentally [8,13–16]. It is presently accepted to be in the region 0.40–0.46. Compared with the old values, the updating experiments converge more toward $\xi\sim 0.42$–0.46 [17–19].

The ultracold fermion atom gas physics offers a plausible perspective in looking for the general nonperturbative statistical field theory methods. In the strongly interacting regime $k_f|a|\gg 1$, especially right at the unitary limit point $a=\pm\infty$, the infrared singular nature of the two-body scattering amplitude $f_0(k,a)=i/k$ (with k being the relative wave-vector magnitude of the colliding particles) rules out any conventional perturbative expansion techniques. It is generally believed that there is not a simple analytical method to compute the universal coefficient ξ, although this topic stimulates many growing arduous theoretical efforts. By noting that an exact many-body theory concerns the behavior at unitarity is presently not available, it is not surprising that the existing analytical results about the universal coefficient ξ differ from each other considerably ($\xi\sim 0.2$–0.6). It is worth noting that there are still uncertain discrepancies among the powerful quantum Monte Carlo works themselves as well as the experimental ones existing in the literature [8,14,15,17–19].

In Ref. [20], we make an almost first principle detour to attack the universal constant ξ with the general Dirac quantum many-body formalism. The analytical circumvention is motivated by the strongly instantaneous Coulomb correlation thermodynamics discussions in a compact nuclear confinement environment [21–23] based on the relativistic continuum nuclear many-body framework [24,25]. The infrared correlation characteristic in the unitary limit promotes us to use the universal strongly instantaneous long range interaction to attack the infinite contact interaction thermodynamics. The obtained analytical result ξ is comparable with other theoretical quantum Monte Carlo attempts. It is derived through a medium dependent interaction by accounting for the interior correlation with a counteracting or anti-correlation external source static potential.

However, the auxiliary quantities, pressure $\widetilde{P}$ and chemical potential $\widetilde{\mu}$, do not obey the profound universal hypothesis in the unitary limit with $T=0$ [4]. According to the universal hypothesis at unitarity with $T=0$, the ratio of pressure to energy density should be still that of noninteracting fermion gas. Therefore, the physical pressure should be $P=\xi P_{\mathrm{ideal}}$ instead of $\widetilde{P}=\frac{1}{6}P_{\mathrm{ideal}}$ obtained in Ref. [20] and consequently affects the important sound speed. The chemical potential should be $\mu=\xi\epsilon_f$ instead of $\widetilde{\mu}=\mu^*+2\pi a_{\mathrm{eff}}n/m=\epsilon_f-\pi^2 n/(mk_f)=\frac{1}{3}\epsilon_f$. Intriguingly, the physical Helmholtz free energy density is not affected. At $T=0$, it can be $f=-\widetilde{P}+\widetilde{\mu}n=\xi\frac{3}{5}n\epsilon_f$. Therefore the underlying energy(-momentum) conservation law is still exactly ensured even with the respective discrepancy of the auxiliary pres-

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sure and chemical potential. This observation instructs us to reanalyze the physical pressure and system chemical potential carefully.

The Hugenholtz–van Hove (HvH) theorem characterizes the fundamental thermodynamic relation between energy density and pressure [26]. It is well known that any effective medium dependent interaction with Brueckner-Bethe-Goldstone (BBG) techniques can easily violate this relation. The standard many-body technique to remove this tortuous difficulty is considering the quantum rearrangement effects of the single particle energy spectrum on the physical chemical potential and pressure [27–30]. The role of the single particle energy spectrum shift will be explored in the strongly interacting unitary limit.

To elucidate the physical chemical potential and pressure (equation of state) at both finite temperature and density transparently with a more concise nonrelativistic recipe for the ultracold atomic physics is the present main motivation. Up to now, quantum statistical field theory of the strongly interacting matter at finite temperature and density is still a challenge. More concretely, the exact determination of the equation of state plays an important role in looking for the detailed knowledge about the collective excitation modes measured with the ultracold atomic physics experiments.

The present work is organized as follows. In Sec. II, the adopted low energy effective theory formalism and corresponding nonperturbative approach for the strongly interacting unitary gas thermodynamics are described. Through taking into account the single particle energy rearrangement effect, the unitary fermion thermodynamics with the dynamically self-consistent Dyson-Schwinger equations is presented in Sec. III. Moreover, the universal thermodynamics at unitarity with $T=0$ as well as the concrete comparisons with existing results is discussed in Sec. IV. Section V includes the discussion and prospective remarks. The summary conclusion is made in Sec. VI.

In this work, the calculations are performed in the nonrelativistic contact interaction formalism. Throughout the paper, the natural units $\hbar=k_B=1$ are employed.

II. FORMALISM

A. In-medium effective action

The usual bare Bethe-Peierls Hamiltonian with a global U(1) symmetry addressing the many-body ground state energy can be found in the literature. The in-medium behavior associated with the many-body correlation characteristic is the key, while a nonperturbative approach is crucial for the strongly interacting unitary regime physics. Motivated by previous works, we work with the medium-scaling universal functional version

$$H=-\int d^3x\psi_\alpha^*(x)\left(\frac{\nabla^2}{2m}-\mu_{r\alpha}[n]\right)\psi_\alpha(x) + \frac{U_{\text{eff}}^*[n]}{2}\int d^3x\psi_\alpha^*(x)\phi_\beta^*(x)\psi_\beta(x)\psi_\alpha(x). \qquad (1)$$

In Eq. (1), m is the bare fermion mass with $\alpha,\beta=\uparrow,\downarrow$ representing the (hyperfine-)spin projection. The physical many-body renormalized interaction potential $U_{\text{eff}}^*[n]$ is a functional of density and temperature, from which one can define the in-medium S-wave scattering length a_{eff}

$$U_{\text{eff}}^*[n,T]\equiv\frac{4\pi a_{\text{eff}}[n,T]}{m}. \qquad (2)$$

The bare interaction matrix $U_{\text{eff}}^*[n,T]\to U_0$ is directly related with the free-space scattering length a with the low-energy Born approximation for the zero range contact interaction.

Considering the medium influence of the surrounding environment on the single particle energy spectrum properties, we have included the rearrangement term $\propto\mu_{r\alpha}[n]$ in Eq. (1). It is introduced through the auxiliary quantity $\tilde{\mu}_\alpha$ indicated by the tilde $\tilde{\mu}_\alpha[n]=\mu_\alpha-\mu_{r\alpha}[n]$. It means that the collective correlation effects are further accounted for as a single particle potential formalism in the spirit of density functional theory [27,31], while the true Lagrange multiplier is μ_α. As a functional of density and temperature, its concrete medium dependence can be derived analytically from the thermodynamic relation for the fixed particle number density.

B. Effective potential determined by the counteracting environment

Theoretically, the derivation of the many-body renormalized effective interaction U_{eff}^* is beyond the bare Bethe-Peierls contact interaction Hamiltonian itself; i.e., it should be derived from the more underlying physical law [27].

From the point of view of the contact interaction fermion physics, it is extremely instructive to recall the primary intermediate vector boson (IVB) hypothesis in weak interaction theory. It is well known that the local intermediate vector boson theory is related with the current-current (CC) contact interaction version through the corresponding connection $g_W^2/m_W^2\equiv\frac{G}{\sqrt{2}}$. In the low-energy limit $k\to0$, the two IVB and CC theories are ingrainedly identical. To model the low-energy unitary limit physics, we "let" $g_W^2/m_W^2\to\infty$ by taking g_W to be an arbitrary large charge or with an arbitrary small mass gap m_W. Consequently, the counteracting environment associated with an additional U(1) conserved "electric charge" can be introduced to attack the challenging infinity. Constrained by the opposite charge neutralizing background, the effective interaction U_{eff}^* is found to appear as an instantaneously antiscreening formalism. This particular analogy and/or discursion makes the coupling constant g_W enter simultaneously in the denominator and numerator of fractions in the relevant thermodynamic expressions [20]. The assumed electric charge g_W is reduced by the physical constraint or charge neutralizing stability condition and its concrete magnitude does not affect the final unitary physical results. Therefore, there are not any regularization or expansion parameters violating the intriguing unitarity in gauging the infrared divergence, which will be further explored in this more concise nonrelativistic recipe.

Moreover, corresponding to counteracting the deviation tendency of charge neutralizing, there exists a similar general Le Chatelier's principle in thermodynamics. This profound stability principle accounts for frustrating an instantaneous departure from equilibrium with an alternating *minus* func-

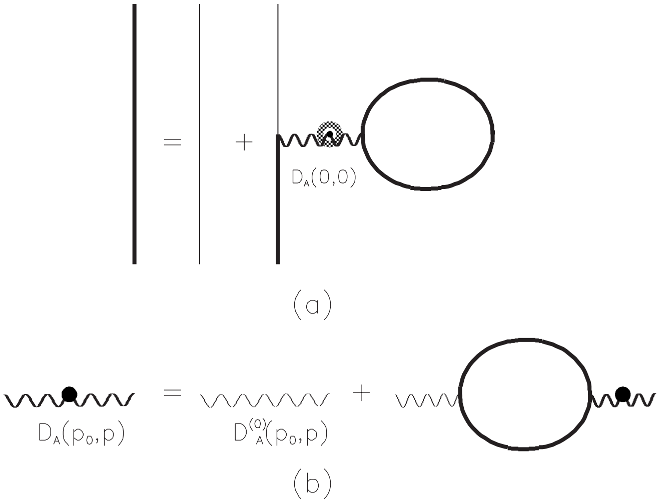

FIG. 1. Generalized self-consistent Dyson-Schwinger equations incorporating the Landau pole contributions associated with the Fermi surface (anti)correlation instability [20]. (a) The full fermion propagator determined with antiscreening interaction potential in instantaneous approximation $U^*_{\text{eff}}[n] \equiv U_0/[1+\Pi^*(0,|\mathbf{p}|\to 0)U_0]$. (b) The boson propagating calculated with the full fermion propagator.

tion of the surrounding environment [32]. First of all, as we will see, the counteracting-antiscreening formalism makes it possible for us to use the algebra equations to characterize the classical thermodynamics with the significant quantum correlation effects, but of crucial importance is that the many-body renormalization effects are introduced through a mirror background with an alternating minus sign to avoid the theoretical double counting trouble.

The following calculations based on Eq. (1) can be characterized by the comprehensive Feynman diagrams with the generalized coupled Dyson-Schwinger equations (DSEs) formalism as indicated by Fig. 1.

The polarization tensor $\Pi^*(\omega_n,\mathbf{p})$ is calculated through the nonrelativistic random phase approximation (RPA) with the full fermion propagator $\mathcal{G}(\omega_n,\mathbf{k})$ as described by Fig. 1(b)

$$\Pi^*(\omega_n,\mathbf{p}) = T\sum_{\omega_1}\int_k \mathcal{G}(\omega_1,\mathbf{k})\mathcal{G}(\omega_1+\omega_n,\mathbf{k}+\mathbf{p})$$

$$= -\int_k \frac{f_{\mathbf{k+p}}-f_{\mathbf{k}}}{i\omega_n - \left(\frac{(\mathbf{k}+\mathbf{p})^2}{2m}-\frac{\mathbf{k}^2}{2m}\right)}, \qquad (3)$$

with $\int_k = \int d^3\mathbf{k}/(2\pi)^3$. Here, $f_{\mathbf{k}}$ is the Fermi-Dirac distribution function given explicitly in Eq. (16).

After employing the analytical continuation for the Matsubara frequency ω_n according to $i\omega_n \to \omega + i\eta$, $\Pi^*(\omega,\mathbf{p})$ determines the dispersion relation of the excitation mode A (the composite boson or dimer) in-medium propagating. The polarization function obtained in this way is the standard in-medium Lindhard density-density or spin-spin response function for the one component fermion gas [12]. In this work, we care about the low-energy long wavelength thermodynamics of the fully pairing correlation occasion, which is related to the static infrared limit

$$\Pi^*(0,|\mathbf{p}|\to 0) = -\frac{m_D^2(T,\lambda)}{2} = \frac{1}{T}\left(\frac{mT}{2\pi}\right)^{3/2}\text{Li}_{1/2}[y]. \qquad (4)$$

The m_D^2 appearing in Eq. (4) is the physical nonrelativistic Debye screening mass squared with unit electric charge and reflects the quantum fluctuation contributions consequently. Its magnitude is the spin-spin or density-density correlation response susceptibility in the static infrared limit. At $T\to 0$, m_D^2 can be reduced to the familiar formalism

$$m_D^2|_{T=0} = \frac{k_f m}{\pi^2}. \qquad (5)$$

Different from the multiplier-system physical chemical potential μ, the symbol λ employed in the above equations is the effective(total) chemical potential. It characterizes the additional many-body collective correlation effects and will be defined explicitly in the following discussion for the full fermion propagator. Here and afterwards, the function identifier $\text{Li}_i[y]$ stands for the standard Fermi integral mathematica polylogarithm function polylog $[i,y]$. Meanwhile, we will use the shortened notation $y=-e^{\lambda/T}$ for the sake of conciseness throughout the paper.

With the polarization function $\Pi^*(0,|\mathbf{p}|\to 0)$, the rearrangement interaction matrix is derived to be

$$U^*_{\text{eff}}[n,T] \equiv \frac{U_0}{1+\Pi^*(0,|\mathbf{p}|\to 0)U_0} = \frac{U_0}{1-\frac{m_D^2}{2}U_0}. \qquad (6)$$

At $T=0$, it can be reduced to

$$U^*_{\text{eff}}[n] = \frac{U_0}{1-N(\epsilon_f)U_0}. \qquad (7)$$

The $N(\epsilon_f)=k_f m/(2\pi^2)$ is the unperturbated density of states on the Fermi surface for one component fermions [12], which is one half of the physical Debye screening mass squared. Without a loss of generality and conformity, the screening mass squared notion m_D^2 instead of the density of states will be used to characterize the general finite temperature thermodynamics.

Compared with the conventional screened interaction customarily applied in the weak coupling perturbative loop diagram ring/ladder resummations

$$U_{\text{ind}}(\omega_n,\mathbf{p}) = \frac{U_0}{1-\Pi^*(\omega_n,\mathbf{p})U_0}, \qquad (8)$$

there is a key dynamically mirrored *minus* sign difference in the denominators of Eqs. (6) and (8), which leads to the quite different physical motivations and calculational details.

III. THERMODYNAMICS WITH GENERALIZED DYSON-SCHWINGER EQUATIONS

A. Full fermion propagator and energy density

The full fermion propagator $\mathcal{G}(\omega_n,\mathbf{k})$ given with the instantaneously antiscreening interaction matrix tells us all the information of the ground state energy or quasiparticle distribution function. It is calculated with the proper self-energy Σ^* through the instantaneous approximation as indicated by Fig. 1(a)

$$\mathcal{G}_{\uparrow} = \mathcal{G}_{0\uparrow} + \mathcal{G}_{0\uparrow}\Sigma^*\mathcal{G}_{\uparrow}, \tag{9}$$

$$\Sigma^*(0,0) = U^*_{\mathrm{eff}}(0,|\mathbf{k}| \to 0)n_{\downarrow} = U^*_{\mathrm{eff}}[n]\frac{n}{2}. \tag{10}$$

Here, n is the total particle number density $n=n_{\uparrow}+n_{\downarrow}=2n_{\uparrow}$ for the unpolarized and fully pairing symmetric trapped dilute atomic Fermi gas.

The self-energy $\Sigma^*(0,0)$ is four-momentum independent in the instantaneous approximation. This means that the self-energy will shift the chemical potential of the particle distribution function; i.e., the correlation contributions to thermodynamics are characterized through the phase space energy-momentum distribution deformation.

From the full fermion propagator with Eq. (1), the particle number and energy densities are given by [33]

$$\frac{N}{V} = T\int_k \sum_n e^{i\omega_n \eta}\mathrm{tr}\mathcal{G}(\omega_n,\mathbf{k}), \tag{11}$$

$$\frac{E}{V} = T\int_k \sum_n e^{i\omega_n \eta}\frac{1}{2}\left(i\omega_n + \frac{\mathbf{k}^2}{2m} + \tilde{\mu}\right)\mathrm{tr}\mathcal{G}(\omega_n,\mathbf{k}). \tag{12}$$

Although we once again consider the linear contact formalism, the lowest order diagrams characterize the thermodynamics with the highly nonlinear/turbulent contributions of the density and temperature fluctuations due to taking the medium dependent interaction potential [27].

After summing the Matsubara discrete frequency ω_n with the residual theorem, the particle density $n=N/V$ and energy density $\epsilon=E/V$ are reduced to the compact formalisms

$$n = 2\int_k f_{\mathbf{k}} = -2\left(\frac{mT}{2\pi}\right)^{3/2}\mathrm{Li}_{3/2}[y], \tag{13a}$$

$$\begin{aligned}\epsilon &= 2\int_k \frac{\mathbf{k}^2}{2m}f_{\mathbf{k}} + \frac{\pi a_{\mathrm{eff}}}{m}n(T,\lambda)^2 \\ &= -3T\left(\frac{mT}{2\pi}\right)^{3/2}\mathrm{Li}_{5/2}[y] + \frac{\pi a_{\mathrm{eff}}}{m}n^2,\end{aligned} \tag{13b}$$

with the entropy density $s=S/V$

$$\begin{aligned}s &= -2\int_k [f_{\mathbf{k}}\ln f_{\mathbf{k}} + (1-f_{\mathbf{k}})\ln(1-f_{\mathbf{k}})] \\ &= \frac{1}{T}\left(\frac{mT}{2\pi}\right)^{3/2}(2\lambda\mathrm{Li}_{3/2}[y] - 5T\mathrm{Li}_{5/2}[y]).\end{aligned} \tag{14}$$

The in-medium effective scattering length is

$$a_{\mathrm{eff}} = \frac{a}{1 - \frac{2\pi m_D^2}{m}a}. \tag{15}$$

In deriving the above equations, we have defined the quasiparticle Fermi-Dirac distribution function with the total (effective) chemical potential λ through Fig. 1(a) with Eq. (9)

$$f_{\mathbf{k}} = \frac{1}{e^{\beta(\mathbf{k}^2/2m-\lambda)}+1}, \tag{16}$$

$$\lambda = \tilde{\mu} - \Sigma^*(0,0), \tag{17}$$

with $\beta=1/T$ being the inverse temperature.

The expressions for the entropy density Eq. (14) and the first energy density term in Eq. (13b) are similar to those of the noninteracting fermion gas. The total chemical potential λ is determined for the given particle number density n and temperature T. The above algebra equations are completely closed.

B. Physical pressure and system chemical potential with rearrangement

From the Helmholtz free energy density $F/V=f=\epsilon-Ts$, the pressure is given by

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = -\left(\frac{\partial\frac{F}{N}}{\partial\frac{V}{N}}\right)_T = n^2\left(\frac{\partial\frac{f}{n}}{\partial n}\right)_T. \tag{18}$$

At $T=0$ with entropy density $s=0$, P is reduced to the important formalism

$$P = n^2\frac{\partial}{\partial n}\left(\frac{\epsilon}{n}\right). \tag{19}$$

The equation (19) determines the fundamental relation between the energy density and pressure according to the HvH theorem [26], which plays a vital role for characterizing the symmetric nuclear matter saturation property.

As widely discussed in the nuclear many-body literature, the HvH theorem constraint will lead to the rearrangement terms in the pressure and chemical potential dealt with any kind of in-medium interaction potential [27–30]. Presently, their exact analytical expressions can be derived in a thermodynamic way instead of doing the numerical interpolation [34].

Let us further address the pressure P. From Eq. (18) given with the medium-dependent interaction potential, one can have

$$P = \frac{\pi a_{\mathrm{eff}}}{m}n^2 + \mathcal{C}(T,\lambda)\left(\frac{2\pi a_{\mathrm{eff}}}{m}\right)^2 n^3 + P_{\mathrm{ideal}}(T,\lambda), \tag{20}$$

with

$$P_{\text{ideal}}(T,\lambda) = -2T\left(\frac{mT}{2\pi}\right)^{3/2}\text{Li}_{5/2}[y]. \tag{21}$$

In Eq. (20), the first term represents the expected direct contribution of the induced interaction, while the last one results from the Fermi kinetic energy. The essential second term is due to the rearrangement contributions, which can be easily neglected. It is worth noting that this rearrangement term is vanishing within the standard mean field theory approach [34].

The discussion for the physical system chemical potential/multiplier is in line with that of pressure

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = \left(\frac{\partial \frac{F}{V}}{\partial \frac{N}{V}}\right)_{T} = \left(\frac{\partial(\epsilon - Ts)}{\partial n}\right)_{T}. \tag{22}$$

Its analytical expression can be further reduced to

$$\mu = \frac{2\pi a_{\text{eff}}}{m}n + \mathcal{C}(T,\lambda)\left(\frac{2\pi a_{\text{eff}}}{m}\right)^2 n^2 + \lambda. \tag{23}$$

There is an additional (second) term in the system chemical potential resulting from the intriguing rearrangement effect of the single particle degrees of freedom analogous to that of pressure.

Explicitly, the rearrangement terms in the pressure Eq. (20) and chemical potential (23) are similar. For the general finite temperature occasion, the rearrangement factor $\mathcal{C}(T,\lambda)$ in Eq. (20) and Eq. (23) is given by

$$\mathcal{C}(T,\lambda) \equiv m_D\left(\frac{\partial m_D}{\partial n}\right)_T = \frac{\text{Li}_{-1/2}[y]}{2T\text{Li}_{1/2}[y]}. \tag{24}$$

From the pressure and chemical potential analytical expressions, one can see that the rearrangement effects play the positive "repulsive" role. These effects prevent the system from collapsing in the strongly coupling attractive unitary limit. Furthermore, the in-medium spontaneously generated rearrangement effects are exactly counteracted (easily neglected) in the Helmholtz free energy density $f=\epsilon-Ts=-p+\mu n$. This reflects an important physical fact that the in-medium off-shell (spontaneous Lorentz violation) formalism can approach the quantum correlation effects on the universal thermodynamics, while the thermodynamics first law constraint ensures the energy conservation. Exactly, the classical thermodynamic consistencies are satisfied with the generalized self-consistent DSEs.

IV. UNIVERSAL THERMODYNAMICS IN THE UNITARY LIMIT WITH $T=0$

From the general analytical expressions, one can give the $T=0$ results with the in-medium interaction

$$U_{\text{eff}}^{*} = \frac{U_0}{1 - \frac{k_f m}{2\pi^2}U_0} \rightarrow a_{\text{eff}} = \frac{a}{1 - \frac{2k_f a}{\pi}}. \tag{25}$$

With $n=k_f^3/(3\pi^2)$ and the rearrangement factor

$$\mathcal{C}(\epsilon_f) = m_D\frac{\partial m_D}{\partial n} = \frac{1}{4\epsilon_f}, \tag{26}$$

the physical internal energy density (13b), pressure (20), and system chemical potential (23) are

$$\frac{E}{N\epsilon_f} = \frac{1}{3}\frac{2ak_f}{\pi - 2ak_f} + \frac{3}{5}, \tag{27a}$$

$$\frac{PV}{N\epsilon_f} = \frac{1}{3}\frac{2ak_f}{\pi - 2ak_f} + \frac{1}{9}\left(\frac{2ak_f}{\pi - 2ak_f}\right)^2 + \frac{2}{5}, \tag{27b}$$

$$\frac{\mu}{\epsilon_f} = \frac{2}{3}\frac{2ak_f}{\pi - 2ak_f} + \frac{1}{9}\left(\frac{2ak_f}{\pi - 2ak_f}\right)^2 + 1. \tag{27c}$$

In the strongly interacting universal unitary limit $|a|\rightarrow\infty$, the ratios of energy density, pressure, and chemical potential to those of noninteracting fermion gas are $\xi=\frac{4}{9}$. This $\xi=\frac{4}{9}$ is exactly consistent with the existing theoretical results [8,13,15] and in agreement with the updated experimental ones [17–19]. In the weak coupling limit, the analytical thermodynamic expressions readily return to the lowest order mean field theory results which neglect the quantum fluctuation/correlation contributions.

At unitarity and with $T=0$, the pressure is

$$P = \frac{4}{45m}(3\pi^2)^{2/3}n^{1+2/3} = \xi P_{\text{ideal}}, \tag{28}$$

with which the sound speed is calculated to be

$$v = \sqrt{\frac{\partial P}{m\partial n}} = \frac{2}{3}v_{\text{FG}} = \sqrt{\xi}v_{\text{FG}}. \tag{29}$$

The $v_{\text{FG}}=\sqrt{\frac{1}{3}}v_f$ is that for the ideal noninteracting fermion gas with $v_f=k_f/m$ being the Fermi speed. Detailed information of the sound speed of the homogeneous system is very crucial for determining the frequencies of collective modes in the realistic trapped systems.

With $T=0$, the ratio of pressure to binding energy is still $\frac{2}{3}$ of the noninteracting fermion gas in the unitary limit as pointed out in the literature [4]. The HvH theorem ensures the thermodynamic consistency dealt with the in-medium interaction potential in addressing the strongly interacting physics. The same ratio is counterintuitive to some extent, which indicates that the universal thermodynamic properties with the infrared long range quantum fluctuation/correlation in the strongly interacting limit are extremely profound.

The numerical results for $T\neq 0$ can be readily given in the unitary limit regime with the explicit analytical expressions, from which one can see that the collective strongly correlating and rearrangement effects influence the chemical potential and pressure simultaneously. However, the internal energy and Helmholtz free energy densities do not directly rely on the intriguing rearrangement effect even with a careful numerical check.

V. DISCUSSION AND PROSPECTIVE

In addressing the unitary fermion physics, the essential task is how to incorporate the significant collective density and temperature fluctuation/correlation contributions into the thermodynamics in a reasonably controllable way.

The physical connotations are crystal clear in this analytical attempt: On the one hand, the many-body surrounding environment modulates the bare two-body interaction, which is characterized by the in-medium interaction matrix $U^*_{\text{eff}}[n,T]$. On the other hand, the collective correlation effects further shift the single particle energy spectrum, which is described by the corresponding single particle correlation potential $\mu_r[n,T]$. The crucial energy-momentum conservation law is guaranteed by the simultaneous consideration of these two collective correlation effects. The fundamental thermodynamic consistency is the basic validity requirement of any scientific theoretical approaches, which can be easily neglected.

The generic infrared singularity of the bare two-body scattering amplitude at unitarity provides the scaling basis, which leads to the thermodynamic universal hypothesis. This scaling property implies that the thermodynamic expressions can be very simple in principle; i.e, the analytical formulas should be similar to those for the ideal noninteracting fermion gas. In turn, this infrared singularity makes any potential theoretical approaches nontrivial. Essentially, the unitary characteristic rules out any naive expansions or conventional perturbative loop diagram ring and ladder resummation calculations.

In this work, we have obtained the $T=0$ dimensionless universal coefficient $\xi=\frac{4}{9}$ for the $D=3$ unitary fermion gas. It is in the regime of the reasonable simulation and experimental values 0.42–0.46. However, as discussed recently in the literature [35,36], it is yet far from the ending of looking for its realistic exact value.

The exact determination of this fundamental constant will influence the detailed knowledge about the collective excitation modes measured in the realistic ultracold unitary atomic fermions. Consequently, more and more theoretical/experimental efforts are in fact most expected to make it certain precisely. To understand the comprehensive thermodynamic properties exhibited by these novel trapped strongly interacting quantum systems, there is still a faraway route before us to constitute a consistent microscopic theory and corresponding analytical calculational framework.

VI. CONCLUSION

In summary, based on the Dyson-Scwhinger equations formalism, the ground state of the $D=3$ ultracold unitary fermion gas has been calibrated in detail in terms of the quantum Ising-like Bethe-Peierls contact interaction model. The universal factor is calculated to be $\xi=\frac{4}{9}$.

In addressing the universal resonant thermodynamics, the medium-dependent interaction through a dynamically twisted minus sign caused by an assumed mirror background makes it possible for us to incorporate the Landau pole contributions with the Fermi surface instability. With the spontaneously generated rearrangements due to the single particle energy spectrum shift, the additional instantaneous correlation potential ensures the classical thermodynamic consistency. Finally, we want to emphasize that there are not any infrared–ultraviolet cutoffs or regularization–expansion parameters obviously violating the significant unitarity; i.e., the bare U_0 drops out in the unitary limit.

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